Theory of Groups (MATH 6270)

Solutions to Additional Problems 1 and 2

Additional Problem 1. Let $H \leq \operatorname{Sym} X$ be a permutation group acting primitively on a finite set X, and let N, L be nontrivial normal subgroups of H such that $N \cap L = 1$. Prove that

(1) N and L are isomorphic centerless groups that act regularly on X, and the action of NL on X is similar to the following action of $N \times N$ on N:

$$N \times N \to \operatorname{Sym} N, \quad (\lambda, \nu) \mapsto \rho_{\lambda,\nu} \colon x \mapsto \lambda^{-1} x \nu = x^{\lambda}(\lambda^{-1}\nu).$$

(2) N and L are minimal normal subgroups of H, H has no other minimal normal subgroups, and NL acts primitively on X if and only if N is a nonabelian simple group.

Solution. (1) N and L act transitively on X (by Theorem 1, Handout 6), and since $[N, L] \leq N \cap L = 1$, they centralize each other. Thus, for arbitrary elements $x, y \in X$, there exists $\sigma \in L$ such that $y = x\sigma$, and hence $N_y = N_{x\sigma} = (N_x)^{\sigma}$. Since σ commutes with all elements of N, we get that $(N_x)^{\sigma} = N_x$. Thus $N_x = N_y$ for all $x, y \in X$, which implies that $N_x = 1$ for all $x \in X$. Hence N acts regularly on X. Similarly, L also acts regularly on X.

Now we fix $a \in X$, and let $\Phi \colon N \to X$, $\pi \mapsto a\pi$ be the bijection from Lemma 3 on Handout 6. The mapping

$$\widetilde{\ } \colon H \to \operatorname{Sym} N, \quad \chi \mapsto \widetilde{\chi} := \Phi \chi \Phi^{-1}$$

is an injective homomorphism, and hence yields an isomorphism between H and the subgroup $\widetilde{H} := \{\widetilde{\chi} : \chi \in H\}$ of $\operatorname{Sym} N$. It follows that Φ and $\widetilde{\ }$ establish a similarity between H and $\widetilde{H}$. Thus $\widetilde{H}$ is primitive on N, its subgroups $\widetilde{N}$ and $\widetilde{L}$ are normal, they centralize each other, and are regular. Moreover, as it was shown in Lemma 3 on Handout 6, each $\widetilde{\nu} \in \widetilde{N}$ ($\nu \in N$) is right multiplication by ν.

Next we will prove that for each $\sigma \in L$, $\widetilde{\sigma}$ is left multiplication by the element $\lambda := 1^{\widetilde{\sigma}}$ of N. Indeed, for each $\nu \in N$,

$$\nu^{\widetilde{\sigma}} = 1^{\widetilde{\nu}\widetilde{\sigma}} = 1^{\widetilde{\sigma}\widetilde{\nu}} = \lambda^{\widetilde{\nu}} = \lambda\nu.$$

Since $\widetilde{L}$ is transitive on N, it follows that $\widetilde{L}$ is the group of all left multiplications by elements $\lambda \in N$, that is, $\widetilde{L}$ is the left regular representation of N. Hence $N \cong \widetilde{L} \cong L$. Left and right multiplication by an element $\gamma \in N$ are the same permutations if and only if $\gamma \in \zeta N$. Therefore $\widetilde{N} \cap \widetilde{L} = 1$ implies that $\zeta N = 1$. Thus N and L are centerless. Finally, the similarity induced by Φ maps NL to $\widetilde{N}\widetilde{L}$, the join of the left and right regular representations of N. This proves that under the similarity induced by Φ, the action of NL on X is similar to the action of $N \times N$ on N described in the problem.

(2) Because of the similarity discussed above, it suffices to prove the statements in (2) for $\widetilde{H}$, $\widetilde{N}$, and $\widetilde{L}$ instead of H, N, and L. Every normal subgroup of $\widetilde{H}$ is transitive, because $\widetilde{H}$ is primitive. Since $\widetilde{N}$, $\widetilde{L}$ are regular, they have no transitive proper subgroups. Hence they are minimal normal subgroups of $\widetilde{H}$. The proof above shows that every minimal normal

subgroup of $\widetilde{H}$ other than $\widetilde{N}$ must be equal to $\widetilde{L}$. Therefore $\widetilde{N}$, $\widetilde{L}$ are the only minimal normal subgroups of $\widetilde{H}$.

To prove the statement on primitivity, let $\widetilde{K} = \widetilde{N}\widetilde{L} = \{x \mapsto x^\lambda \nu \in \operatorname{Sym} N : \lambda, \nu \in N\}$. Clearly, the stabilizer of $1 \in N$ is $\widetilde{K}_1 = \{x \mapsto x^\lambda \in \operatorname{Sym} N : \lambda \in N\}$. We will show that there is a bijection between the subgroups S of $\widetilde{K}$ containing $\widetilde{K}_1$, and the normal subgroups T of N. If $T \triangleleft N$, then clearly,

$$T^* := \langle K_1, \{x \mapsto x\nu : \nu \in T\}\rangle$$

is a subgroup of K containing K_1, while if $K_1 \leq S \leq K$, then

$$S^\circ := \{\nu \in N : x \mapsto x\nu \text{ belongs to } S\}$$

is normal subgroup of N; closure under conjugation follows by observing that for each $\nu \in S^\circ$, $x \mapsto x\nu^\lambda = ((x^{\lambda^{-1}})\nu)^\lambda$ is a composition of the following three permutations: $x \mapsto x^{\lambda^{-1}}$ in K_1, $x \mapsto x\nu$, and $x \mapsto x^\lambda$ in K_1. It is straightforward to check that for $T \triangleleft N$ we have $T^* = \{x \mapsto x^\lambda \nu : \lambda \in N, \nu \in T\}$, hence $(T^*)^\circ = T$. If $K_1 \leq S \leq K$, then for each $x \mapsto x^{\lambda_0}\nu$ in S we have that $x \mapsto x^\lambda \nu = (x^{\lambda\lambda_0^{-1}})^{\lambda_0}\nu$ belongs to S for all $\lambda \in N$. Hence $S = (S^\circ)^*$. This proves that * and $^\circ$ are bijections that are inverses of each other. It follows in particular that K_1 is a maximal subgroup of K if and only if N is simple. (N is nonabelian, because it is centerless by part (1).) Since the condition that K acts primitively on N is equivalent to K_1 being maximal in K, the proof is complete.

Additional Problem 2. Let $M = \mathrm{PSL}(n+1, q)$ with its usual permutation action on the set X of points of the projective geometry $\mathcal{PG}(n, q)$. Prove the equality $N_{\operatorname{Sym} X}(M) = \mathrm{P\Gamma L}(n+1, q)$ for $n \geq 2$.

Solution. We will start by discussing a general fact about the normalizer $N_{\operatorname{Sym} Y}(G)$ of a permutation group $G \leq \operatorname{Sym} Y$. For arbitrary $\pi \in \operatorname{Sym} Y$, if $\gamma \in G$ and $a\gamma = b$ $(a, b \in Y)$, then $(a\pi)\gamma^\pi = b\pi$. Therefore, if the orbit of $a \in Y$ under the action of G is O, then the orbit of $a\pi \in X$ under the action of G^π $(= \pi^{-1}G\pi)$ is $O\pi$. In particular, if $G^\pi = G$, that is, if $\pi \in N_{\operatorname{Sym} Y}(G)$, then for each orbit O of G, $O\pi$ is also an orbit of G.

To prove that $N_{\operatorname{Sym} X}(M) \leq \mathrm{P\Gamma L}(n+1, q)$ it suffices to show (by the Fundamental Theorem of Projective Geometry) that $M = \mathrm{PSL}(n+1, q)$ $(n \geq 2)$ preserves collinearity. Let Y be the set of 3-element subsets of X. The group M acts on Y, and by Theorem 8 on Handout 5 one of the orbits is the set O of those 3-element subsets, which consist of points in general position, that is, points that are not collinear. For any two distinct points $x, y \in X$ the number of points collinear with x, y is $q - 1$ (since each line has $q + 1$ points), and the number of points that are not collinear with x, y is $|X| - (q+1) = q^n + q^{n-1} + \cdots + q^2$. Since $|X| - (q+1) \geq q^2 > 2(q-1)$, more than half of the 3-element sets in Y belong to the orbit O, and hence no other orbit of M has the same size as O. By the observation in the previous paragraph it follows that each $\pi \in N_{\operatorname{Sym} X}(M)$ maps O to O; that is, each $\pi \in N_{\operatorname{Sym} X}(M)$ preserves non-collinearity, and hence it also preserves collinearity.

$\mathrm{P\Gamma L}(n+1, q)$ is defined as the action of $\Gamma\mathrm{L}(n+1, q)$ on X, the set of 1-dimensional subspaces of $(\mathbb{F}_q)^{n+1}$. Furthermore, $\Gamma\mathrm{L}(n+1, q)$ is generated by $\mathrm{GL}(n+1, q)$ and the permutations $\hat{\sigma}$ $(\sigma \in \operatorname{Aut} \mathbb{F}_q)$. (See Handout 5.) Since $\mathrm{SL}(n+1, q) \triangleleft \mathrm{GL}(n+1, q)$ and each $\hat{\sigma}$ $(\sigma \in \operatorname{Aut} \mathbb{F}_q)$ normalizes $\mathrm{SL}(n+1, q)$ we get that $\mathrm{SL}(n+1, q) \triangleleft \Gamma\mathrm{L}(n+1, q)$. Hence the same holds for their action on X, proving that $M = \mathrm{PSL}(n+1, q) \triangleleft \mathrm{P\Gamma L}(n+1, q)$. This implies that $N_{\operatorname{Sym} X}(M) \geq \mathrm{P\Gamma L}(n+1, q)$, completing the proof.