Theory of Groups (MATH 6270)

HOMEWORK ASSIGNMENT 2

(January 23, 2008)

Read Sections 1.4 (except for the subsections on direct products and direct limits) and 1.5 (except for the subsection on semidirect products).

Problems:
Section 1.4: Exercise 6
Section 1.5: Exercises 1, 2, 3, 5, <u>6</u>, 7, <u>15</u>, 16
Additional Problems 1, <u>2</u>, 3 (see below)

Turn in underlined problems
Due Date: January 30, 2008

Additional Problems:

1. Show that if a G-set $(X;G)$ is simple, then either it is transitive, or else $|X| = 2$ and G acts trivially: $xg = x$ for all $x \in X$ and $g \in G$.

Definition. A *primitive G-set* is a simple G-set that is transitive.

2. Let G be a group, let $(X;G)$ be a transitive G-set, $a \in X$ a fixed element, and let $G_a = \mathrm{St}_G(a) = \{g \in G : ag = a\}$ be the stabilizer of a in G. Prove the following statements.

(i) The lattice of congruences of the G-set $(G;G)$ (where G acts on itself by right multiplication) is isomorphic to the lattice of subgroups of G.
(*Hint:* An equivalence relation $\sim$ on G is a congruence of $(G;G)$ if and only if $G/{\sim} = \{Hx : x \in G\}$ for some subgroup H of G.)

(ii) The lattice of congruences of $(X;G)$ is isomorphic to the lattice of subgroups of G that contain G_a.

(iii) The following conditions are equivalent:
  (a) $(X;G)$ is a primitive G-set,
  (b) G_a is a maximal subgroup of G,
  (c) X has no proper subset Y such that $|Y| > 1$ and
    $(*)\quad Y = Yg$ or $Y \cap Yg = \emptyset$ holds for each $g \in G$.
(*Hint:* $Y \subseteq X$ satisfies $(*)$ if and only if Y is an equivalence class of a congruence of $(X;G)$.)

Definition. A proper subset Y of a G-set $(X;G)$ such that $|Y| > 1$ and Y satisfies $(*)$ is called a *block* or a *domain of imprimitivity* of $(X;G)$.

3. Find a G-set $(X;G)$ which has two distinct congruences $\sim$ and $\approx$ such that $a/{\sim} = a/{\approx}$ for some $a \in X$.