Theory of Groups (MATH 6270)

HOMEWORK ASSIGNMENT 11
(April 16, 2008)

Read Sections 7.4, 10.1, and Handout 5 (pp. 4–5).

Problems:
Section 7.4: Exercises 2, 3, 4.
Section 10.1: Exercises 1, 6, 14.
Additional Problems 1, 2, 3 (see below)

Turn in underlined problems
Due Date: April 23, 2008

Additional Problems:

1. Let $q = p^{2m}$ where p is an odd prime and m is a positive integer. Prove that

$$T(q) = \left\{ \frac{xa+b}{xc+d} \ : ad - bc \text{ is a square in } \mathbb{F}_q \right\}$$

is a subgroup of index 2 in both $L(q)$ and $M(q)$, and $T(q)$ is isomorphic to $\mathrm{PSL}(2, q)$.

2. Let H be an abelian subgroup of finite index $|G : H| = n$ in G. Prove that
(1) if $\varphi\colon G \to H$ is a homomorphism and $\varphi|_H\colon H \to H$ is its restriction to H, then the transfer of $\varphi|_H$ is

$$G \to H, \quad x \mapsto (x^{\varphi})^n;$$

(2) if H has a normal complement K and $\psi\colon G = HK \to H$ is the natural homomorphism $hk \mapsto h$ ($h \in H$, $k \in K$), then the transfer of G into H is

$$G \to H, \quad x \mapsto (x^{\psi})^n.$$

(3) Use part (2) to determine all integers $m \geq 2$ such that the transfer of S_m into its subgroup $H = \langle (1\,2) \rangle$ is the trivial homomorphism.

3. Let G be a torsion-free group that has a cyclic subgroup of finite index. Prove that G is cyclic.

Hint: First use the transfer to show that
($*$) if $\tilde{G}$ is torsion-free and its center contains a cyclic subgroup that is of finite index in $\tilde{G}$, then $\tilde{G}$ is cyclic.
To prove the statement in the problem, consider a torsion-free group G that has a proper cyclic subgroup H of finite index, and use statement ($*$) to find a cyclic subgroup K of G such that $H < K \leq G$.