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A NOTE ON ALGEBRAIC EQUIVALENCE OF WHITE'S TEST AND A VARIATION OF THE GODFREY/BREUSCH–PAGAN TEST FOR HETEROSCEDASTICITY

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Two, recently proposed tests for heteroscedasticity are examined. Under certain conditions and a modification of one of the tests already proposed in the literature, they are found to be algebraically equivalent.

Recently, Godfrey (1978), and Breusch and Pagan (1979) have independently proposed testing for heteroscedasticity based on squared least squares residuals. The squared residuals divided by the mean-squared residual are regressed on a set of regressors chosen by the investigator, and the test statistic (GBP hereafter) is one-half the explained sum of squares from this regression. The test statistic has asymptotically the chi-squared distribution.

In another paper, White (1980) develops a test also based on squared least squares residuals. Here, the vector of squared residuals is again the regressand, while the regressors are the squares and crossproducts of the regressors in the initial regression. The test statistic is the sample size times the squared multiple correlation coefficient from this regression. This test statistic also has asymptotically the chi-squared distribution. In each case, the number of degrees of freedom is equal to the number of regressors in the artificial regression minus one (a constant must be included).

Both test statistics have unit power asymptotically but are different in finite samples and sometimes give very different results in simulation settings. [1] However, several authors [Bickel (1978), Koenker (1981), Koenker and Bassett (1982)] have suggested that the GBP test statistic is sensitive to slight departures from normality. Since choosing between heteroscedastic Gaussian disturbances and thick-tailed non-Gaussian disturbances in the population is difficult in any case and probably unjustifiable in the large majority of economic models, this non-robustness would appear to be important. In particular the apparent size of the test is not the actual size when disturbances are not normal. These authors suggest a simple modification of the GBP statistic which then has the correct size for any disturbance law. We consider this modified GBP statistic below. [2]

Although Godfrey and Breusch–Pagan leave to the researcher the choice of regressors in the artificial regression, in both papers the squared regressor from the model is used in their simulations. It is perhaps unlikely that an investigator, having formulated a model for the conditional expectation of the dependent variable, has at his disposal additional information concerning the disturbance variance not already included in the conditional expectation function. Hence it would seem that a good choice for the set of artificial regressors in the GBP test will be the same as in White's test.

White (1980, p. 825), citing an unpublished paper, remarks that his test '... resembles a Lagrange multiplier test... considered by Godfrey...'. Koenker (1981, p. 111) observes that the modified GBP test equals the sample size times the uncentered coefficient of determination. In fact, employing the constant-adjusted R^2, we demonstrate below that the modified GBP test, with choice of regressors as in White's test, is algebraically equivalent to White's test.

Preliminaries. For N observations on the dependent variable and K independent variables we have the usual linear regression model

$$y = X\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

[1] We performed some limited small sample investigations, under the assumption of normality, to compare the two tests. Although generalizations are difficult due to the number and variety of possible environments, our tentative conclusions are that *in the case of strict normality* the GBP test is often but not always more powerful than White's test. We hope to report at a later date an extensive set of simulation results.

[2] Solving the size problem, however, does not guarantee a powerful test. Koenker and Bassett (1982) find that the asymptotic relative efficiency (ARE) of the modified GBP test to their robust test based on regression quantiles, deteriorates rapidly as disturbances depart from normality.

where $\boldsymbol{\epsilon}$ is a possibly heteroscedastistic random vector. Define $M = I - X(X'X)^{-1}X'$ so that the vector of least squares residuals is $\boldsymbol{e} = M\boldsymbol{y}$. Let $u_i = e_i^2$, $i = 1,\ldots, N$ and $\boldsymbol{u} = (u_1, u_2,\ldots, u_N)'$. Define $\hat{\sigma}^2 = N^{-1}\boldsymbol{e}'\boldsymbol{e}$ and $\boldsymbol{g} = \hat{\sigma}^{-2}\boldsymbol{u}$. Let Z be the N by $K(K+1)/2$ matrix whose columns are the squares and crossproducts of $\boldsymbol{x}_1, \boldsymbol{x}_2,\ldots, \boldsymbol{x}_K$, the columns of X. That is, let $Z_{is} = x_{ik}x_{il}$ ($s = 1,\ldots, K(K+1)/2$; $k = 1,\ldots, K$; $l = 1,\ldots, k$) for $i = 1,\ldots, N$. Let z_s be the $N \times 1$ vector with typical element z_{is}. Then $Z = (z_1, z_2,\ldots, z_{K(K+1)/2})$. It is required that $z_1 = (1, 1,\ldots 1)' = \boldsymbol{i}$.

The test statistics. The GBP test statistic, T_{GBP}, is given by [Godfrey (1978, eq. (25)), Breusch and Pagan (1979, p. 1289)]

$$T_{\mathrm{GBP}} = \left[\boldsymbol{g}'Z(Z'Z)^{-1}Z'\boldsymbol{g} - N^{-1}(\boldsymbol{i}'\boldsymbol{g})^2\right]/2.$$

Since $\boldsymbol{i}'\boldsymbol{g} = \hat{\sigma}^{-2}\boldsymbol{i}'\boldsymbol{u} = \hat{\sigma}^{-2}\hat{\sigma}^2 N = N$, this can be rewritten

$$T_{\mathrm{GBP}} = \left[\boldsymbol{u}'Z(Z'Z)^{-1}Z'\boldsymbol{u} - N\hat{\sigma}^4\right]/2\hat{\sigma}^4.$$

If the ϵ_i are normally distributed, $V(\epsilon_i^2) = 2\sigma^4$, which is well estimated above by $2\hat{\sigma}^4$. A more robust alternative [Koenker (1981, p. 111), Koenker and Bassett (1982, p. 51)] is to replace $2\hat{\sigma}^4$ in the denominator with $N^{-1}(\boldsymbol{u} - \hat{\sigma}^2\boldsymbol{i})'(\boldsymbol{u} - \hat{\sigma}^2\boldsymbol{i})$, the second sample moment of the squared residuals.

Since $N^{-1}\boldsymbol{i}'\boldsymbol{u} = \hat{\sigma}^2$, then $\boldsymbol{u} - \hat{\sigma}^2\boldsymbol{i} = A\boldsymbol{u}$, where $A = I - N^{-1}\boldsymbol{i}\boldsymbol{i}'$, a symmetric, idempotent matrix which removes the mean. The modification, then, amounts to replacing $2\hat{\sigma}^4$ with $\boldsymbol{u}'A\boldsymbol{u}/N$, and hence the modified statistic, T_{MGBP} becomes

$$T_{\mathrm{MGBP}} = N\left[\boldsymbol{u}'Z(Z'Z)^{-1}Z'\boldsymbol{u} - N\hat{\sigma}^4\right]/\boldsymbol{u}'A\boldsymbol{u}.$$

White's statistic [White (1980, eq. (3))] is NR^2, where R^2 is the constant-adjusted squared multiple correlation coefficient from the regression of $\boldsymbol{u}$ on Z. from Theil (1971, p. 176) we have $1 - R^2 = \boldsymbol{u}'M\boldsymbol{u}/\boldsymbol{u}'A\boldsymbol{u}$. Therefore

$$T_{\mathrm{W}} = N(1 - \boldsymbol{u}'M\boldsymbol{u}/\boldsymbol{u}'A\boldsymbol{u}).$$

Combining terms and factoring yields

$$T_{\mathrm{W}} = N\boldsymbol{u}'(A - M)\boldsymbol{u}/\boldsymbol{u}'A\boldsymbol{u}.$$

But $A - M = (I - N^{-1}\boldsymbol{ii}') - (I - Z(Z'Z)^{-1}Z') = Z(Z'Z)^{-1}Z' - N^{-1}\boldsymbol{ii}'$ so that the numerator of T_{W} becomes $N(\boldsymbol{u}'Z(Z'Z)^{-1}Z'\boldsymbol{u} - N^{-1}\boldsymbol{u}'\boldsymbol{ii}'\boldsymbol{u})$. Since $\boldsymbol{u}'\boldsymbol{i} = \boldsymbol{i}'\boldsymbol{u} = N\hat{\sigma}^2$, the second term in this numerator is equal to $N\hat{\sigma}^4$. Hence $T_{\mathrm{W}} = N(\boldsymbol{u}'Z(Z'Z)^{-1}Z'\boldsymbol{u} - N\hat{\sigma}^4)/\boldsymbol{u}'A\boldsymbol{u} = T_{\mathrm{MGBP}}$.

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