Theory of Groups (MATH 6270)

HOMEWORK ASSIGNMENT 6
(February 20, 2008)

Read Sections 6.2 and 4.1

Problems:
Section 6.2: Exercises 3, 5, 6, <u>9</u>, 10
Section 4.1: Exercises 2, 3, 4, <u>6</u>
Additional Problems 1, <u>2</u>, 3 (see below)

Turn in underlined problems
Due Date: February 27, 2008

Additional Problems:

1. Let $\mathcal{V}$ be a nontrivial variety of groups.[1] For a set X let $F_{\mathcal{V}}(X)$ denote the $\mathcal{V}$-free group with free generating set X.
Prove that for arbitrary sets X and Y, $F_{\mathcal{V}}(X) \cong F_{\mathcal{V}}(Y)$ if and only if $|X| = |Y|$.

Hint for proving "$\Rightarrow$": For a prime p let $\mathcal{A}_p$ denote the variety of elementary abelian p-groups (abelian groups satisfying the equation $x^p \approx 1$).
(i) Show that $\mathcal{A}_p \subseteq \mathcal{V}$ for some prime p.
(ii) Prove that for every set X the $\mathcal{A}_p$-free group with free generating set X is isomorphic to $F_{\mathcal{V}}(X)/W\big(F_{\mathcal{V}}(X)\big)$ where $W = \{[x,y], x^p\}$.
(iii) Argue that any isomorphism $F_{\mathcal{V}}(X) \to F_{\mathcal{V}}(Y)$ induces an isomorphism $F_{\mathcal{V}}(X)/W\big(F_{\mathcal{V}}(X)\big) \to F_{\mathcal{V}}(Y)/W\big(F_{\mathcal{V}}(Y)\big)$, and use (ii) to deduce that $|X| = |Y|$.

2. Show that the following conditions on an abelian group G are equivalent:
(a) G is divisible;
(b) Every quotient G/N ($N \neq G$) of G is infinite.
(c) G has no maximal proper subgroups.

3. Prove that the following groups are all isomorphic:
- $\mathbb{R}/\mathbb{Z}$ (additive),
- $\mathbb{R} \oplus \mathbb{Q}/\mathbb{Z}$ (additive),
- $\operatorname*{Cr}_{q \text{ prime}} P_q$ where P_q is a quasicyclic q-group,
- $\{\varepsilon \in \mathbb{C} : |\varepsilon| = 1\}$ (multiplicative),
- $\mathbb{C} \setminus \{0\}$ (multiplicative).

[1] A variety $\mathcal{V}$ is called trivial, if $\mathcal{V} \models x \approx y$, that is, each member of $\mathcal{V}$ has only one element.