Theory of Groups (MATH 6270)

HOMEWORK ASSIGNMENT 4
(February 6, 2008)

Read Sections 1.6 (subsection 'Standard Wreath Products and Sylow Subgroups of the Symmetric Group'), 2.1, 2.2 (except 'Finitely Presented Groups' and 'The Word Problem'), and 2.3 (subsection 'Free Abelian Groups').

Problems:
Section 1.6: Exercises <u>17</u>
Section 2.1: Exercises 4, 5, 6, 7
Section 2.2: Exercises 2, 3, 4, <u>5</u>[1]
Additional Problems 1, <u>2</u>, 3 (see below)

Turn in underlined problems
Due Date: February 13, 2008

Additional Problems: Let $H \le \operatorname{Sym} X$ and $K \le \operatorname{Sym} Y$.

1. Show that the following definition yields an action of the complete wreath product[2] $H \,\bar{\wr}\, K$ on the set $X^Y = \{(x_y)_{y\in Y} : x_y \in X\}$ (the set of all functions $Y \to X$): for arbitrary $(x_y)_{y\in Y} \in X^Y$ and $\big(\kappa, (\gamma_y)_{y\in Y}\big) \in H \,\bar{\wr}\, K$,

$$(x_y)_{y\in Y}\big(\kappa, (\gamma_y)_{y\in Y}\big) \stackrel{\text{def}}{=} (x_{y\kappa^{-1}}\gamma_y)_{y\in Y}.$$

Definition. This is called *the product action* of $H \,\bar{\wr}\, K$ on X^Y.

2. Prove that the product action of $H \wr K$ is primitive if
(a) H acts primitively but not regularly on X,
(b) Y is finite, and
(c) K acts transitively on Y.

Hint: Fix $a \in X$, and let $\mathbf{a} = (a, \dots, a) \in X^Y$.
(i) Show that the stabilizer of $\mathbf{a}$ in $H \wr K$ (in its product action) is $H_a \wr K$ where H_a is the stabilizer of a in H.
(ii) Deduce from condition (a) that $\gamma^{-1} H_a \gamma \ne H_a$ holds for all $\gamma \in H \setminus H_a$.
(iii) Use this to argue that if conditions (a)–(c) are met, then $H_a \wr K$ is a maximal subgroup of $H \wr K$.

3. Show that if any one of conditions (a)–(c) fails, then the product action of $H \,\bar{\wr}\, K$ is imprimitive.

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[1] *Hint:* Think of one of the following groups: the group of isometries of the plane, $\mathrm{AGL}(2, \mathbb{R})$, or $\mathrm{AGL}(1, \mathbb{C})$. For a positive integer d and a field F, $\mathrm{AGL}(d, F)$ denotes the *affine general linear group* that consists of all affine transformations $Ax + b$ of the vector space F^d where $A \in \mathrm{GL}(d, F)$ and $b \in F^d$.

[2] The same definition also yields an action of the subgroup $H \wr K$ of $H \,\bar{\wr}\, K$ on X^Y.