# Clones and Relational Clones on Finite Sets

Let $A$ be a fixed set. The set of all $n$-ary operations on $A$ will be denoted by $\mathrm{Op}^{(n)}$, and the set of all $n$-ary relations on $A$ will be denoted by $\mathrm{Rel}^{(n)}$. We will use the symbols Op and Rel to denote the graded set of all finitary operations and the graded set of all finitary relations on $A$, respectively.

For an operation $f$ and a relation $\rho$ on $A$ we say that $f$ **preserves** $\rho$, or $\rho$ **is compatible with** $f$, and write $f \perp \rho$, if $f(\rho, \rho, \ldots) \subseteq \rho$. Equivalently, $f$ preserves $\rho$ iff $\rho$ is a subalgebra of the algebra $\langle A; f\rangle^n$ where $n$ is the arity of $\rho$. Compatibility of operations and relations defines a **Galois connection** between subsets of Op and Rel as follows:

$$\begin{array}{rcl} \mathrm{Op} & \rightleftharpoons & \mathrm{Rel} \\ F & \mapsto & F^\perp = \{\rho \in \mathrm{Rel} : f \perp \rho \text{ for all } f \in F\} \\ \{f \in \mathrm{Op} : f \perp \rho \text{ for all } \rho \in R\} = R^\perp & \leftarrow\!\shortmid & R \end{array}$$

that is,

(1) $F_1 \subseteq F_2 \;\Rightarrow\; F_1^\perp \supseteq F_2^\perp$ for all $F_1, F_2 \subseteq \mathrm{Op}$,
(2) $R_1 \subseteq R_2 \;\Rightarrow\; R_1^\perp \supseteq R_2^\perp$ for all $R_1, R_2 \subseteq \mathrm{Rel}$,
(3) $F \subseteq F^{\perp\perp}$ for all $F \subseteq \mathrm{Op}$, and
(4) $R \subseteq R^{\perp\perp}$ for all $R \subseteq \mathrm{Rel}$.

Hence also

(5) $F^{\perp\perp\perp} = F^\perp$ for all $F \subseteq \mathrm{Op}$, and
(6) $R^{\perp\perp\perp} = R^\perp$ for all $R \subseteq \mathrm{Rel}$,

implying that

(7) $F \mapsto F^{\perp\perp}$ is a closure operator on Op,
i.e., (3) holds and $(F^{\perp\perp})^{\perp\perp} = F^{\perp\perp}$ for all $F \subseteq \mathrm{Op}$, and
(8) $R \mapsto R^{\perp\perp}$ is a closure operator on Rel.
i.e., (4) holds and $(R^{\perp\perp})^{\perp\perp} = R^{\perp\perp}$ for all $R \subseteq \mathrm{Op}$.

A set $C \subseteq \mathrm{Op}$ of operations is called **Galois closed** if $C = C^{\perp\perp}$, and a set $K \subseteq \mathrm{Rel}$ of relations is called **Galois closed** if $K = K^{\perp\perp}$.

(9) $F \subseteq \mathrm{Op}$ is Galois closed if and only if $C = R^\perp$ for some $R \subseteq \mathrm{Rel}$, and
(10) $K \subseteq \mathrm{Rel}$ is Galois closed if and only if $K = F^\perp$ for some $F \subseteq \mathrm{Op}$.

Moreover,

(11) The assignments $C \mapsto C^\perp$ and $K \mapsto K^\perp$ define order reversing bijections between the family of Galois closed sets of operations and the family of Galois closed sets of relations, which are inverses of each other.

Now we will discuss how the Galois closed sets of operations and relations can be characterized internally, that is, by closure under certain operations on Op and Rel, respectively. This characterization is especially simple if $A$ is finite. Since this is the case that is relevant to CSP, we will state the characterization theorems only for finite $A$.

**Definition 1.** For arbitrary set $A$ we define a multi-sorted algebra $\mathbf{Op}$ on Op which has the following operations:

- Distinguished elements (0-ary operations): the projection operations $p_i^{(n)} \in \mathrm{Op}^{(n)}$ $(1 \le i \le n)$ defined by $p_i^{(n)}(\overline{a}) = a_i$ for all $\overline{a} = (a_1, \dots, a_n) \in A^n$.
- Compositions: $\mathrm{Op}^{(m)} \times (\mathrm{Op}^{(n)})^m \to \mathrm{Op}^{(n)}$, $(f, g_1, g_2 \dots, g_m) \mapsto f(g_1, g_2, \dots, g_m)$ where $f(g_1, g_2, \dots, g_m)$ is defined by
$$f(g_1, g_2, \dots, g_m)(\overline{a}) = f(g_1(\overline{a}), g_2(\overline{a}), \dots, g_m(\overline{a})) \qquad \text{for all } \overline{a} \in A^n.$$

The subalgebras of $\mathbf{Op}$ are called *clones* (of operations) on $A$.

**Theorem 1.** *Let $A$ be a finite set. TFAE for arbitrary $C \subseteq \mathrm{Op}$:*
(a) *$C$ is Galois closed;*
(b) *$C$ is a clone;*
(c) *$C$ is the clone* $\mathrm{Clo}\,\mathbf{A}$ *of term operations of an algebra $\mathbf{A} = \langle A; F \rangle$.*

**Definition 2.** For a finite set $A$ we define a multi-sorted algebra $\mathbf{Rel}$ on Rel which has the following operations:

- Distinguished element (0-ary operation): the equality relation $=$ in $\mathrm{Rel}^{(2)}$.
- Cartesian product: $\mathrm{Rel}^{(m)} \times \mathrm{Rel}^{(n)} \to \mathrm{Rel}^{(m+n)}$, $(\rho, \sigma) \mapsto \rho \times \sigma$.
- Projection onto a list $(i_1, \dots, i_n)$ of coordinates (where $i_1, \dots, i_n$ are distinct elements of $\{1, \dots, m\}$): $\mathrm{Rel}^{(m)} \to \mathrm{Rel}^{(n)}$, $\rho \mapsto \pi_{i_1,\dots,i_n}(\rho)$ where
$$\pi_{i_1,\dots,i_n}(\rho) = \{(a_{i_1}, \dots, a_{i_n}) \in A^n : (a_1, \dots, a_m) \in \rho\}.$$
If $m = n$, then the operation $\pi_{i_1,\dots,i_n}()$ performs a permutation of coordinates.
- Intersection of relations of the same arity $n$: $(\mathrm{Rel}^{(n)})^2 \to \mathrm{Rel}^{(n)}$, $(\rho, \sigma) \mapsto \rho \cap \sigma$.

The subalgebras of $\mathbf{Rel}$ are called *relational clones* on $A$.

**Theorem 2.** *Let $A$ be a finite set. TFAE for arbitrary $K \subseteq \mathrm{Rel}$:*
(a) *$K$ is Galois closed;*
(b) *$K$ is a relational clone;*
(c) *$K$ contains every relation $\rho \in \mathrm{Rel}$ such that $\rho$ is definable by a pp-formula in the relational structure $\langle A, K \rangle$.*

A *pp-formula* (*positive primitive formula*) in the language of $\langle A, K \rangle$ is a first order formula involving only the logical symbols $\exists$, $\wedge$, $=$, and symbols for the relations in $K$.