Undecidable Problems in Algebra

Turing Machines and Computability

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This is the first talk in a series. The goal of the series is to address certain undecidable problems in algebra by associating to each Turing machine an algebraic structure. This talk will cover the requisite background information on Turing machines, computability, and the halting problem. No prior knowledge of computability theory will be assumed.

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- $\mathbb{A}(\mathcal{T})$ has property **P** if and only if $\mathcal{T}$ halts.
- Since determining if $\mathcal{T}$ halts is an undecidable problem, we will have shown that determining if an algebra has property **P** is undecidable.

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$\cdots$	0	1	1	0	0	0	1	0	0	1	1	0	0	1	1	0	0	0	1	1	0	0	$\cdots$

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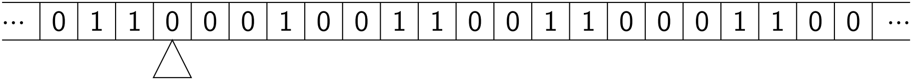
2. a movable "head" that can read the contents of whatever cell it is currently placed over and write a 0 or 1 to that cell;



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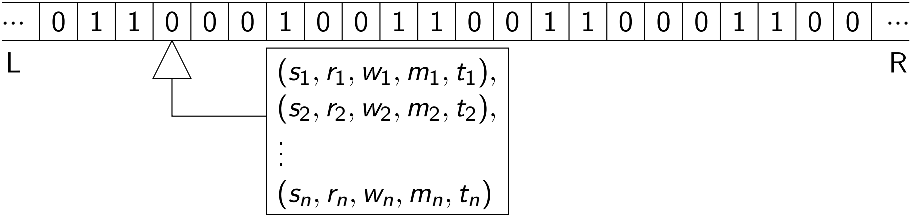
2. a movable "head" that can read the contents of whatever cell it is currently placed over and write a 0 or 1 to that cell; and
3. a program, consisting of a (finite) sequence 5-tuples, (s, r, w, m, t), meant to be interpreted as "if in state s and reading r, then write w, move m, and enter state t.



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- If the Turing machine eventually stops computing, it is said to have *halted*.
- If a Turing machine has halted on some input, the contents of the tape are called the *output*. We enumerate the output. $(\varphi(n) = m)$

Examples

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(\mathrm{START}, & 0, & 1 & \mathrm{R}, & s_1) \\
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Church-Turing Thesis

Any effectively calculable function is a computable function.

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Since a Turing machine consists of a finite list of instructions, the set of all Turing machines is countable. We enumerate them as φ_i, $i \in \mathbb{N}$.

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Answer: No!

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$$\Psi(x, i) = \begin{cases} \varphi_i(x) + 1 & \text{if } \eta(i, x) = 1 \\ 1 & \text{otherwise} \end{cases}$$

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$$\varphi_y(y) = \psi(y) = \varphi_y(y) + 1,$$

a contradiction. □

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  then we have shown that there is no algorithm to decide if $\mathcal{T}$ has $\mathbf{P}$.

Thank you.