UNDECIDABLE PROBLEMS IN ALGEBRA: FROM TURING MACHINES TO ALGEBRAS

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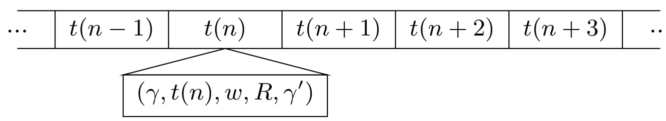
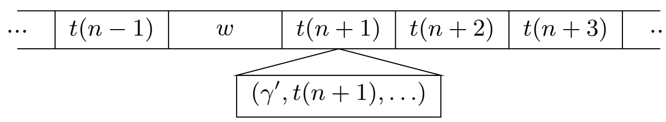
Definition. Given a Turing machine, $\mathcal{T}$, a *configuration* of $\mathcal{T}$ is a triple $\mathcal{Q} = (t, n, \gamma)$, where $t : \mathbb{Z} \to \{0, 1\}$ is a tape, $n \in \mathbb{Z}$ is the position of the reading head on the tape, and γ is a state of $\mathcal{T}$. If the line $(\gamma, t(n), w, D, \gamma')$ appears in the program of $\mathcal{T}$, we write $\mathcal{T}(\mathcal{Q}) = \langle t', n \pm 1, \gamma' \rangle$, where t' is the modified tape and $n \pm 1$ is determined by D.





Let $\mu_0, \mu_1, \ldots, \mu_k$ be the states of $\mathcal{T}$. μ_0 is considered to be the halting state, and μ_1 is considered to be the initial or starting state.

$$U = \{1, 2, H\} \qquad W = \{C, D, \overline{C}, \overline{D}\} \quad A = \{0\} \cup U \cup W$$

$$V_{ir}^{w} = \{C_{ir}^{w}, D_{ir}^{w}, M_i^r, \overline{C_{ir}^{w}}, \overline{D_{ir}^{w}}, \overline{M_i^r}\} \text{ for } 0 \le i \le k \text{ and } r, s \in \{0, 1\}$$

$$V_{ir} = V_{ir}^{0} \cup V_{ir}^{1}, \quad V_i = V_{i0} \cup V_{i1}, \qquad V = \bigcup_i V_i$$

$$\mathrm{A}(\mathcal{T}) = A \cup V$$

We encode the configurations in a subdirect product of $\mathrm{A}(\mathcal{T})$. Let $n \in N$, $\mathcal{Q} = (t, n, \mu_i)$ be a configuration, and $\eta \in \{0, 1\}^X$ any function. Define an element $\beta = \beta(\mathcal{Q}) \in \mathbf{B} \le \mathrm{A}(\mathcal{T})^X$ by

$$\beta(x) = \begin{cases} C_{it(n)}^{\eta(x)} & \text{when } x \in X_L \\ C_{it(n)}^{t(j)} & \text{when } x \in X_j, j < n \\ M_i^{t(n)} & \text{when } x \in X_n \\ D_{it(n)}^{t(j)} & \text{when } x \in X_j, j > n \\ D_{it(n)}^{\eta(x)} & \text{when } x \in X_R \end{cases}$$

Note that $\beta(\mathcal{Q})$ encodes t (restricted to N), μ_i, $t(n)$, and n as $\beta(x) = M_i^{t(n)}$ when $x \in X_n$.

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- Encoding the initial input: define the unary operation I on $\mathrm{A}(\mathcal{T})$ by
$$I(x) = \begin{cases} C_{10}^0 & \text{if } x = 1 \\ M_1^0 & \text{if } x = H \\ D_{10}^0 & \text{if } x = 2 \\ 0 & \text{otherwise} \end{cases}$$
Then
$$I(1, \dots, 1, H, 2, \dots, 2) = (C_{10}^0, \dots, C_{10}^0, M_1^0, D_{10}^0, \dots D_{10}^0) = \beta(\overline{0}, n, \mu_1).$$
- Encoding the Turing Program: for each instruction (μ_i, r, w, L, μ_j) and (μ_i, r, w, R, μ_j) in the program of $\mathcal{T}$, and for each $s \in \{0, 1\}$, define the 3-ary operations L_{irs} and R_{irs} on $\mathrm{A}(\mathcal{T})$ by
$$L_{irs}(x, y, z) = \begin{cases} C_{js}^{w'} & \text{if } x = y = 1, z = C_{js}^{w'} \text{ for some } w' \\ M_j^s & \text{if } x = H, y = 1, z = C_{ir}^w \\ D_{js}^w & \text{if } x = 2, y = H, z = M_i^r \\ D_{js}^{w'} & \text{if } x = y = 2, z = D_{ir}^{w'} \text{ for some } w' \\ \overline{v} & \text{if } z \in V \text{ and } L_{irs}(x, y, \overline{z}) = v \in V \\ 0 & \text{otherwise} \end{cases}$$
and
$$R_{irs}(x, y, z) = \begin{cases} C_{js}^{w'} & \text{if } x = y = 1, z = C_{js}^{w'} \text{ for some } w' \\ C_{js}^w & \text{if } x = H, y = 1, z = M_i^r \\ M_j^s & \text{if } x = 2, y = H, z = D_{ir}^w \\ D_{js}^{w'} & \text{if } x = y = 2, z = D_{ir}^{w'} \text{ for some } w' \\ \overline{v} & \text{if } z \in V \text{ and } R_{irs}(x, y, \overline{z}) = v \in V \\ 0 & \text{otherwise} \end{cases}$$
These operations emulate the operation of $\mathcal{T}$ when it is in state μ_i reading r and the square to the left/right of the head contains an s.

$\beta = \beta(t, n, \mu_i)$, $(\mu_i, 0, 1, R, \mu_j) \in \mathcal{T}$,
$\mathcal{T}(t, n, \mu_i) = (t', n+1, \mu_j)$, $\beta' = \beta(t', n+1, \mu_j)$

		$n-2$	$n-1$	n	$n+1$	$n+2$	
t	$\cdots$	0	1	0	1	1	$\cdots$

	X_L		X_{n-2}	X_{n-1}	X_n	X_{n+1}	X_{n+2}		X_R
f_{n-1}	1	$\cdots$	1	H	2	2	2	$\cdots$	2
f_n	1	$\cdots$	1	1	H	2	2	$\cdots$	2
f_{n+1}	1	$\cdots$	1	1	1	H	2	$\cdots$	2
β	C_{i0}^η	$\cdots$	C_{i0}^0	C_{i0}^1	M_i^0	D_{i0}^1	D_{i0}^1	$\cdots$	D_{i0}^η
β'	C_{j1}^η	$\cdots$	C_{j1}^0	C_{j1}^1	C_{j1}^1	M_j^1	D_{j1}^1	$\cdots$	D_{j1}^η

$$R_{i01}(f_n, f_{n+1}, \beta) = \beta'$$