of the form $f_U(x,y) \approx f_V(x,y)$ so that $\mathcal{B}(f)$ has no closed, proper, nonempty lattice filter.

The term p from Example 2.13 is a Hobby–McKenzie term, and therefore also a Taylor term. To see this, suppose that $\mathcal{F}$ is a closed, proper, nonempty lattice filter of $\mathcal{B}(p)$. Then since $\mathcal{F}$ is a nonempty filter it contains the top element N. Since $\mathcal{F}$ is closed it must contain $\{1\}$ ($\equiv_E N$) and also $\{3\}$ ($\equiv_E N$). Since $\mathcal{F}$ is a lattice filter, it contains $\{1\} \cap \{3\} = \emptyset$. But any lattice filter containing $\emptyset$ is improper.

2.5. The Term Condition

Let $\mathbf{A} = \langle A; F\rangle$ be an algebra. An n-ary relation $R \subseteq A^n$ is **compatible** if it is a subalgebra of $\mathbf{A}^n$. If $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, then the **restriction** of a relation $R \subseteq A^n$ to $\mathbf{B}$ is $R|_{\mathbf{B}} := R \cap B^n$. If δ is a congruence on $\mathbf{A}$, then $R/\delta := \{(a_1/\delta, \ldots, a_n/\delta) \mid (a_1, \ldots, a_n) \in R\}$. Both $R|_{\mathbf{B}}$ and R/δ are compatible if R is. If δ is a congruence, then R is **δ-closed** if $R = \delta \circ R \circ \delta$. (I.e., if $\mathbf{a}\ \delta\ \mathbf{b}\ R\ \mathbf{c}\ \delta\ \mathbf{d}$ implies $\mathbf{a}\ R\ \mathbf{d}$.)

A compatible, reflexive, symmetric binary relation is called a **tolerance**. We will usually denote tolerances by upper case italic letters: $R, S, T, \ldots$. A compatible equivalence relation (i.e., a transitive tolerance) is a **congruence**, and congruences will usually be denoted by lower case Greek letters: $\alpha, \beta, \gamma, \ldots$. The tolerance or congruence generated by set $X \subseteq A \times A$ is usually denoted by $\mathrm{Tg}^{\mathbf{A}}(X)$ or $\mathrm{Cg}^{\mathbf{A}}(X)$ respectively, although if X contains only a few pairs then we may write, for example, $\mathrm{Cg}^{\mathbf{A}}(a,b)$ or $\mathrm{Cg}^{\mathbf{A}}((a,b),(c,d))$ instead. If T is a tolerance on $\mathbf{A}$, then a maximal subset $B \subseteq A$ such that $B \times B \subseteq T$ is called a **block** of T. If T is in fact a congruence, then a block is the same thing as a congruence class. A tolerance or congruence is **trivial** if it is the equality relation and **nontrivial** otherwise.

The collection of congruences on $\mathbf{A}$, ordered by inclusion, is an algebraic lattice which is denoted $\mathbf{Con}(\mathbf{A})$. Its least and largest elements are denoted 0 and 1. Meet and join are denoted $\wedge$ and $\vee$ and are computed by $\alpha \wedge \beta = \alpha \cap \beta$ and $\alpha \vee \beta = \mathrm{tr.cl.}(\alpha \cup \beta)$ where tr.cl. represents transitive closure.

An m-ary **polynomial operation** of $\mathbf{A}$ is an operation $f\colon A^m \to A$ such that $f(x_1, \ldots, x_m) = t^{\mathbf{A}}(x_1, \ldots, x_m, \mathbf{a})$ for some $(m+n)$-ary term t and some tuple $\mathbf{a} \in A^n$.

If S and T are tolerances on $\mathbf{A}$, then an **S,T-matrix** is a 2×2 matrix of elements of $\mathbf{A}$ of the form

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} f(\mathbf{a},\mathbf{u}) & f(\mathbf{a},\mathbf{v}) \\ f(\mathbf{b},\mathbf{u}) & f(\mathbf{b},\mathbf{v}) \end{bmatrix}$$

where $f(\mathbf{x}, \mathbf{y})$ is an $(m+n)$-ary polynomial of $\mathbf{A}$, $\mathbf{a}\ S\ \mathbf{b}$, and $\mathbf{u}\ T\ \mathbf{v}$. The set of all S, T-matrices is denoted $M(S, T)$.

Since tolerances are compatible with all polynomial operations, any two elements in the same row of an S, T-matrix are T-related and any two elements in the same column are S-related.

The fact that S and T are symmetric relations implies that $M(S, T)$ is closed under interchanging rows or columns:

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M(S,T) \Leftrightarrow \begin{bmatrix} r & s \\ p & q \end{bmatrix} \in M(S,T) \Leftrightarrow \begin{bmatrix} s & r \\ q & p \end{bmatrix} \in M(S,T)\,.$$

If $S = T$, then $M(S, T) = M(T, T)$ is also closed under transpose, as one sees by interchanging the roles of $\mathbf{x}$ and $\mathbf{y}$ in the polynomial $f(\mathbf{x}, \mathbf{y})$ that defines a given matrix.

Definition 2.18. Let S and T be tolerances on an algebra $\mathbf{A}$, and let δ be a congruence on $\mathbf{A}$. If $p \equiv_\delta q$ implies that $r \equiv_\delta s$ whenever

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M(S,T), \tag{2.5}$$

then we say that $\mathbf{C}(S, T; \delta)$ **holds**, or S **centralizes** T **modulo** δ.

By interchanging the rows of matrices one sees that $\mathbf{C}(S, T; \delta)$ holds if and only if

$$p \equiv_\delta q \quad \Longleftrightarrow \quad r \equiv_\delta s$$

for every S, T-matrix in (2.5).

The S, T**-term condition** is the condition $\mathbf{C}(S, T; 0)$. There are other similar conditions called term conditions that we will meet later, but this is the original one.

When establishing that the implication defining $\mathbf{C}(S, T; \delta)$ holds, or when making use of the fact, we may use underlining to highlight places in equations or expressions where changes are to be made. For example, we may write the implication defining $\mathbf{C}(S, T; \delta)$ in the following form: If

$$f(\underline{\mathbf{a}}, \mathbf{u}) \equiv_\delta f(\underline{\mathbf{a}}, \mathbf{v}),$$

then

$$f(\underline{\mathbf{b}}, \mathbf{u}) \equiv_\delta f(\underline{\mathbf{b}}, \mathbf{v})\,.$$

The relation $\mathbf{C}(\ \ ,\ \ ;\ \ )$ is called the **centralizer relation**. The reason that this terminology is used is that when $\mathbf{A}$ is a group and S, T and δ are congruences on $\mathbf{A}$, then $\mathbf{C}(S, T; \delta)$ holds if and only if $[S, T] \leq \delta$ (see Chapter 1 of [**19**]).

The basic properties of the centralizer relation are enumerated in the following theorem.

THEOREM 2.19. *Let* $\mathbf{A}$ *be an algebra with tolerances* S, S', T, T' *and congruences* $\alpha, \alpha_i, \beta, \delta, \delta', \delta_j$. *The following are true.*

(1) (Monotonicity in the first two variables) *If* $\mathbf{C}(S, T; \delta)$ *holds and* $S' \subseteq S$, $T' \subseteq T$, *then* $\mathbf{C}(S', T'; \delta)$ *holds.*
(2) $\mathbf{C}(S, T; \delta)$ *holds if and only if* $\mathbf{C}(\mathrm{Cg}^{\mathbf{A}}(S), T; \delta)$ *holds.*
(3) $\mathbf{C}(S, T; \delta)$ *holds if and only if* $\mathbf{C}(S, \delta \circ T \circ \delta; \delta)$ *holds.*
(4) *If* $T \cap \delta = T \cap \delta'$, *then* $\mathbf{C}(S, T; \delta)$ *holds if and only if* $\mathbf{C}(S, T; \delta')$ *holds.*
(5) (Semidistributivity in the first variable) *If* $\mathbf{C}(\alpha_i, T; \delta)$ *holds for all* $i \in I$, *then* $\mathbf{C}(\bigvee_{i \in I} \alpha_i, T; \delta)$ *holds.*
(6) *If* $\mathbf{C}(S, T; \delta_j)$ *holds for all* $j \in J$, *then* $\mathbf{C}(S, T; \bigwedge_{j \in J} \delta_j)$ *holds.*
(7) *If* $T \cap (S \circ (T \cap \delta) \circ S) \subseteq \delta$, *then* $\mathbf{C}(S, T; \delta)$ *holds.*
(8) *If* $\beta \wedge (\alpha \vee (\beta \wedge \delta)) \leq \delta$, *then* $\mathbf{C}(\alpha, \beta; \delta)$ *holds.*
(9) *Let* $\mathbf{B}$ *be a subalgebra of* $\mathbf{A}$. *If* $\mathbf{C}(S, T; \delta)$ *holds in* $\mathbf{A}$, *then* $\mathbf{C}(S|_{\mathbf{B}}, T|_{\mathbf{B}}; \delta|_{\mathbf{B}})$ *holds in* $\mathbf{B}$.
(10) *If* $\delta' \leq \delta$, *then the relation* $\mathbf{C}(S, T; \delta)$ *holds in* $\mathbf{A}$ *if and only if* $\mathbf{C}(S/\delta', T/\delta'; \delta/\delta')$ *holds in* $\mathbf{A}/\delta'$.

PROOF. Item (1) follows from the fact that $M(S', T') \subseteq M(S, T)$.

For (2), $\mathbf{C}(\mathrm{Cg}^{\mathbf{A}}(S), T; \delta) \Longrightarrow \mathbf{C}(S, T; \delta)$ follows from (1), since $S \subseteq \mathrm{Cg}^{\mathbf{A}}(S)$. For the reverse implication (and also for the proof of item (5)), we will argue that if S_i, is a tolerance, $\mathbf{C}(S_i, T; \delta)$ holds for all $i \in I$, and $\alpha :=$ tr.cl. $\left(\bigcup_{i \in I} S_i\right)$, then $\mathbf{C}(\alpha, T; \delta)$. (To complete the proof of (2) we need this only when $|I| = 1$, while in (5) we need it only when the S_i are congruences.)

Choose any matrix in $M(\alpha, T)$. If it is

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} f(\mathbf{a}, \mathbf{u}) & f(\mathbf{a}, \mathbf{v}) \\ f(\mathbf{b}, \mathbf{u}) & f(\mathbf{b}, \mathbf{v}) \end{bmatrix},$$

then $\mathbf{a}$ is related to $\mathbf{b}$ by tr.cl. $\left(\bigcup_{i \in I} S_i\right)$, so there exist tuples $\mathbf{a} = \mathbf{a}_0 \; S_{i_1} \; \mathbf{a}_1 \; S_{i_2} \; \cdots \; S_{i_n} \; \mathbf{a}_n = \mathbf{b}$. These tuples determine matrices

$$\begin{bmatrix} p_k & q_k \\ p_{k+1} & q_{k+1} \end{bmatrix} := \begin{bmatrix} f(\mathbf{a}_k, \mathbf{u}) & f(\mathbf{a}_k, \mathbf{v}) \\ f(\mathbf{a}_{k+1}, \mathbf{u}) & f(\mathbf{a}_{k+1}, \mathbf{v}) \end{bmatrix} \in M(S_{i_{k+1}}, T).$$

We must show that $p \equiv_\delta q$ implies $r \equiv_\delta s$, so assume that $p \equiv_\delta q$. This is the same as $p_0 \equiv_\delta q_0$, and so by induction (using that $\mathbf{C}(S_{i_k}, T; \delta)$ holds for all k) we get that $p_k \equiv_\delta q_k$ for all k. Therefore $r = p_n \equiv_\delta q_n = s$. This completes the proofs of (2) and (5).

For (3), the implication $\mathbf{C}(S, \delta \circ T \circ \delta; \delta) \Longrightarrow \mathbf{C}(S, T; \delta)$ follows from (1), since $T \subseteq \delta \circ T \circ \delta$. For the reverse implication, assume that

$\mathbf{C}(S,T;\delta)$ holds, that

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} = \begin{bmatrix} f(\mathbf{a},\mathbf{u}) & f(\mathbf{a},\mathbf{v}) \\ f(\mathbf{b},\mathbf{u}) & f(\mathbf{b},\mathbf{v}) \end{bmatrix} \in M(S, \delta \circ T \circ \delta),$$

and that $p \equiv_\delta q$. There exist tuples $\mathbf{u}'$ and $\mathbf{v}'$ such that $\mathbf{u}\ \delta\ \mathbf{u}'\ T\ \mathbf{v}'\ \delta\ \mathbf{v}$. The matrix

$$\begin{bmatrix} p' & q' \\ r' & s' \end{bmatrix} = \begin{bmatrix} f(\mathbf{a},\mathbf{u}') & f(\mathbf{a},\mathbf{v}') \\ f(\mathbf{b},\mathbf{u}') & f(\mathbf{b},\mathbf{v}') \end{bmatrix}$$

is an S,T-matrix. Moreover,

$$p' = f(\mathbf{a},\mathbf{u}')\ \delta\ f(\mathbf{a},\mathbf{u}) = p\ \delta\ q = f(\mathbf{a},\mathbf{v})\ \delta\ f(\mathbf{a},\mathbf{v}') = q'.$$

Since $\mathbf{C}(S,T;\delta)$ holds, it follows that $r' \equiv_\delta s'$. Hence

$$r = f(\mathbf{b},\mathbf{u})\ \delta\ f(\mathbf{b},\mathbf{u}') = r'\ \delta\ s' = f(\mathbf{b},\mathbf{v}')\ \delta\ f(\mathbf{b},\mathbf{v}) = s,$$

or $r \equiv_\delta s$. This establishes $\mathbf{C}(S, \delta \circ T \circ \delta; \delta)$.

For (4), recall that elements in the same row of an $M(S,T)$ are T-related. Therefore, if $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M(S,T)$, then since $T \cap \delta = T \cap \delta'$ we get that

$$p \equiv_\delta q \iff p \equiv_{T\cap\delta} q \iff p \equiv_{T\cap\delta'} q \iff p \equiv_{\delta'} q,$$

and

$$r \equiv_\delta s \iff r \equiv_{T\cap\delta} s \iff r \equiv_{T\cap\delta'} s \iff r \equiv_{\delta'} s.$$

Therefore the implication $p \equiv_\delta q \Longrightarrow r \equiv_\delta s$ is equivalent to $p \equiv_{\delta'} q \Longrightarrow r \equiv_{\delta'} s$.

For (6), assume that $\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M(S,T)$. If $p \equiv q \pmod{\bigwedge \delta_j}$, then $p \equiv q \pmod{\delta_j}$ for all j. Since $\mathbf{C}(S,T;\delta_j)$ holds for all j we get that $r \equiv s \pmod{\delta_j}$ for all j, or equivalently that $r \equiv s \pmod{\bigwedge \delta_j}$. This shows that $\mathbf{C}(S,T;\bigwedge_{j\in J}\delta_j)$ holds.

For (7), choose an S,T-matrix $M = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$. Assume that $p \equiv_\delta q$. Since the elements in the same row of M are T-related and the elements in the same column are S-related, we have $r\ S\ p\ T\cap\delta\ q\ S\ s$. Moreover, $r\ T\ s$ since these elements belong to the same row. Together this yields that $r\ T \cap (S \circ (T \cap \delta) \circ S)\ s$. By the assumption in (7), this implies that $r \equiv_\delta s$. This proves (7).

For item (8), if $\beta \wedge (\alpha \vee (\beta \wedge \delta)) \leq \delta$, then $\beta \cap (\alpha \circ (\beta \cap \delta) \circ \alpha) \leq \delta$, so $\mathbf{C}(\alpha, \beta; \delta)$ holds by (7).

Item (9) holds because any instance of the implication in Definition 2.18 defining $\mathbf{C}(S|_{\mathbf{B}}, T|_{\mathbf{B}}; \delta|_{\mathbf{B}})$ in $\mathbf{B}$ is an instance of the implication defining $\mathbf{C}(S,T;\delta)$ in $\mathbf{A}$.

For item (10), it suffices to observe that, when $\delta' \leq \delta$,

$$\begin{bmatrix} p'/\delta' & q'/\delta' \\ r'/\delta' & s'/\delta' \end{bmatrix} \in M(S/\delta', T/\delta')$$

if and only if there exist $p \equiv_{\delta'} p'$, $q \equiv_{\delta'} q'$, $r \equiv_{\delta'} r'$, and $s \equiv_{\delta'} s'$ with

$$\begin{bmatrix} p & q \\ r & s \end{bmatrix} \in M(S, T),$$

and that $p \equiv_{\delta} q \Leftrightarrow p'/\delta' \equiv_{\delta/\delta'} q'/\delta'$ and $r \equiv_{\delta} s \Leftrightarrow r'/\delta' \equiv_{\delta/\delta'} s'/\delta'$. $\square$

Definition 2.20. The **commutator** of S and T, denoted by $[S, T]$, is the least congruence δ such that $\mathbf{C}(S, T; \delta)$ holds. T is **abelian** if $[T, T] = 0$. An algebra $\mathbf{A}$ is **abelian** if its largest congruence is.

By Theorem 2.19 (6), the class of all δ such that $\mathbf{C}(S, T; \delta)$ holds is closed under complete meet, so there is a least such δ. This implies that $[S, T]$ exists for any two tolerances S and T.

It is a well known fact, easily derivable from the definitions, that $\mathbf{A}$ is abelian if and only if the diagonal of $A \times A$ is a class of a congruence of $\mathbf{A} \times \mathbf{A}$.

Definition 2.21. The **centralizer of T modulo δ**, denoted by $(\delta : T)$, is the largest congruence α on $\mathbf{A}$ such that $\mathbf{C}(\alpha, T; \delta)$ holds.

By Theorem 2.19 (5), the class of all α such that $\mathbf{C}(\alpha, T; \delta)$ holds is closed under complete join, so there is a largest such α. This implies that $(\delta : T)$ exists for every δ and T. By Theorem 2.19 (2), the centralizer $(\delta : T)$ contains every tolerance S such that $\mathbf{C}(S, T; \delta)$ holds.

2.6. Congruence Identities

If $\mathcal{V}$ is a variety of algebras, then any lattice identity that holds in the class $\{\mathbf{Con}(\mathbf{A}) \mid \mathbf{A} \in \mathcal{V}\}$ of congruence lattices of algebras in $\mathcal{V}$ is called a **congruence identity** of $\mathcal{V}$. The **congruence variety** of $\mathcal{V}$, denoted CON($\mathcal{V}$), is the subvariety of $\mathcal{L}$ generated by $\{\mathbf{Con}(\mathbf{A}) \mid \mathbf{A} \in \mathcal{V}\}$, or alternatively is the variety lattices axiomatized by the congruence identities that hold in $\mathcal{V}$. Similarly, a lattice quasi-identity that holds in congruence lattices of members of $\mathcal{V}$ is a **congruence quasi-identity** of $\mathcal{V}$.

The following theorem will be used in several places in this monograph.

Theorem 2.22. *(Cf. [**6**]) Let Q be a quasi-identity satisfying* (W). *The class of varieties satisfying Q as a congruence quasi-identity is definable by a set of idempotent Maltsev conditions.*