Spaces and equations

by

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Dedicated to the memory of Garrett Birkhoff (1911–1996)

Abstract. It is proved, for various spaces A, such as a surface of genus 2, a figure-eight, or a sphere of dimension $\neq 1, 3, 7$, and for any set Σ of equations, that Σ cannot be modeled by continuous operations on A unless Σ is undemanding (a form of triviality that is defined in the paper).

0 Introduction.

A celebrated theorem of Adams [1] (and others) asserts that the only spheres that are H-spaces are S^1, S^3 and S^7. In other words, if $n \neq 1, 3, 7$, then it is impossible to have a continuous operation $F : S^n \times S^n \longrightarrow S^n$ and a point $e \in S^n$ such that the system of equations

$$e \cdot x \quad \approx \quad x \cdot e \quad \approx \quad x, \tag{1}$$

is satisfied up to homotopy on S^n. The result is known for many other spaces (see e.g. Novikov [30, pp. 191–194] or Dieudonné [9, pp. 236–238]), but not for other systems Σ of equations. The aim of this paper is to extend the result (for known spaces) to many other Σ. In fact, our result is best-possible as far as Σ is concerned.

Let Σ be a set of equations involving operation symbols F_t ($t \in T$). In other words, for each t, F_t is a symbol for a finitary operation, i.e., a function of $n(t)$ variables for some finite $n(t)$ (which may be zero). We say that Σ is *undemanding* (or *easily satisfied*), iff there exists a set A of more than one

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element, and an interpretation of each F_t as either a constant function or a projection function (from $A^{n(t)}$ to the ${k_t}^{\text{th}}$ factor, for some $k_t \leq n(t)$), which satisfies the equations Σ on A. (Notice that for finite Σ there is a simple algorithm for checking if Σ is undemanding or not. Indeed it suffices to try the $n(t)+1$ interpretations of each F_t by projections or a constant.)

For example, if each equation in Σ has the form $\sigma \approx \tau$, with σ and τ each a composite term (i.e., not a variable standing alone), then Σ is undemanding. That is, if we interpret all the F_t as constant operations with the same value, then the equations of Σ are obviously satisfied (albeit somewhat trivially). Clearly the theory of commutative semigroups (whose axioms are the associative and commutative laws) is in this category. For a second example, the theory of idempotent semigroups, axiomatized by this[1] Σ:

$$\begin{aligned} x\cdot(y\cdot z) &\approx (x\cdot y)\cdot z \\ x\cdot x &\approx x, \end{aligned}$$

is also undemanding, by using projections (but not by using constants). In fact, in this example the product $x\cdot y$ may be interpreted as either x (first co-ordinate projection) or as y (second co-ordinate projection).

On the other hand, if Σ is given by

$$xx \approx x \tag{2}$$
$$xy \approx yx, \tag{3}$$

then Σ is demanding. (Equation (2) rules out a constant for xy, and Equation (3) rules out both projections.) Thus the familiar theory of semilattices is also demanding, for its axioms contain Equations (2) and (3) along with the associative law. Along the same lines, it is easy to see that Equations (1) — defining H-spaces — form a demanding set.

For a space A and operations $\overline{F_t}\colon A^{n(t)} \longrightarrow A$ we say that the operations $\overline{F_t}$ *satisfy* Σ and write

$$(A, \overline{F}_t)_{t\in T} \models \Sigma, \tag{4}$$

if for each equation $\sigma \approx \tau$ in Σ, both σ and τ evaluate to the same function when the operations $\overline{F_t}$ are substituted for the symbols F_t appearing in σ

[1] In simple and familiar cases, we may dispense with the formal $F_t(\cdots)$ notation. In this case we have $T = \{0\}$, and we write $F_0(x, y)$ as $x\cdot y$

and τ. (For the sake of our proofs, this notion will be defined more precisely in §1.2.) Given a space A and a set of equations Σ, we write

$$A \models \Sigma, \tag{5}$$

and say that A and Σ are *compatible*, iff there exist *continuous* operations $\overline{F_t}$ on A satisfying Σ. (It will be a consequence of Theorem 1 that Σ is undemanding iff it is compatible with every space — see §2.2.)

As we indicated at the start of this introduction, Adams' result actually ruled out the possibility of satisfaction of Equations (1) up to homotopy. Operations $\overline{F_t}$ are said to satisfy an equation $\sigma \approx \tau$ *up to homotopy* if, when we substitute $\overline{F_t}$ for each F_t, the functions associated to σ and τ are homotopic to each other (although not necessarily equal as functions). In a similar way, one speaks of *compatibility up to homotopy*, and so on. Theorem 1 will be stated and proved for satisfaction up to homotopy.

For example, it is an easy exercise that if Σ axiomatizes lattice theory, then Σ is compatible with an interval $I = [a, b]$, whereas the axioms of group theory are not compatible with I. Up to homotopy, of course, both theories are compatible with I. Further examples (mostly of incompatibility) are scattered in the mathematical literature; some of them have been collected in [39]. For an example of immediate concern to this paper, the equation-set (1), which defines H-spaces, is compatible with spheres S^1, S^3 and S^7 (using multiplication of unimodular complex numbers, quaternions and Cayley numbers, respectively). According to the cited theorem of Adams, et al., however, (1) is *not* compatible with spheres S^n ($n \neq 1, 3, 7$), not even up to homotopy. Our main theorem, which we now state, has a similar conclusion for any demanding theory.

Theorem 1 *Let A be a path-connected space satisfying one of the following seven hypotheses. If A is compatible with Σ up to homotopy, then Σ is undemanding.*

1. *A is homeomorphic to the sphere S^n ($n \neq 1, 3, 7$).*

2. *A has fundamental group isomorphic to a non-Abelian free group of finite rank.*

3. *A has cohomology ring (over some field) isomorphic to the cohomology ring of an* **even-dimensional sphere**.

4. *A has cohomology ring (over some field) isomorphic to the cohomology ring of the* **orientable surface of genus** 2*.*

5. *A has cohomology ring (over the prime field of characteristic* 2*) isomorphic to the cohomology ring of the* **Klein bottle***.*

6. *A has cohomology ring (over the prime field of characteristic* 2*) isomorphic to the cohomology ring of n-dimensional* **real projective space***, with $n+1$ not a power of* 2*.*

7. *A has cohomology ring (over a field of characteristic $\neq 2$) isomorphic to the cohomology ring of the* **figure-eight space***.*

In referring to the part of this theorem that infers the conclusion from the k^{th} hypothesis, we will simply say "Part k," without always mentioning the theorem itself. Some further theorems (variations on this one) will be stated in §2.3 and §11.3.

The rest of the paper is devoted mostly to the proofs of Theorem 1 and of Theorem 2 (of §2.3). After a brief development of homotopy and free groups in §§3–4, the proof of Part 2 of Theorem 1 is completed in §5. After a brief development of the cohomology ring in §6.3, the proofs of Parts 3–7 are completed in §7. After a brief development of degrees and the Hopf invariant in §8, the proof of Part 1 is completed in §9. The proof of Theorem 2 is sketched in §10.

It is our experience in speaking about this material that few mathematicians are conversant in both the requisite equational logic (§1) and the requisite algebraic topology (§3, §6, §8). Those who are conversant in either of these subjects will find the corresponding sections elementary, and can obviously move on, after possibly learning our notation.

In spite of the requisite attention to detail, the main spirit of the paper is still categorical. We rely heavily on the functorial properties of the homotopy group and the cohomology ring. See also the final remarks in §11.2, where we give an alternate explication of our results in terms of abstract clone theory (algebraic theories), which is a branch of category theory.

The main results here were announced on web pages at Vanderbilt University (`http://atlas.math.vanderbilt.edu/~jsnow/universal_algebra/`) and at York University (`http://at.yorku.ca/i/d/e/a/89.htm`).

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1 Terms, equations and satisfaction.

1.1 Terms and interpretations.

The material of §1.1 is elementary but subtle. In particular, we need to distinguish carefully between a term (symbolic composite operation) τ and various composite operations such as $\overline{\tau}$, $\tau^\star$ and τ' that are patterned after τ. The operations $\overline{\tau}$ are essential to a precise understanding and a precise mathematical definition of the identical satisfaction relation (4); and the recursive construction of $\overline{\tau}$, $\tau^\star$ and τ' is essential to the inductive arguments that are needed in our proofs. The reader who is familiar with this material can read quickly, while pausing to take in our notation; although a little fussier than usual, it is essential to the remainder of the article.

As in §0, we begin with an indexed collection of operation symbols F_t $(t \in T)$. Attached to each $t \in T$ is a non-negative integer $n(t)$ called the *arity* of F_t. An *interpretation* of F_t on a non-empty set A is an $n(t)$-ary operation on A, i.e., a function

$$\overline{F_t} \colon A^{n(t)} \longrightarrow A.$$

(Thus when we say "The operations $\overline{F}_t$ interpret the function symbols F_t," the only real assertion is that each $\overline{F}_t$ has the correct domain $A^{n(t)}$.) In many cases of interest, there are only one or two operations F_t, having traditional designations like $+$, $\cdot$, $\wedge$, $\vee$, etc. We will use these familiar designations when they are available. Sometimes one omits the bar from $\overline{F_t}$, allowing the context to differentiate the symbolic operation from the concrete operation; this practice is especially widespread in the case of $+$, $\wedge$, etc.

A *term* is a symbolic expression that is recursively defined to be either a variable x_i (for some $i = 0, 1, 2, \ldots$), or $F_t(\tau_1, \ldots, \tau_{n(t)})$ for some $t \in T$ and some simpler terms τ_j. An *equation* is an ordered pair of terms (σ, τ). This pair is usually written $\sigma \approx \tau$, with the bent equal-sign emphasizing the role of equality in the interpretation of $\sigma \approx \tau$, which we describe presently. Nevertheless, it should be remembered that "$\sigma \approx \tau$" merely symbolizes an equation as a linguistic entity; by itself it makes no assertion. On the other hand, "$\sigma = \tau$" does make an assertion: it asserts that σ and τ are precisely the same term.

Our proofs about terms are usually by induction. One way to say this is that we induct over the well-founded order defined on the set of all terms by always taking τ_j to lie below $F_t(\tau_1, \cdots, \tau_{n(t)})$. A more elementary plan — which we adopt — is to assume we have $|\tau| \in \omega$ for every term τ, with $|\tau_j|$ always less than $|F_t(\tau_1, \cdots, \tau_{n(t)})|$, and then to carry out an elementary inductive proof relative to the quantity $|\tau|$. There are many possible ways to define $|\tau|$, such as the number of function symbols in τ.

If the operations $\overline{F_t}$ interpret the symbols F_t on a set A, then every term τ has an associated interpretation $\overline{\tau} \colon A^\omega \longrightarrow A$ which is defined[2] recursively on A via

$$\overline{x_i}(a) = a_i \qquad (6)$$
$$\overline{\tau}(a) = \overline{F_t}(\overline{\tau_1}(a), \ldots, \overline{\tau_{n(t)}}(a)) \qquad (7)$$

where

$$\tau = F_t(\tau_1, \ldots, \tau_{n(t)}) \qquad (8)$$

Notice that the bar notation is not essential to the construction described in Equations (6) and (7). In §3 we will interpret the function symbols F_t with operations $F_t^\star$, and in §5 and §7 with function symbols F_t'. In those contexts, each term τ will have corresponding interpretations $\tau^\star$ and τ'.

In §7 we will need the *N-restricted interpretation* $\overline{\tau}^N \colon A^N \longrightarrow A$. This will be defined only when the variables appearing in τ are among $\{x_i : i < N\}$. In fact $\overline{\tau}^N$ is also defined by Equations (6–8) that define $\overline{\tau}$, but with the domain changed to A^N. In §7 it will be helpful to have Equation (7) recast

[2] Recall that $\omega = \{0, 1, 2, 3, \ldots\}$. We adopt the convention that if $a \in A^\omega$, then a_i denotes the i^{th} component of a. In other words, $a = \langle a_0, a_1, a_2, \ldots \rangle$.

as

$$\overline{\tau}^N \;=\; \overline{F_t} \circ \widehat{\tau}^N \tag{9}$$

where $\widehat{\tau}^N : A^N \longrightarrow A^{n(t)}$ is specified by the equations

$$\pi_i^{n(t)} \circ \widehat{\tau}^N \;=\; \overline{\tau_i}^N \tag{10}$$

for $1 \leq i \leq n(t)$. (Where $\pi_i^{n(t)}$ denotes the i^{th} coordinate projection from $A^{n(t)}$ onto A.)

It will also be useful to be able to compare $\overline{\tau}^N$ and $\overline{\tau}^M$ for $M > N$. For this purpose, we use the N-fold projection operations

$$\begin{aligned} \Pi_N^M : A^M &\longrightarrow A^N \\ \Pi_N : A^\omega &\longrightarrow A^N \end{aligned}$$

which are defined by

$$\begin{aligned} \pi_i^N \circ \Pi_N \quad &= \quad \pi_i \\ \pi_i^N \circ \Pi_N^M \quad &= \quad \pi_i^M \end{aligned} \tag{11}$$

for $i < N$. (In other words (11) says that the i^{th} component of $\Pi_N^M(a)$ is the i^{th} component of a.) It seems obvious that

$$\begin{aligned} \overline{\tau}^N \circ \Pi_N \quad &= \quad \overline{\tau} \\ \overline{\tau}^N \circ \Pi_N^M \quad &= \quad \overline{\tau}^M \end{aligned} \tag{12}$$

for $M > N$, and moreover these equations have an easy inductive proof involving (9) (which we omit).

One easily proves by induction that $\overline{\tau}(a)$ (or $\overline{\tau}^N(a)$) depends only on the variables appearing in τ, i.e., that $\overline{\tau}(a) = \overline{\tau}(b)$ if $a_i = b_i$ for each $i \in \omega$ with x_i appearing in τ. If these variables are x_{i_0}, x_{i_1}, … , one sometimes writes $\overline{\tau}(a_{i_0}, a_{i_1}, \ldots)$ in place of $\overline{\tau}(a)$.

1.2 Identical satisfaction.

An interpretation $\overline{F_t}$ ($t \in T$) on A is said to *model* or *identically satisfy* an equation $\sigma \approx \tau$ iff $\overline{\sigma} = \overline{\tau}$ (as functions defined on A^ω). (The word "identically" can be be omitted in a context such as this one, where identical satisfaction is the main topic. The reader is, however, advised that, in general, satisfaction is a more elaborate topic.) Satisfaction has the notation

$$(A, \overline{F_t})_{t\in T} \ \models \ \sigma \approx \tau, \tag{13}$$

which relates the set A, the operations $\overline{F_t}$, and the formal terms σ and τ. Sometimes we say instead that $\sigma \approx \tau$ *is an identity of* $(A, \overline{F_t})_{t\in T}$. If (13) holds for every equation $\sigma \approx \tau$ in Σ, we write

$$(A, \overline{F_t})_{t\in T} \ \models \ \Sigma, \tag{14}$$

and say that the interpretation $\overline{F_t}$ $(t \in T)$ *models* or *identically satisfies* Σ.

A tuple of the form $(A, \overline{F_t})_{t\in T}$, i.e., a set with operations, is an *algebra*, and so quite often one reads (14) as saying that *the algebra* $(A, \overline{F_t})_{t\in T}$ *models, or satisfies identically the equations* Σ. That terminology is less useful in this paper, since our main point is to prove that *no* interpretation models Σ.

The notation (14) is of course still valid for A the underlying set of a topological space,[3] and this is the notation that we used in (4) of the introduction. As noted there, if in addition the operations $\overline{F_t} : A^{n(t)} \longrightarrow A$ are continuous (with respect to the usual product topology on $A^{n(t)}$), then we say that the operations $\overline{F_t}$ *model* Σ *continuously on* A. (We may also say that $(A, \overline{F_t})_{t\in T}$ is a *topological algebra satisfying* Σ.) If there are any operations $\overline{F_t}$ continuously modeling Σ on A, then we say that Σ is *continuously modelable (satisfiable)* on the space A, or that A *supports* Σ, or simply that A and Σ are *compatible* (as noted in the introduction).

An interpretation $\overline{F_t}$ is said to *model* an equation $\sigma \approx \tau$ *up to homotopy* if $\overline{\sigma}$ and $\overline{\tau}$ are homotopic (as functions $A^\omega \longrightarrow A$). We say that A and Σ are *compatible up to homotopy* if there exist continuous operations $\overline{F_t}$ on A such that each equation of Σ is satisfied up to homotopy. Until §11 we make no real distinction between compatibility and compatibility up to homotopy. In fact, all the topological tools that we use (the homotopy groupoid, the cohomology ring, degrees of maps) are homotopy invariants. Hence our proofs are automatically strong enough to accommodate satisfaction up to homotopy. The reader who chooses to disregard or de-emphasize homotopy in the statement of Theorem 1 will suffer very little loss of understanding.

Let us call an algebra $(A, \overline{F_t})_{t\in T}$ *trite*[4] iff each $\overline{F_t}$ is either a constant operation or a projection operation. To paraphrase what we said in the

[3] In this article, we denote a space and its underlying set by the same letter.

[4] We thank B. Banaschewski for suggesting this terminology.

introduction, a theory Σ is *undemanding* if it has a trite model based on a set B of more than one element. The reader may easily check that the set B itself is hardly relevant: if Σ is undemanding, then for *every* set B, Σ has a trite model based on B.

(Equivalently, Σ is undemanding if it is possible to augment Σ with equations of the form

$$F_t(x_1, \cdots, x_{n(t)}) \approx x_{k_t} \tag{15}$$

or

$$F_t(x_1, \cdots, x_{n(t)}) \approx c \tag{16}$$

(one equation for each $t \in T$) with a consistent outcome. This means that for each equation $\sigma \approx \tau$ from Σ, if one reduces σ and τ each to a variable or constant via Equations (15–16), then σ and τ reduce both to the constant c or both to the same variable.)

Obviously, for finite Σ, it is easy in principle to check if Σ is demanding or not. One need only try all possible combinations of k_t and c (as in (15) and (16)), for t appearing in Σ. Jan Mycielski has shown [private communication] that the problem is NP-complete, and hence most likely the time needed to check whether Σ is demanding grows exponentially with the size of T.

It is easily checked that if each equation in Γ is a consequence of Σ — denoted $\Sigma \vdash \Gamma$ — then every space compatible with Σ is compatible with Γ. It follows readily that if Σ_1 and Σ_2 axiomatize the same equational theory, then Σ_1 and Σ_2 have exactly the same compatibilities. Clearly a similar situation holds for trite models of Σ and Γ, and hence if Σ_1 and Σ_2 axiomatize the same equational theory, then Σ_1 is undemanding iff Σ_2 is undemanding. These remarks should enhance one's understanding of the material, but are not needed in our proofs. Hence we omit any detailed treatment of the consequence relation. In this article, in fact, we work with formal equations $\sigma \approx \tau$ in only one way: we invoke the definition of satisfaction to obtain $\overline{\sigma} = \overline{\tau}$ (or $\overline{\sigma}$ homotopic to $\overline{\tau}$) for some interpretation $\overline{F_t}$ $(t \in T)$.

2 The theorems.

Our main result, Theorem 1, was stated in the Introduction. Theorem 2 — which gives a relatively straightforward extension to finite Cartesian powers of the spaces in Theorem 1 — will be stated in §2.3. Four relatively straightforward results, Theorems 41, 42, 45, 46, will be stated and proved in §11.3.

2.1 Background to Theorem 1.

As we mentioned in the introduction, the specialization of Part 1 to the Equations (1) is an important classical result in algebraic topology, whose proof has a long and influential history. The classical result on H-spaces was easier to prove for some values of the dimension n than for certain other values. The most difficult cases ($n = 2^k - 1$, $k \geq 4$) were completed by J. F. Adams around 1960 [1]. The classical proof, which will be outlined in §8, revolved around the notions of *degree of a map* and the *Hopf invariant of a map.* Our proof of Part 1 will also make essential use of these notions. As we shall see in §2.2, our Part 1 is the best possible extension[5] of the classical result to other theories.

We asked in 1986 — see page 38 of [39] — whether there is any simple space A for which the conclusion of Theorem 1 holds. (We knew about, but were not content with, the very complicated space of Cook that is mentioned in §2.2.) In particular, we asked whether Theorem 1 holds for A taken to be a figure-eight space or the 2-sphere S^2. (These two spaces stood out because they had no history of compatibility with any demanding Σ, and because they were known to be incompatible with many Σ's.) Problem 9.4 on page 83 of [39] in fact pointed out that for almost any space $(A, \mathcal{T})$ that one can name, it is open whether A is compatible (5) with any non-trivial[6] Σ. This article represents the first time that we can answer[7] the question negatively for any relatively simple space $(A, \mathcal{T})$.

Parts 1, 2 and 3 of Theorem 1 have long been known for idempotent Σ. (A set Σ of equations is called *idempotent* if the equation

$$F_t(x, \ldots, x) \;=\; x$$

is a consequence of Σ for each $t \in T$.) The idempotent specialization of Part 2 follows immediately from Theorems 3.1 and 5.1 of [37]; the idempotent specialization of Part 3 is a special case of Theorem 2.8 of [38]; the idempotent

[5] The reader should note that we are not offering a slick new general-algebra proof of a classical result. In fact, our proof of Part 1 applies the same deep methods of algebraic topology — notably Theorem 38 below — that were used for the H-spaces result.

[6] The concept of "undemanding" was not developed at that time.

[7] I am happy to say also that Problem 9.1 [*loc. cit.*] has been solved affirmatively by Vera Trnková. There do exist spaces A and B that are the same at the first clone level, but whose clones satisfy different first-order sentences at higher levels. See recent articles by V. Trnková [41] [42] [43] [44], and by J. Sichler and V. Trnková [32].

specialization of Part 1 is a special case of Corollary 3.2 of [38]. (By the same token, the difficult and unmotivated tensor-algebra calculations on pages 80–85 of [38] are outmoded by the easier calculations in this article.) At the homotopy level (Part 2), one methodological advance over the earlier paper [37] is in the use of the full homotopy groupoid (§3). (The homotopy group sufficed in the idempotent case.)

2.2 Remarks on Theorem 1.

Notice that Parts 1 and 3 of the theorem overlap, in that they both cover the even-dimensional spheres. However, Part 1 is obviously stronger, and Part 3 is more general, since its hypotheses are only about the cohomology ring of A. Similarly Parts 2 and 7 overlap on the figure-eight space.

Projection functions and constant functions are always continuous; hence a trite algebra (§1.2) can be topologized in any way one likes. One immediately sees that if Σ is undemanding, then Σ is compatible with every space A. As we mentioned in [39], the converse statement — if Σ is compatible with every space A, then Σ is undemanding — follows immediately from the existence of a space A with the extravagant property that *every* topological algebra based on A is trite. Such a space, the *continuum of Cook*,[8] has been known since 1967. It now seems (to the author) more intuitive and accessible to base the converse on Theorem 1: if Σ is compatible with every space, then Σ is compatible with each A in Theorem 1, and hence undemanding.

The preceding remarks make it clear that each part of Theorem 1 decides the question of compatibility for every Σ, and hence is a best-possible incompatibility result for its space A. From other perspectives, e.g. the perspective of fixing a consistent demanding theory Σ and characterizing those spaces compatible with Σ, there is nothing even close to a best-possible result.

It is important to realize that the conclusion to Theorem 1 is a conclusion about Σ, and not about any particular operations modeling Σ on A. For example, if Σ consists of the commutative law

$$F(x, y) \approx F(y, x),$$

then on the sphere S^2 there are many ways to model Σ other than with a constant operation — e.g., $\overline{F}(x, y) = \phi(d(x, y))$, where d represents Euclidean distance, and ϕ is any continuous function from R to S^2. Part 3

[8]Constructed by H. Cook in a series of articles culminating in [8]; a self-contained exposition occupies a long appendix to [31].

obviously cannot preclude the existence of this $\overline{F}$. Rather, its proof will use the cohomology functor to construct a new operation F' from $\overline{F}$, in such a way that F' is either a projection or a constant, and F' also satisfies Σ. (In other words, Theorem 1 asserts the possibility of constructing a trite algebra from $(S^2, \overline{F})$, but does not assert that $(S^2, \overline{F})$ itself must be trite.)

The results here work a little differently than those found in 1977 [37], in 1981 [38] or in 1986 [39]. In those earlier articles, some general properties of the space $(A, \mathcal{T})$ (such as non-commutativity of its fundamental group) were used to rule out the compatibility relation (5) for certain Σ's (such as Σ defining group theory). In this article, we narrow our focus down almost to a single space $(A, \mathcal{T})$ (e.g. by specifying the isomorphism type of its fundamental group), and then prove that in this context (5) fails for all Σ (except for undemanding Σ).

2.3 Extension of Theorem 1 to q^{th} powers.

Our second theorem requires a mild extension of the notion of an undemanding Σ. We call a set Σ of equations *easily satisfied in q^{th} powers*, or *q-undemanding*, iff there is a set A of more than one element, which is a q^{th} power (i.e. $A = B^q$ for some set B), and there are operations $\overline{F_t}\colon A^{n(t)} \longrightarrow A$, such that

$$(A, F_t)_{t \in T} \;\models\; \Sigma$$

and such that, for each $t \in T$, and for $i = 1, \ldots, q$, the composite map

$$A^{n(t)} = B^{qn(t)} \quad \xrightarrow{\overline{F_t}} \quad A = B^q \quad \xrightarrow{\pi_i} \quad B$$

is either a projection or a constant.

For an example, consider Σ consisting of the single equation

$$F(F(x, y), F(y, x)) \approx y. \tag{17}$$

Clearly Σ is demanding. Nevertheless Σ can be satisfied on $A = B^2$ by defining[9]

$$F((b_1, b_2), (b_3, b_4)) \;=\; (b_2, b_3),$$

and so Σ is 2-undemanding.

[9]For an elementary, but very rich, exposition of some of the many possible ways to define operations on $A = B^q$ in terms of their components in B—and the associated varieties — see Evans [10].

One nice thing about interpretations of this type is that they respect the product topology on B^q. That is, if B is a topological space, and A is given the q-fold product topology of B^q, then (obviously) $\overline{F_t}$ is a continuous operation on A. Thus, if Σ is q-undemanding, then Σ is compatible with every space A that is homeomorphic to a direct power B^q (for any space B). The converse again follows from the existence of a Cook space: if B is a Cook space and $A = B^q$, then all operations on A are of the desired type; hence any Σ compatible with A is q-undemanding. As in our earlier theorems, we here present simpler q-th power spaces that are compatible only with q-undemanding sets of equations.

Theorem 2 *If the space $A = B^q$, $1 \leq q < \omega$, where B is as in Theorem 1, and if A is compatible with Σ up to homotopy, then Σ is q-undemanding.*

2.4 A lemma for all parts of Theorem 1.

Although the topological methods vary, there is one simple lemma that unites our proofs for the seven parts of Theorem 1. In each case, given operations $\overline{F}_t$ on A satisfying Σ, we will supply a construction (which depends on the situation) of a trite algebra $(B, F'_t)_{t\in T}$. It will then be our job to show that the constructed algebra $(B, F'_t)_{t\in T}$ also satisfies Σ. That is the conclusion of Lemma 3.

Since we are discussing satisfaction, the lemma naturally refers to the term operations $\overline{\tau}$ (built from the operations $\overline{F_t}$) and τ' (built from the operations F'_t.

Lemma 3 *Given operations $\overline{F}_t$ defined on A for $t \in T$, and operations F'_t defined on B for $t \in T$. If it is possible to define each term operation $\tau' : B^\omega \longrightarrow B$ directly from the term operation $\overline{\tau} : A^\omega \longrightarrow A$ (i.e.,* without reference to the syntax of *τ), and if the operations $\overline{F}_t$ satisfy Σ, then the operations F'_t also satisfy Σ.*

In the topological context, if it is possible to define each term operation τ' directly from the homotopy class of the term operation $\overline{\tau}$, and if the operations $\overline{F}_t$ satisfy Σ up to homotopy, then the operations F'_t also satisfy Σ.

The same conclusions hold if there is an algorithm defining τ' directly from $\overline{\tau}^N$ (or its homotopy class) whenever N is bigger than the subscript of any variable appearing in τ.

Proof. (We prove the first assertion only.) Consider an equation $\sigma \approx \tau$ from Σ. Since the operations $\overline{F}_t$ satisfy $\sigma \approx \tau$, the term operations $\overline{\sigma}$ and $\overline{\tau}$ are identical (as operations defined on A^ω). It follows from our hypothesis that the term operations σ' and τ' are identical (as operations defined on B^ω). In other words, the operations F'_t satisfy $\sigma \approx \tau$. Since this was an arbitrary equation from Σ, we see in fact that the operations F'_t satisfy Σ. ∎

Much of the work that follows, therefore, has to do with establishing the hypothesis of Lemma 3 in each of our various contexts. Each of Lemmas 14, 15, 26, 35 and 40 shows how to define τ' from $\overline{\tau}$ (or from a $\tau^\star$ that is readily obtained from $\overline{\tau}$). As one might imagine, those lemmas are proved by induction on $|\tau|$.

3 Path groupoids and fundamental groups.

To every topological space A there is associated an algebraic object known as its *path groupoid* or *fundamental groupoid*, denoted $\Pi(A)$. The *fundamental group* or *first homotopy group* of A will appear (in Lemma 4) as a subgroup of $\Pi(A)$. (This definition of the path groupoid agrees with the one on page 139 of [22]. See also [37].) The elements of $\Pi(A)$ are the equivalence classes of continuous maps (or *paths*) $\gamma : [0,1] \longrightarrow A$ with respect to the relation (denoted $\sim$) of homotopy with endpoints fixed. (The $\sim$-class of γ is denoted $[\gamma]$.)

A binary operation ("product") is defined on the set $\Pi(A)$ as follows. If γ and δ are paths, as defined above, and if $\gamma(1) = \delta(0)$, then the product $\gamma \cdot \delta : [0,1] \longrightarrow A$ is defined by

$$\gamma \cdot \delta(x) \;=\; \begin{cases} \gamma(2x) & 0 \le x \le \frac{1}{2} \\ \delta(2x-1) & \frac{1}{2} \le x \le 1. \end{cases}$$

It is not hard to check that the homotopy class of $\gamma \cdot \delta$ depends only on the homotopy classes of γ and δ, and hence $[\gamma] \cdot [\delta]$ can unambiguously be defined as $[\gamma \cdot \delta]$. It is also not hard to prove that if γ^{-1} is defined by

$$\gamma^{-1}(x) \;=\; \gamma(1-x), \tag{18}$$

then

$$\gamma \cdot \gamma^{-1} \quad \sim \quad \gamma(0) \tag{19}$$
$$\gamma^{-1} \cdot \gamma \quad \sim \quad \gamma(1) \tag{20}$$

(with the right-hand sides denoting constant maps).

For $a \in A$, we define the set of *loops at* a to be the subset of $\Pi(A)$:

$$\Pi_a(A) \;=\; \{[\gamma] \in \Pi(A) \;:\; \gamma(0) = a = \gamma(1)\,\}. \tag{21}$$

Lemma 4 $(\Pi_a(A), \cdot, ^{-1})$ *is a group (whose unit element is the constant path with value a). (This group is frequently known as the* **fundamental group** *of A or the* **first homotopy group of** *A.)*

The reader can easily prove that if there is a path from a to b in A, then $\Pi_a(A) \cong \Pi_b(A)$. See also Lemma 10 below. Thus in path-connected spaces, all the fundamental groups are isomorphic.

At the abstract level, a category is called a *groupoid* if for each γ there exists γ^{-1} satisfying Equations (19) and (20). The construction of $\Pi_a(A)$ from $\Pi(A)$ has a counterpart in category theory of forming the monoid of self-maps of a given object. If the category is a groupoid, then the individual monoid is a group. Since they are not necessary for our work in this paper, we omit the precise form of these abstract statements. The reader may consult any basic work on category theory, or, for example, §3.6 of [22]. Also see Higgins [14] and Brown [5] and [6].

With no further assumptions, $\Pi(A)$ might consist only of constant paths, or indeed it might be the case that there are many non-constant paths, but that any two paths with the same endpoints are homotopic to each other. (The first possibility occurs for a totally disconnected space like the Cantor set or the rational line; the second occurs e.g. for Euclidean space R^n.) In such extreme cases $\Pi(A)$ contains no useful information. The fortunate fact is that some spaces A — such as A a figure-eight — have highly complex and non-trivial $\Pi(A)$. In fact this A has $\Pi_a(A)$ a free group on two generators (regardless of the choice of a).

Now suppose that our space A is equipped with some continuous operations $\overline{F_t}$ for $t \in T$, in other words, that $(A, \overline{F_t})_{t\in T}$ is a topological algebra based on A. We first observe that paths can be subjected to the operations $\overline{F_t}$, simply by performing the operations pointwise. In other words, we extend the operations $\overline{F_t}$ to paths $\gamma_i : [0,1] \longrightarrow A$ as follows:

$$\overline{F_t}(\gamma_1, \ldots, \gamma_{n(t)})(x) \;=\; \overline{F_t}(\gamma_1(x), \ldots, \gamma_{n(t)}(x)) \tag{22}$$

for $t \in T$ and $0 \leq x \leq 1$. In order to define counterparts of $\overline{F_t}$ on $\Pi(A)$, we next need to consider the homotopy relation.

Lemma 5 *The homotopy relation $\sim$ is a congruence relation on the algebra of all paths. In other words, If $\gamma_i \sim \delta_i$ for $1 \leq i < n(t)$, then*

$$\overline{F_t}(\gamma_1, \ldots, \gamma_{n(t)}) \;\sim\; \overline{F_t}(\delta_1, \ldots, \delta_{n(t)})$$

Now, in the usual way, one can form the *quotient algebra* with respect to homotopy of the algebra of paths under the operations $\overline{F_t}$ (for $t \in T$). By §3 its universe is $\Pi(A)$, and so we have constructed an algebra

$$(\Pi(A), F_t^\star)_{t \in T}$$

whose operations

$$F_t^* : \Pi(A)^{n(t)} \longrightarrow \Pi(A)$$

are defined via

$$F_t^\star([\gamma_1], \ldots, [\gamma_{n(t)}]) \;=\; [\overline{F_t}(\gamma_1, \ldots, \gamma_{n(t)})].$$

We skip the proof of the following lemma, which involves a fairly obvious induction on $|\tau|$.

Lemma 6 *Let τ be a term in the operation symbols F_t ($t \in T$). Let $\overline{\tau}$ be the term-operation defined by τ in $(A, \overline{F_t})_{t \in T}$, and let $\tau^\star$ be the term-operation defined by τ in $(\Pi(A), F_t^\star)_{t \in T}$. Then*

$$\tau^\star([\gamma_0], [\gamma_1], \ldots) \;=\; [\overline{\tau}(\gamma_0, \gamma_1, \ldots)]$$

(where the right-hand side denotes the homotopy class of the indicated curve from $[0, 1]$ to A).

Lemma 7, which follows, is the conceptual underpinning of our proof of Part 2, for it allows us to apply the hypothesis that A is compatible with Σ. It is for this purpose that we require the fundamental groupoid; there is no counterpart to Lemma 7 for the fundamental group.

Lemma 7 *Let Σ be a set of equations in the operation symbols F_t ($t \in T$). If continuous operations $\overline{F_t}$ model Σ on A up to homotopy, then the operations $F_t^\star$ model Σ on $\Pi(A)$. In other words*

$$\textit{if} \quad (A, \overline{F_t})_{t \in T} \;\models\; \Sigma, \qquad \textit{then} \quad (\Pi(A), F_t^\star)_{t \in T} \;\models\; \Sigma.$$

Proof. Immediate from Lemma 6. ∎

Lemmas 5–7 do not in themselves imply any particular advantage to the path-algebra $\Pi(A)$. The real advantage of this algebra comes in the combination of Lemma 7 with the following lemma.

Lemma 8 *The operations $F_t^\star$ $(t \in T)$ commute with the multiplication of paths in $\Pi(A)$. In other words, if $[\gamma_i], [\delta_i] \in \Pi(A)$ $(1 \leq i \leq n(t))$, and if $[\gamma_i] \cdot [\delta_i]$ is defined for each i, then*

$$F_t^\star([\gamma_1] \cdot [\delta_1], \cdots, [\gamma_{n(t)}] \cdot [\delta_{n(t)}]) \;=\; F_t^\star([\gamma_1], \cdots, [\gamma_{n(t)}]) \,\cdot\, F_t^\star([\delta_1], \cdots, [\delta_{n(t)}]). \tag{23}$$

Lemma 8 may be summarized by saying that *each operation $F_t^\star$ is a groupoid-homomorphism.* We mostly use the following specialization of Lemma 8 to the case where the γ_i and δ_i are loops at a single point a. The full path-groupoid is useful as a context for establishing satisfaction of Σ (as we did in Lemma 7); on the other hand, for calculations about homomorphisms, it is more useful to work at the level of $\Pi_a(A)$, since we have a whole theory of group homomorphisms to draw on (see §4 below).

Lemma 9 *Let $t \in T$, let $a_1, \ldots, a_{n(t)} \in A$, and let $b = \overline{F}_t(a_1, \ldots, a_{n(t)})$. The operation $F_t^\star$ maps the group $\Pi_{a_1}(A) \times \cdots \times \Pi_{a_{n(t)}}(A)$ to the group $\Pi_b(A)$. Moreover the resulting map, which we also denote*

$$F_t^\star : \Pi_{a_1}(A) \times \cdots \times \Pi_{a_{n(t)}}(A) \longrightarrow \Pi_b(A),$$

is a homomorphism of groups.

Lemma 9 interests us especially in the case where the fundamental groups $\Pi_a(A)$ are free on two (or more) generators, since, as we shall see in §4, the homomorphisms between such groups are few and limited.

Lemma 10 *If A is pathwise connected, the homomorphism in Lemma 9 is independent, up to isomorphisms, of the choice of $a_1, \cdots, a_{n(t)}$. More precisely, for all $a_1, \cdots, a_{n(t)}, c_1, \cdots, c_{n(t)} \in A$, for $b = \overline{F_t}(a_1, \cdots, a_{n(t)})$, and for $d = \overline{F_t}(c_1, \cdots, c_{n(t)})$ there are group isomorphisms*

$$\begin{aligned} \lambda_i : \Pi_{a_i}(A) &\longrightarrow \Pi_{c_i}(A) \\ \mu : \Pi_b(A) &\longrightarrow \Pi_d(A) \end{aligned}$$

for $1 \leq i \leq n(t)$, *such that*

$$F_t^\star(\lambda_1([\gamma_1]), \cdots, \lambda_{n(t)}([\gamma_{n(t)}])) \;=\; \mu(F_t^\star([\gamma_1], \cdots, [\gamma_{n(t)}])) \qquad (24)$$

for all $[\gamma_i] \in \Pi_{a_i}(A)$ $(1 \leq i \leq n(t))$.

4 Operations on non-commutative free groups.

In this section we consider maps from a finite power of a finitely generated free group, to another finitely generated free group. We assume that the reader has some background in the general subject of free groups. (See for instance Hall [13] or Magnus, Karrass and Solitar [23]. There is also a short development of free groups on pages 119–120 of [22].)

The only thing one needs to carry from §4 to the rest of the paper is Lemma 13, which rather strictly curtails the homomorphisms that are possible from a power G^n of a free group to G. The application of Lemma 13 (in §5) will be to the group homomorphisms described in Lemma 9 above.

Our first lemma is actually a rather deep theorem in free-group theory. We will not include a proof. We quote Hall's version of the statement.

Lemma 11 *A free group* F_r *with a finite number* r *of generators is freely generated by any set of* r *elements which generate it.*

Proof. See Theorem 7.3.3 on page 109 of Hall [13], or Corollary 2.13.1 on page 110 of Magnus, Karrass and Solitar [23]. ▮

Lemma 12 *If* G *is a free group on* k *generators (for some* $k < \omega$*), and* $f : G \longrightarrow G$ *maps onto* G*, then* f *is one-to-one.*

Proof. Let $g_1, \ldots, g_k$ be free generators of G. Since f is onto, $f(g_1), \ldots, f(g_k)$ generate G; by Lemma 11, they freely generate G. Hence there exists a homomorphism $h : G \longrightarrow G$ such that

$$h(f(g_j)) \;=\; g_j \qquad (1 \leq j \leq k)$$

Since the g_i generate G, $h \circ f$ is the identity map, and hence f is one-to-one.

▮

We thank the referee for suggestions leading to a simplification of the proof of Lemma 13.

Lemma 13 *If G is a free group on k generators (for some k with $2 \leq k < \omega$), and if $\overline{F}\colon G^n \longrightarrow G$ maps onto G, then there exists r $(1 \leq r \leq n)$ and there exists an automorphism ϕ of G such that*

$$\overline{F}(x_1, \cdots, x_n) \;=\; \phi(x_r) \tag{25}$$

for all $x_1, \ldots, x_n \in G$.

Proof. Let G be freely generated by $g_1, \ldots, g_k$. Since F is onto, there exist elements $a_{ij} \in G$ $(1 \leq i \leq k,\ 1 \leq j \leq n)$ such that

$$\overline{F}(a_{i1}, \cdots, a_{in}) \;=\; g_i \tag{26}$$

for $1 \leq i \leq k$.

Now consider the $k \times n$ matrix M whose i, j-entry is

$$m_{ij} \;=\; \overline{F}(1, \cdots, 1, a_{ij}, 1, \cdots, 1) \tag{27}$$

with a_{ij} in the j^{th} place. From the fact that $\overline{F}$ is a group homomorphism, one easily sees that

$$m_{ij} \quad \text{commutes with} \quad m_{rs} \quad \text{for} \quad j \neq s, \tag{28}$$

and, using Equation (26), that the elements in the i^{th} row of M have product g_i. In other words

$$m_{i1}\, m_{i2} \,\cdots\, m_{in} \;=\; g_i \tag{29}$$

for each i. It readily follows from (28) and (29) that, for each i, the elements of the i^{th} row, namely $m_{i1}, \ldots, m_{in}$ all lie in a commutative subgroup of G containing g_i. From elementary free-group theory we know that the only such subgroup is the subgroup generated by g_i. Thefore, we may write

$$m_{ij} \;=\; g_i^{n_{ij}} \tag{30}$$

for some appropriate integers n_{ij}.

We claim that all non-unit m_{ij} must either lie in one column (i.e. have a single value for i) or in one row (a single value for j). For suppose not: then we have $m_{ij} \neq 1 \neq m_{rs}$ with $i \neq r$ and $j \neq s$. By (30), m_{ij} does not commute with m_{rs}. By Equation (28), m_{ij} does commute with m_{rs}. This contradiction establishes our claim that all non-unit m_{ij} lie in a single row or a single column.

Equation (29) tells us that each row contains at least one non-unit m_{ij}; hence it must be that all non-unit m_{ij} lie in a single column, say the r^{th} column. In other words,

$$m_{ij} \;=\; 1 \qquad \text{unless} \qquad j \;=\; r. \tag{31}$$

From Equations (26) and (31), we deduce that

$$\overline{F}(1, \cdots, 1, a_{ir}, 1, \cdots, 1) \;=\; g_i \tag{32}$$

for each i.

We now define $\phi : G \longrightarrow G$ via

$$\phi(x) \;=\; \overline{F}(1, \cdots, 1, x, 1, \cdots, 1), \tag{33}$$

with x appearing in the r^{th} position. By Equation (32), ϕ maps G onto G, and so ϕ is an automorphism of G, by Lemma 12. (Notice that Equation (33) is a special case of the desired Equation (25).)

We next observe, for $b_1, \cdots, b_n \in G$, that if $b_r = 1$, then $(b_1, \cdots, b_n)$ commutes with $(1, \cdots, 1, x, 1, \cdots, 1)$ in the group G^n. By Equation (33), $\overline{F}(b_1, \cdots, b_n)$ commutes with every $\phi(x)$, and hence with every element of G, since ϕ is onto. In a non-commutative free group, the only element commuting with every element is the unit element. Thus

$$\overline{F}(b_1, \cdots, b_n) \;=\; 1 \tag{34}$$

whenever $b_r = 1$. (Another special case of the desired Equation (25).)

Our final calculation is now immediate from Equations (33) and (34):

$$\begin{aligned} \overline{F}(x_1, \cdots, x_n) &= \overline{F}(1, \cdots, 1, x_r, 1, \cdots, 1) \cdot \overline{F}(x_1, \cdots, x_{r-1}, 1, x_{r+1}, \cdots, x_n) \\ &= \phi(x_r) \cdot 1 \\ &= \phi(x_r). \end{aligned}$$

This proves the lemma. ▮

5 The proof of Part 2 of Theorem 1.

As in the statement of the theorem, we let A be a path-connected topological space whose fundamental group is free on k generators $(2 \leq k < \omega)$ — for instance, a figure-eight space.

We assume that A is homotopy-compatible with a set Σ of equations. That is, we are given continuous operations $\overline{F_t}\colon A^{n(t)} \longrightarrow A$ such that

$$(A, \overline{F_t})_{t\in T} \models \Sigma$$

up to homotopy. Our objective is to prove that Σ is undemanding. In other words, we need to find special operations F'_t modeling Σ, i.e. operations F'_t on a set B with more than one element, such that

$$(B, F'_t)_{t\in T} \models \Sigma$$

and such that each F'_t is either a constant operation or a projection operation.

According to §3, we have operations $F^\star_t$ defined on the groupoid $\Pi(A)$ such that

(i) $(\Pi(A), F^\star_t)_{t\in T} \models \Sigma$ (Lemma 7).

(ii) The subset $\Pi_a(A)$ has the structure of a free group on k generators (Lemma 4).

(iii) For arbitrary $a_1, \cdots, a_{n(t)} \in A$, and for $b = \overline{F_t}(a_1, \cdots, a_{n(t)})$, the restriction of $F^\star_t$ is a group-homomorphism from $\Pi_{a_1}(A) \times \cdots \times \Pi_{a_{n(t)}}(A)$ to $\Pi_b(A)$ (Lemma 9). When we say that $F^\star_t$ is *onto*, we mean that this restricted map has $\Pi_b(A)$ as its image.

(iv) The homomorphism of (iii) is onto or not, independently of the choice of $a_1, \cdots, a_{n(t)}$ (Lemma 10).

Definition of the operations F'_t: We now define the set B and the operations F'_t on B. In fact B can be taken as any set with more than one element. We then let c be any element of B, and define the operations F'_t as follows:

(A) If $F^\star_t$ is not onto (see (iii) above; by (iv) this condition is independent of $a_1, \cdots, a_{n(t)}$), we define

$$F'_t(x_1, \cdots, x_{n(t)}) \;=\; c.$$

(B) If $F_t^\star$ is onto, then by Lemma 13 we have $F_t^\star(x_1, \cdots, x_{n(t)}) = \phi(x_i)$ for some i and some automorphism ϕ. It is an easy application of Lemma 10 to see that in fact i does not depend on the choice of $a_1, \cdots, a_{n(t)}$. In this case we define

$$F'_t(x_1, \cdots, x_{n(t)}) = x_i.$$

This completes the definition of the operations F'_t for $t \in T$. Evidently each F'_t is either a constant or a projection operation. What remains is to show that they model Σ.

In Equations (6) and (7) we saw how an interpretation of symbols F_t by operations $\overline{F_t}$ leads to an interpretation of any term τ by a function $\overline{\tau} : A^\omega \longrightarrow A$. It is merely a change of notation to do the same thing for the interpretations $F_t^\star$: they lead in the same way to an associated interpretation $\tau^\star : \Pi(A)^\omega \longrightarrow \Pi(A)$. And, much as before, for any sequence $a_0, a_1, \cdots \in A$, the restriction of $\tau^\star$ to the group $\Pi_{a_0}(A) \times \Pi_{a_1}(A) \times \cdots$ is a homomorphism from that group to $\Pi_{\overline{\tau}(a_0,\cdots)}(A)$. When we say that $\tau^\star$ is onto (in the statements and proofs of Lemmas 14 and 15 below), we are referring to the surjectivity of this restricted homomorphism. An obvious analog of Lemma 10 tells us that $\tau^\star$ is onto or not, regardless of the choice of $a_0, a_1, \cdots$.

Similarly, interpretations F'_t of the operation symbols lead to an interpretation τ' of each term τ. In these terms, our plan for the rest of §5 can be expressed as follows: we are given that $\sigma^\star = \tau^\star$ for an equation $\sigma \approx \tau$ of Σ; we need to prove that $\sigma' = \tau'$.

Lemma 13 and (B) are special cases of the next Lemma.

Lemma 14 *If $\tau^\star$ is onto, then $\tau^\star(x) = \phi(x_j)$ for some j and some automorphism ϕ of the fundamental group. In this case*

$$\tau'(x) = x_j$$

Proof. The proof is by induction on $|\tau|$. If τ is a variable, the conclusion clearly holds (with ϕ taken as the identity map). Otherwise, by Equations (6–8), τ is formed as $F_t(\tau_1, \cdots, \tau_{n(t)})$, and

$$\tau^\star(x) = F_t^\star(\tau_1^\star(x), \cdots, \tau_{n(t)}^\star(x)). \tag{35}$$

Since $\tau^\star$ was assumed to be onto, we know that $F_t^\star$ must also be onto. Hence, by Lemma 13,

$$F_t^\star(x_1, \cdots, x_{n(t)}) = \psi(x_i) \tag{36}$$

for some i and some automorphism ψ. By (B), we have

$$F'_t(x_1, \cdots, x_{n(t)}) \;=\; x_i$$

for all x. It follows immediately that

$$\tau'(x) \;=\; \tau'_i(x). \tag{37}$$

Now from Equations (35) and (36) we immediately deduce that

$$\tau^\star_i \;=\; \psi^{-1} \circ \tau^\star$$

and hence that $\tau^\star_i$ is onto. Therefore, by induction,

$$\tau^\star_i(x) \;=\; \lambda(x_j) \tag{38}$$

for some j and some automorphism λ, and moreover

$$\tau'_i(x) \;=\; x_j. \tag{39}$$

Now by Equations (35), (36) and (38), we have

$$\begin{aligned} \tau^\star(x) \;&=\; F^\star_t(\tau^\star_1(x), \cdots, \tau^\star_n(x)) \\ &=\; \psi(\tau^\star_i(x)) \;=\; \psi(\lambda(x_j)) \;\;=\; \phi(x_j) \end{aligned}$$

(where $\phi = \psi \circ \lambda$). And by Equations (37) and (39), $\tau'(x) \;=\; x_j$. ∎

Lemma 15 *If $\tau^\star$ is not onto, then $\tau'(x) = c$.*

Proof. The proof is by induction on $|\tau|$. Clearly τ is not a variable, so

$$\tau \;=\; F_t(\tau_1, \cdots, \tau_{n(t)}) \tag{40}$$

for some terms $\tau_1, \ldots, \tau_{n(t)}$.

Case 1: $F^\star_t$ is not onto. Then $F'_t(x_1, \cdots, x_{n(t)}) = c$, by (A). Clearly then τ' is the same constant, and the proof is complete in this case.

Case 2: $F^\star_t$ is onto. In this case, by (B),

$$F^\star_t(x_1, \cdots, x_{n(t)}) \;=\; \phi(x_i) \tag{41}$$

for some i and some automorphism ϕ, and moreover

$$F'_t(x_1, \cdots, x_{n(t)}) \;=\; x_i.$$

From Equations (40) and (41) we have

$$\tau^\star(x) \;=\; F^\star_t(\tau^\star_1(x), \cdots, \tau^\star_{n(t)}(x)) \;=\; \phi(\tau^\star_i(x))$$

and so $\tau^\star_i$ is not onto. By induction, $\tau'_i(x) = c$, and so

$$\tau'(x) \;=\; F'_t(\tau'_1(x), \cdots, \tau'_n(x)) \;=\; \tau'_i(x) \;=\; c.$$

$\blacksquare$

Completion of the proof of Part 2. We begin by establishing the hypothesis of Lemma 3 (from §2.4). Clearly Lemma 7 (or Lemma 6) implies that the operation $\tau^\star$ depends only on the homotopy class of the operation $\overline{\tau}$, and clearly Lemmas 14 and 15 define τ' from the operation $\tau^\star$. All in all, we have τ' defined from the homotopy class of the operation $\overline{\tau}$, and so the hypothesis of Lemma 3 is satisfied. Thus the operations F'_t satisfy Σ, and hence Σ is undemanding. This completes the proof of Part 2.

6 CGR's and cohomology.

In §6.3 below, we will summarize the needed facts about the (absolute) *cohomology ring* $H^\star(A, R)$, with coefficients in a fixed commutative ring R with unit. We preface that section with two purely algebraic sections.

6.1 Commutative graded rings.

Let R be a commutative ring with unit. All of §6.1 (a list of definitions) makes sense in this general context, Many of the proofs that come later (notably Lemmas 16 and 25, and Theorem 21) require further assumptions on R — either that it is a field or at least an integral domain. To make matters simpler, therefore, until §8, we will assume that R is a field. In some of our work, such as Lemma 28 of §15, it will be necessary to assume that the characteristic of R is not 2. On the other hand, in §7.4 (the Klein bottle) and §7.5 (real projective space), we will work with the field $Z/2$ of integers modulo 2. Then, in defining *degrees* in §8 below, we will take R to be Z, the ring of integers. Lemma 43 below also requires integral coefficients.

A *graded ring* over R is an associative bilinear algebra H over R with unit (see page 15 of [22]), which has designated R-submodules H_i ($i \in \omega$) such that

(i) $H \;=\; \bigoplus_{i\in\omega} H_i$

(ii) $H_i H_j \;\subseteq\; H_{i+j}$.

It follows, of course, that the unit element 1 lies in H_0. A *commutative graded ring over* R (to which we will refer as an R-CGR) is a graded ring over R that also satisfies

(iii) $xy \;=\; (-1)^{pq}\, yx$

for $x \in H_p$ and $y \in H_q$.

In fact, the reader of this article need only be concerned with the specific CGR's that are defined (rather simply) in Lemmas 20, 27, 29, 31 and 33, and finite tensor powers of these CGR's.

A *homomorphism* from an R-CGR H to an R-CGR K is a homomorphism $f\colon H \longrightarrow K$ of bilinear algebras that also satisfies

(i) $f[H_p] \;\subseteq\; K_p$

(ii) $f(hk) \;=\; f(h)f(k)$,

for $p, q \in \omega$, $h \in H_p$ and $k \in K_q$.

The *tensor product* $H \otimes K$ of R-CGRs H and K is the R-CGR with the following presentation. Its generators are all ordered pairs $(h, k) \in H \times K$. Such a pair, in the context of the tensor product, is traditionally denoted $h \otimes k$. The relators for the presentation are

(i) all relations of R-multilinearity:

$$\begin{aligned} r(h \otimes k) \;&=\; (rh) \otimes k \;=\; h \otimes (rk) \\ (h_1 + h_2) \otimes k \;&=\; (h_1 \otimes k) + (h_2 \otimes k) \\ h \otimes (k_1 + k_2) \;&=\; (h \otimes k_1) + (h \otimes k_2) \end{aligned}$$

(ii)

$$(h_1 \otimes k_1) \cdot (h_2 \otimes k_2) \;=\; (-1)^{pq}(h_1h_2 \otimes k_1k_2).$$

for[10] $h_2 \in H_p$ and $k_1 \in H_q$.

Finally,

(iii) $H \otimes K$ is made into a graded algebra by defining

$$(H \otimes K)_p \;=\; \{\, h \otimes k \;:\; (\exists s \leq p)(h \in H_s \text{ and } k \in K_{p-s}) \,\}$$

The reader may easily check that the mapping

$$\eta_1 : h \longmapsto h \otimes 1 \tag{42}$$

is a homomorphism $\eta_1 : H \longrightarrow H \otimes K$ — called the *first-coordinate injection* — and that

$$\eta_2 : k \longmapsto 1 \otimes k \tag{43}$$

is a homomorphism $\eta_2 : K \longrightarrow H \otimes K$ — *second-coordinate injection.* Moreover, categorically speaking, the diagram

$$\begin{tikzcd} H \arrow[dr, "\eta_1"] & \\ & H \otimes K \\ K \arrow[ur, "\eta_2"'] & \end{tikzcd}$$

is a *co-product.* In other words, given R-CGR homomorphisms $f_1 : H \longrightarrow G$, $f_2 : K \longrightarrow G$, there is a unique homomorphism $f : H \otimes K \longrightarrow G$ such that $f \circ \eta_i = f_i$ (for $i = 1, 2$). In other words, the diagram

$$\begin{tikzcd} & H \arrow[dl, "f_1"'] \arrow[dr, "\eta_1"] & \\ G & & H \otimes K \arrow[ll, "f"'] \\ & K \arrow[ul, "f_2"] \arrow[ur, "\eta_2"'] & \end{tikzcd}$$

commutes. (The interested reader may prove this for himself.)

[10] For the proof of Part 3 the reader may disregard the minus signs that crop up for elements of odd degree, both here and in the foregoing definition of commutativity. In fact, the R-CGR for the proof of Part 3 has $H_{2n+1} = \{0\}$ for $n \geq 0$. Nevertheless, the minus sign explains why the proof of Part 3 does not extend to odd dimensions. Moreover, the minus sign is used in an essential way in the (omitted) proof of Lemma 17 below. In §7.4 and §7.5, the minus sign will not appear; the characteristic is 2.

6.2 Some technical lemmas on CGR's.

We present, mostly without proof, some particular results on R-CGR's. They will be useful in the proofs of Parts 4, 5 and 7 in §7 below. (Lemma 17 appears in the proofs of Lemma 34 and Lemma 30, and Lemma 19 appears in the proof of Lemma 28.) On a first reading one may be well advised to skip §6.2 and proceed directly to §6.3. Then one could focus first on the proof of Part 3, which epitomizes the cohomological method, without requiring the technicalities of §6.2.

As noted at the start of §6.1, we are assuming that R is a field.

Lemma 16 *Suppose that H is an R-CGR, and that $a, c \in H_p$, with $a \neq 0$. Then there exists $\lambda \in R$ such that, for all $b, d \in H_q$, if*

$$a \otimes b \; = \; c \otimes d, \tag{44}$$

then $b = \lambda d$. Moreover, if either $b \neq 0$ or $d \neq 0$, then Equation (44) implies also that $c = \lambda a$.

For an R-CGR H, we will call H_p a *prime homogeneous component* of H if $p = m_1 + \cdots + m_n$ (with $n > 1$ and each $m_i > 0$) implies that $H_{m_i} = \{0\}$ for some i. For example, H_1 is always a prime homogeneous component.

Lemma 17 *Suppose that p is odd, and that H_p is a prime homogeneous component of an R-CGR H. Suppose that z_1, z_2 lie in this component of $\bigotimes^n H$, i.e., $z_1, z_2 \in (\bigotimes^n H)_p$. If $z_1 z_2 = 0$, then either (a) the space generated by z_1 and z_2 is one-dimensional, or (b) there exists i, with $1 \leq i \leq n$, such that*

$$z_k \; = \; 1 \otimes \cdots \otimes 1 \otimes a_k^i \otimes 1 \otimes \cdots \otimes 1 \tag{45}$$

for $k = 1, 2$.

Lemma 18 *Suppose that H, p, z_1 and z_2 are as in Lemma 17, with $z_1 z_2 = 0$. If*

$$z_1 \; = \; 1 \otimes \cdots \otimes 1 \otimes a_1^i \otimes 1 \otimes \cdots \otimes 1$$

for some non-zero $a_1^i \in H_p$, then

$$z_2 \; = \; 1 \otimes \cdots \otimes 1 \otimes a_2^i \otimes 1 \otimes \cdots \otimes 1$$

for some a_2^i (possibly 0) in H_p.

Lemma 19 *Suppose that p is odd, and that H_p is a prime homogeneous component of an R-CGR H. Suppose that z_1, z_2, z_3, z_4 lie in this component of $\bigotimes^n H$, i.e., $z_1, z_2, z_3, z_4 \in (\bigotimes^n H)_p$. If*

$$z_1 z_3 \;=\; z_1 z_4 \;=\; z_2 z_3 \;=\; z_2 z_4 \;=\; 0, \tag{46}$$

then either (a) $z_1 = z_2 = 0$, or (b) $z_3 = z_4 = 0$, or (c) z_1, z_2, z_3, z_4 all lie in a single one-dimensional subspace of $(\bigotimes^n H)_p$, or (d) there exist $i \in \{1, \ldots, n\}$ and $a_k^i \in H_p$ $(k = 1, 2, 3, 4)$ such that

$$z_k \;=\; 1 \otimes \cdots \otimes 1 \otimes a_k^i \otimes 1 \otimes \cdots \otimes 1 \tag{47}$$

for $k = 1, 2, 3, 4$.

Proof. If (a) and (b) are both false, then we may assume, without loss of generality, that $z_1 \neq 0$ and $z_3 \neq 0$.

Case 1. Each of the pairs $\{z_2, z_3\}$ $\{z_3, z_1\}$ $\{z_1, z_4\}$ is linearly dependent. From this condition and the fact that both z_1 and z_3 are non-zero, it is easy to see that all four z_i lie in a single one-dimensional subspace.

Case 2. One of the pairs $\{z_2, z_3\}$ $\{z_3, z_1\}$ $\{z_1, z_4\}$ is linearly independent. We will look in detail at one subcase:

Subcase 2a. $\{z_2, z_3\}$ is linearly independent. Since $z_2 z_3 = 0$, it follows from Lemma 17 that there exists i, with $1 \leq i \leq n$ such that

$$z_k \;=\; 1 \otimes \cdots \otimes 1 \otimes a_k^i \otimes 1 \otimes \cdots \otimes 1$$

for $k = 2, 3$. Obviously $a_3^i \neq 0$, since $z_3 \neq 0$. Since $z_3 z_1 = 0$, we have

$$z_1 \;=\; 1 \otimes \cdots \otimes 1 \otimes a_1^i \otimes 1 \otimes \cdots \otimes 1,$$

by Lemma 18. Since $z_1 \neq 0$ and $z_1 z_4 = 0$, we have

$$z_4 \;=\; 1 \otimes \cdots \otimes 1 \otimes a_4^i \otimes 1 \otimes \cdots \otimes 1,$$

by another application of Lemma 18. This completes the proof for Subcase 2a.

Subcases 2b, 2c. $\{z_3, z_1\}$ is linearly independent, $\{z_1, z_4\}$ is linearly independent. The proofs in these two cases are similar to the proof for Subcase 2a, and hence may be omitted. ∎

6.3 The cohomology ring of a topological space.

We will deal with $H^\star(A; R)$, the absolute[11] cohomology ring of a space A, with coefficients from a ring R with unit. Generally speaking, we take R to be a fixed field (except in §8, where $R = Z$), we suppress mention of R, and we merely write $H^\star(A)$.

In fact, until §8, our proofs require only three basic understandings about $H^\star$:

- The fact that $H^\star$ is a functor, from topological spaces and homotopy classes of maps, to the dual of the category of R-CGR's.
- A certain version of the Künneth Theorem (see Theorem 21 below), which relates the cohomology of A^n to that of A.
- For each space A of interest, a description of the isomorphism type of $H^\star(A)$. Typically, such a description is formally given by a *presentation*. For the five cohomology rings of interest to us in this paper, we supply (without proof) presentations in Lemmas 20, 27, 29, 31, and 33 below.

The reader who knows little or no cohomology theory can simply take these three points as given. As we shall see, in combination they can have powerful consequences. On the other hand, it is a non-trivial task to construct $H^\star$ from scratch, so as to satisfy our three points (or to satisfy the more traditional axiom system for $H^\star$). The construction of $H^\star$, and the derivation of its many properties, may be found in standard sources (e.g. [26], [15], [33]). Except for our use of Čech cohomology in Lemma 43 of §11.3 below, our spaces are simplicial complexes, and any elementary version of cohomology theory will work.

As an illustration of $H^\star$, we include here one of the presentations mentioned in the third point just above. The others are in §§7.3–7.6.

Lemma 20 *The R-CGR $H^\star(S^2, R) = H$ has the presentation*

$$\langle\, a \,:\, a \in H_2,\ a^2 \;=\; 0 \,\rangle. \tag{48}$$

[11] Of course, relative cohomology plays a background role, especially in §8, but we do not need to mention it expressly.

In other words, H_0 is the R-module consisting of all R-multiples of the unit element 1, and H_2 is the R-module consisting of all R-multiples of a. All other H_j are $\{0\}$, and all products are zero, except for

$$\begin{aligned}(r1)\cdot(s1) &= (rs)1\\ (r1)\cdot(sa) &= (rs)a,\end{aligned}$$

and linear consequences of these products.

We conclude §6.3 with some elementary consequences of functoriality. Some of these equations will be used later in our recursive analysis of the terms appearing in Σ. The fact that $H^\star$ is a functor to the *dual* category (a so-called *contravariant functor*) causes some of the equations to seem somewhat non-intuitive. Along the way, we will state the version that we need of Künneth's Theorem.

In order to investigate continuous operations on a space A, we must be able to examine maps defined on finite powers A^n. We begin by considering the i^{th}-coordinate projection maps $\pi_i\colon A^n \longrightarrow A$. Their $H^\star$-images are

$$H^\star(\pi_i)\colon H^\star(A) \longrightarrow H^\star(A^n) \tag{49}$$

for $i \leq i \leq n$. As in the coproduct of two factors discussed above, we have n copower injections

$$\eta_i\colon H^\star(A) \longrightarrow H^\star(A)\otimes\cdots\otimes H^\star(A)$$

given by

$$\eta_i(x) \;=\; 1\otimes\cdots\otimes 1\otimes x\otimes 1\otimes\cdots\otimes 1, \tag{50}$$

with the x appearing in the i^{th} position in the righthand side of (50). Since the maps η_i define a copower, there is a unique map

$$\times_n\colon H^\star(A)\otimes\cdots\otimes H^\star(A) \longrightarrow H^\star(A^n) \tag{51}$$

such that

$$\times_n\circ\eta_i \;=\; H^\star(\pi_i) \tag{52}$$

for $i \leq i \leq n$.

Consider the *diagonal map* $\Delta\colon A \longrightarrow A^n$, defined by

$$\Delta\colon a \longmapsto (a,\ldots,a).$$

It too has an image under the function $H^\star$, and so we may form the composite map

$$H^\star(A) \otimes \cdots \otimes H^\star(A) \xrightarrow{\times_n} H^\star(A^n) \xrightarrow{H^\star(\Delta)} H^\star(A).$$

We claim that $H^\star(\Delta) \circ \times_n$ represents[12] multiplication of n factors in the ring $H^\star(A)$ (see (54) below). (Munkres attributes (54) to Lefschetz [26, Theorem 61.3, p. 362]. It will not be needed until the proof of Lemma 36 in §8 below.) To see this, we note that

$$H^\star(\Delta) \circ H^\star(\pi_i) \;=\; \text{identity}$$

(since $\pi_i \circ \Delta =$ identity), and hence, by (52),

$$\begin{aligned} [H^\star(\Delta) \circ \times_n](1 \otimes \cdots \otimes z \otimes \cdots \otimes 1) \;&=\; H^\star(\Delta) \circ \times_n \circ \eta_i](z) \\ &=\; H^\star(\Delta) \circ H^\star(\pi_i)(z) \;=\; z \quad (53) \end{aligned}$$

Taking products of (53), we see that

$$\begin{aligned} [H^\star(\Delta) \circ \times_n](z_1 \otimes \cdots \otimes z_n) \;&=\; \prod_{j=1}^{n} [H^\star(\Delta) \circ \times_n](1 \otimes \cdots \otimes z_j \otimes \cdots \otimes 1) \\ &=\; \prod_{j=1}^{n} z_j. \qquad (54) \end{aligned}$$

Under certain conditions of finite-dimensionality (which hold for S^2 and the other spaces of this paper), the Theorem of Künneth has the corollary [26, Theorem 61.6, p. 364] [33, §5.6] that $\times_n$ is an *isomorphism.* (In other words, finite products are preserved by $H^\star$.) We state the version of Munkres (who writes $\times$ where we would write $\times_n$).

Theorem 21 *If the graded group $H_\star$ is finitely generated in each dimension, then the cross product $\times$ defines a monomorphism of rings*

$$H^\star(X) \otimes H^\star(Y) \longrightarrow H^\star(X \times Y).$$

If F is a field, it defines an isomorphism of algebras

$$H^\star(X; F) \otimes_F H^\star(Y; F) \longrightarrow H^\star(X \times Y; F).$$

In the case of no torsion, similar results hold for $H^\star(X \times Y, Z)$, with Z the ring of integers. See e.g. Hilton and Wylie [15, 9.4.13, page 377].

[12]This argument shows that, if there is to be any functor from spaces to R-CGRs, then n-fold multiplication in $H^\star(A)$ must be given by $H^\star(\Delta)\circ\times_n$. In other words, it is a sort of uniqueness result for multiplication. Using $H^\star(\Delta)\circ\times_n$ for an *ab initio* construction of multiplication, i.e. for an existence result, is a more complex endeavor.

7 The proofs of Parts 3–7 of Theorem 1.

In each part, as before, we are given continuous operations $\overline{F_t} : A^{n(t)} \longrightarrow A$ such that

$$(A, \overline{F_t})_{t \in T} \models \Sigma$$

up to homotopy, and we need to construct operations F'_t (each a constant or a projection) and prove that they satisfy Σ. As before we will establish this last fact by invoking Lemma 3 of §2: we shall prove (in §§7.2–7.6) that τ' can be defined directly from the homotopy class of $\overline{\tau}$. We begin with a construction and some lemmas that are common to all the proofs of §§7.2–7.6.

7.1 The co-operations $F^\star_t$ and their properties.

Much as in §5 (the proof of Part 2), the desired construction of τ' from $\overline{\tau}$ proceeds via some intermediate operations $F^\star_t$. Since cohomology is a functor into the dual category, the $F^\star_t$ in this section goes in a direction opposite to ordinary operations (see Equation (55) just below). Such reversed operations are sometimes called co-operations.

Definition of the co-operations $F^\star_t$. We begin by letting H stand for $H^\star(A, R)$, and then defining co-operations

$$H \xrightarrow{F^\star_t} \bigotimes{}^{n(t)} H \tag{55}$$

via

$$F^\star_t \;=\; \times^{-1}_{n(t)} \circ H^\star(\overline{F_t}). \tag{56}$$

(For $\times_n$ see Equations (51–52). The existence of $\times^{-1}_n$ comes from Theorem 21 at the end of §6.3 just above.)

Recursive definition of $\tau^\star$. As in Equations (9–10), it is convenient to define, by recursion, a function $\tau^\star$ associated to each term τ. Since the functions $F^\star_t$ are co-operations, so must the $\tau^\star$ be co-operations, and therefore Equations (9–10) cannot be applied in their original form. We remark informally that the following definition is a natural dualization of Equations (9–10) (although this remark is not subject to proof, since we do not have available a precise definition of "natural dualization").

For each term τ, and for N large enough so that $i < N$ if x_i appears in τ, we define $\tau^\star : H \longrightarrow \bigotimes^N H$ as follows.

(i) If $\tau = x_i$, then $\tau^\star = \eta_i$ (defined above).

(ii) If $\tau = F_t(\tau_1, \ldots, \tau_{n(t)})$, then

$$\tau^\star \;=\; \tau^\bullet \circ F_t^\star, \tag{57}$$

where $\tau^\bullet : \bigotimes^{n(t)} H \longrightarrow \bigotimes^N H$ is defined by

$$\tau^\bullet \circ \eta_i \;=\; \tau_i^\star \tag{58}$$

for $1 \leq i \leq n(t)$. (Of course $\tau_i^\star : H \longrightarrow \bigotimes^N H$ is available recursively. Notice that Equation (58) uniquely defines $\tau^\bullet$, by the co-product property of tensor products.)

We shall need the recursive definition of $\tau^\star$ for the inductive proofs of Lemmas 26 and 35 below. On the other hand, Lemma 22 just below gives us an immediate (non-recursive) definition of $\tau^\star$ from the homotopy class of the operation $\overline{\tau}^N$ that is defined in Equation (9) of §1. Later (§§7.2–7.6), in a particular manner for each of Parts 3–7, we will show how to define τ' directly from $\tau^\star$, thereby furthering our objective of establishing the hypotheses of Lemma 3.

Lemma 22 *For any term τ, $\tau^\star \;=\; \times_N^{-1} \circ H^\star(\overline{\tau}^N)$.*

Proof. By induction on $|\tau|$.

Case 1: $\tau = x_i$. Then by (52)

$$\tau^\star \;=\; \eta_i \;=\; \times_N^{-1} \circ H^\star(\pi_i) \;=\; \times_N^{-1} H^\star(\overline{x_i}^N).$$

Case 2: $\tau \;=\; F_t(\tau_1, \ldots, \tau_{n(t)})$. Applying the cohomology functor $H^\star$ to Equation (10) of §1, we obtain

$$H^\star(\overline{\tau_i}^N) \;=\; H^\star(\widehat{\tau}^N) \circ H^\star(\pi_i^{n(t)})$$

for each i. Premultiplying by $\times_N^{-1}$, and again invoking (52), we obtain

$$\begin{aligned} \times_N^{-1} H^\star(\overline{\tau_i}^N) &\;=\; (\times_N^{-1} H^\star(\widehat{\tau}^N) \times_{n(t)})\, (\times_{n(t)}^{-1} H^\star(\pi_i^{n(t)})) \\ &\;=\; (\times_N^{-1} H^\star(\widehat{\tau}^N) \times_{n(t)})\, \eta_i. \end{aligned}$$

By induction, the left-hand side of these equations is equal to $\tau_i^\star$, and so

$$\tau_i^\star \;=\; (\times_N^{-1} H^\star(\widehat{\tau}^N) \times_{n(t)})\, \eta_i$$

for $1 \leq i \leq n(t)$. Since Equation (58) uniquely defines $\tau^\bullet$, the last equation tells us that

$$\tau^\bullet \;=\; \times_N^{-1} H^\star(\widehat{\tau}^N) \times_{n(t)} . \tag{59}$$

It now follows immediately from Equations (57), (59), (56) and (9) that

$$\begin{aligned} \tau^\star &= \tau^\bullet \circ F_t^\star \\ &= \times_N^{-1} H^\star(\widehat{\tau}^N) \times_{n(t)} \times_{n(t)}^{-1} H^\star(\overline{F_t}) \\ &= \times_N^{-1} H^\star(\overline{F_t} \circ \widehat{\tau}^N) \;=\; \times_N^{-1} H^\star(\overline{\tau}^N). \end{aligned}$$

∎

Now the general construction of $\tau^\star$ depended on the arbitrary integer N. We will show that this dependence is not essential. (We obviously need such a result, because we wish to talk about interpretations of all terms, and no single N will simultaneously cover all terms.) To facilitate our exposition, we temporarily append a superscript N to $\tau^\star$ and η_i, to indicate the N that was used in their construction.

Lemma 23 *Let A be a topological space, with H denoting $H^\star(A)$, and suppose that $N < M \in \omega$. There exists a homomorphism*

$$\psi_N^M : \bigotimes^N H \longrightarrow \bigotimes^M H$$

such that

$$\eta_i^M \;=\; \psi_N^M \circ \eta_i^N \tag{60}$$

for all $i < M$, and

$$\tau^{\star M} \;=\; \psi_N^M \circ \tau^{\star N} \tag{61}$$

for all terms τ such that $j < N$ for all x_j appearing in τ.

Proof. We note first that really (60) is a special case of (61) (by taking x_i for τ), and so we need only prove (61). We define

$$\psi_N^M \;=\; \times_M^{-1} \circ H^\star(\Pi_N^M) \circ \times_N, \tag{62}$$

where Π_N^M is defined by Equation (11) of §1. In the following calculation, the first line is by Lemma 22, the second line comes from Equation (12) of §1, and the third line comes from the functorial property of $H^\star$.

$$\begin{array}{rcl}\tau^{\star M} & = & \times_M^{-1} \circ H^\star(\overline{\tau}^M) \\ & = & \times_M^{-1} \circ H^\star(\overline{\tau}^N \circ \Pi_N^M) \\ & = & \times_M^{-1} \circ H^\star(\Pi_N^M) \circ H^\star((\overline{\tau}^N) \\ & = & \left(\times_M^{-1} \circ H^\star(\Pi_N^M) \circ \times_N\right) \circ \left(\times_N^{-1} \circ H^\star(\overline{\tau}^N)\right) \\ & = & \psi_N^M \circ \tau^{\star N},\end{array}$$

and the proof of the lemma is complete. ∎

Lemma 24 *Let A be a topological space, with H denoting $H^\star(A)$, and suppose that $N < M \in \omega$. Let τ be a term such that $j < N$ for each x_j appearing in τ. If*

$$\tau^{\star N}(a) \;=\; \eta_i^N(b)$$

for some $a, b \in H$, then

$$\tau^{\star M}(a) \;=\; \eta_i^M(b)$$

Proof. Immediate from Lemma 23. ∎

With this general framework established, we now attend to the individual spaces (and their cohomology rings) appearing in Parts 3–7.

7.2 Even-dimensional spheres. (Part 3.)

Let A be a space satisfying the conditions of Part 3, for example, the 2-sphere S^2. By Lemma 20 we know that $H = H^\star(A, R)$ is an R-CGR with a single generator a that lies in H_2 and satisfies $a^2 = 0$.

Lemma 25 *For any continuous $\overline{F} : A^n \longrightarrow A$, there exist $\lambda \in R$ and $i \in \{1, \ldots, n\}$ such that*

$$F^\star(a) \;=\; \lambda\,(1 \otimes \cdots \otimes a \otimes \cdots \otimes 1) \;=\; \lambda \eta_i(a)$$

(with a in the i^{th} position and all other entries 1). (In fact, the conclusion holds for any CGR-homomorphism $F^\star : H \longrightarrow \bigotimes^n H$.)

Proof. Let us define

$$e_i \;=\; \eta_i(a) \;=\; 1 \otimes \cdots \otimes a \otimes \cdots \otimes 1$$

with a in the i^{th} position. Since $H_1 = \{0\}$, it is not hard to check that e_i $(1 \leq i \leq n)$ form a basis of $(\bigotimes^n H)_2$, and so we must have

$$F^\star(a) \;=\; \sum_{i=1}^{n} \lambda_i e_i$$

for some scalars $\lambda_i \in R$. Since $a^2 = 0$, we have

$$0 \;=\; F^\star(a^2) \;=\; (F^\star(a))^2 \;=\; \sum_{i \neq j} \lambda_i \lambda_j e_i e_j \tag{63}$$

(where the products $e_i e_i$ are obviously 0, and hence have been eliminated from this sum). It is not hard to check that the products $e_i e_j$ $(j \neq i)$ form a basis of $(\bigotimes^n H)_4$, and hence all coefficients appearing in (63) must be zero. In other words, $\lambda_i \lambda_j = 0$ for $i \neq j$. Since R is an integral domain, this means that for $i \neq j$, either λ_i or λ_j must be zero. In other words, all but one of the λ_i must be zero, and the conclusion of the lemma is immediate. ∎

Definition of the operations F_t': We now define the set B and the operations F_t' on B. In fact B can be taken as any set with more than one element. We then let c be any element of B, and define the operations F_t' as follows:

(A) If $F_t^\star(a) = 0$, then
$$F_t'(x_1, \cdots, x_{n(t)}) \;=\; c.$$

(B) If $F_t^\star(a) \neq 0$, then by Lemma 25 we have $F_t^\star(a) = \lambda \eta_i(a)$ for some i and some λ with $\lambda \neq 0$. In this case we put
$$F_t'(x_1, \cdots, x_{n(t)}) \;=\; x_i.$$

The conditions defining (A) and (B) are independent of N, by Lemma 24. As in §5, we use Equations (6) and (7) to create an associated interpretation τ' for any term τ. We continue to let a denote the generator of $H = H^\star(A)$.

Lemma 26 *For any term τ, if $\tau^\star(a) = 0$, then $\tau'(x) = c$ for any $x \in B^\omega$. If $\tau^\star(a) = \lambda \eta_i(a)$ with $\lambda \neq 0$, then $\tau'(x) = x_i$ for any $x \in B^\omega$.*

Proof. By induction on $|\tau|$.

Case 1: $\tau = x_i$. Then $\tau^\star = \eta_i$, by part (i) of the definition of $\tau^\star$. Therefore $\tau^\star(a) = \eta_i(a) = \lambda e_i$ with $\lambda = 1$. The lemma asserts in this case that $\tau'(x) = x_i$, and indeed this holds by Equation (6).

Case 2: $\tau = F_t(\tau_1, \cdots, \tau_{n(t)})$. Then $F_t^\star(a) = \lambda e_i$ for some λ and some i, by Lemma 25, and again, for this i, we have $\tau_i^\star(a) = \mu e_j$ for some μ and some j. And thus we have, by Equations (57) and (58),

$$\begin{aligned} \tau^\star(a) &= \tau^\bullet F_t^\star(a) \ = \ \tau^\bullet(\lambda e_i) \\ &= \lambda\tau^\bullet\eta_i(a) \ = \ \lambda\tau_i^\star(a) \ = \ \lambda\mu e_j. \end{aligned}$$

Case 2A: $\tau^\star(a) = 0$. This means that $\lambda\mu = 0$. Since the ring R is an integral domain, either $\lambda = 0$ or $\mu = 0$. If $\lambda = 0$, then $F_t^\star(a) = 0$, and $F_t'(x) = c$ for all x by (A) above. Clearly in this case $\tau'(x) = c$, as required by the lemma. On the other hand, if $\lambda \neq 0$ and $\mu = 0$, then $F_t'(x_i, \cdots, x_{n(t)}) = x_i$ (by (B)), and $\tau_i^\star(a) = 0$. Thus $\tau_i'(a) = c$ by induction, and so $\tau'(x) = c$, as is required for the lemma.

Case 2B: $\tau^\star(a) \neq 0$. This means that $\lambda\mu \neq 0$, and hence $\lambda \neq 0$ and $\mu \neq 0$. In this case we have $F_t^\star(a) \neq 0$ and $\tau_i^\star(a) \neq 0$. Thus $F_t'(x) = x_i$ by (B) above, and $\tau_i'(x) = x_j$ by induction. One then easily checks that $\tau'(x) = x_j$, as is required for the lemma. ∎

Completion of the proof of Part 3. We begin by establishing the hypothesis of Lemma 3 (from §2). As we remarked at the time, Lemma 22 defines $\tau^\star$ from the homotopy class of the operation $\overline{\tau}^N$, and clearly Lemma 26 defines τ' from the operation $\tau^\star$. All in all, we have τ' defined from the homotopy class of the operation $\overline{\tau}^N$, and so the hypothesis of Lemma 3 is satisfied. Thus the operations F_t' satisfy Σ, and hence Σ is undemanding. This completes the proof of Part 3.

7.3 The orientable surface of genus 2. (Part 4.)

In fact, the proof for S^2 in §7.2 is valid in this case also, with only minimal changes, and so we will not repeat it in detail. The only needed modifications are, first, that we need a specific presentation of the cohomology ring of this surface (Lemma 27 below), and, second, that we need a lemma to replace Lemma 25, whose proof was specific to even-dimensional spheres. The necessary replacement is Lemma 28. Instead of considering $F^\star(a)$, Lemma

28 considers $F^\star(\Omega)$, where $\Omega = g_1g_2 = g_3g_4$ in the presentation that follows. Similarly, the definition of the operations F'_t should refer to $F^\star(\Omega)$ instead of $F^\star(a)$.

For the cohomology ring of the surface of genus 2, see e.g. Munkres [26], especially Exercise 2(b) of §6 (page 40), §49, pages 293–295, page 298. We state the result without proof.

Lemma 27 *If A is the orientable surface of genus 2, then the R-CGR $H^\star(A) = H$ has the presentation*

$$\begin{array}{ll}\langle\, g_1, g_2, g_3, g_4 \;\; \mid & g_1g_3 = g_1g_4 = g_2g_3 = g_2g_4 = 0; \quad g_1g_2 = g_3g_4; \\ & g_1^2 = g_2^2 = g_3^2 = g_4^2 = 0 \,\rangle,\end{array}$$

with $g_i \in H_1$ (for $1 \le i \le 4$).

As we mentioned above, Ω denotes g_1g_2.

Lemma 28 *Suppose that the ring R has characteristic $\neq 2$. For any continuous $\overline{F}\colon A^n \longrightarrow A$, there exist $\lambda \in R$ and $i \in \{1, \ldots, n\}$ such that*

$$F^\star(\Omega) \;=\; \lambda\,(1 \otimes \cdots \otimes \Omega \otimes \cdots \otimes 1) \;=\; \eta_i(\Omega)$$

(with Ω in the i^{th} position and all other entries 1). (In fact, the conclusion holds for any CGR-homomorphism $F^\star\colon H \longrightarrow \bigotimes^n H$.)

Proof. Let g_k $(k = 1, 2, 3, 4)$ be as in the presentation. Let $z_k = F^\star(g_k)$ $(k = 1, 2, 3, 4)$. Obviously

$$F^\star(\Omega) \;=\; z_1z_2 \;=\; z_3z_4.$$

Clearly the z_k obey the hypotheses of Lemma 19, and so one of the alternative conclusions (a), (b), (c), (d) of that lemma holds. Clearly conclusions (a) and (b) imply that $F^\star(\Omega) = z_1z_2 = z_3z_4 = 0$, and our proof is complete. Conclusion (c) also implies, in characteristic $\neq 2$, that $F^\star(\Omega) = z_1z_2 = z_3z_4 = 0$. Finally condition (d) implies that there exist $i \in \{1, \ldots, n\}$ and $a^i_k \in H_p$ $(k = 1, 2)$ such that

$$z_1z_2 \;=\; 1 \otimes \cdots \otimes 1 \otimes a^i_1a^i_2 \otimes 1 \otimes \cdots \otimes 1.$$

Since H_2 is in fact a one-dimensional space generated by Ω, we in fact have

$$z_1z_2 \;=\; \lambda\,(1 \otimes \cdots \otimes 1 \otimes \Omega \otimes 1 \otimes \cdots \otimes 1),$$

and the proof of the lemma is complete. ∎

7.4 The Klein bottle. (Part 5.)

Again we can use the proof that has already appeared in §7.2 and §7.3, with only minimal changes, and so we will not repeat it in detail. The only needed modifications are, first, that we need a specific presentation of the cohomology ring of the Klein bottle (Lemma 29 below), and, second, that we need a lemma to replace Lemmas 25 and 28, whose proofs were specific to spheres and the surface of genus 2. The necessary replacement is Lemma 30. This new lemma again considers $F^\star(\Omega)$, but this time $\Omega = g_1^2 = g_2^2$ in the presentation that follows. Again, the definition of the operations F_t' should reflect this change.

For the Klein bottle, it is useful to have cohomology with coefficients taken from $Z/2$, the prime field of characteristic 2. Notice that in the special case of characteristic $= 2$, it does not follow that $x^2 = 0$ for elements x of odd degree. The following presentation of the Klein bottle's cohomology can be found on page 296 of Munkres [26].

Lemma 29 *(Coefficient ring = Z_2, the field of integers modulo 2.) If A is the Klein bottle, then the Z_2-CGR $H^\star(A) = H$ has the presentation*

$$\langle g_1, g_2 \mid g_1 g_2 = 0;\ g_1^2 = g_2^2;\ g_1^3 = g_2^3 = 0\rangle,$$

with $g_i \in H_1$ (for $1 \le i \le 2$).

We will let Ω denote the product $g_1^2 = g_2^2$. Obviously $\Omega \in H_2$. The following lemma is the counterpart for the Klein bottle to Lemma 25 (which applied to $A =$ the 2-sphere) and to Lemma 28 (which applied to $A =$ the surface of genus 2).

Lemma 30 *Suppose that R is the ring $Z/2$ of integers modulo 2. For any continuous $\overline{F}\colon A^n \longrightarrow A$, there exist $\lambda \in Z/2$ and $i \in \{1, \ldots, n\}$ such that*

$$F^\star(\Omega) \;=\; \lambda\,(1 \otimes \cdots \otimes \Omega \otimes \cdots \otimes 1) \tag{64}$$

(with Ω in the i^{th} position and all other entries 1). In other words, either $F^\star(\Omega) = 0$ or

$$F^\star(\Omega) \;=\; 1 \otimes \cdots \otimes \Omega \otimes \cdots \otimes 1$$

(In fact, the conclusion holds for any CGR-homomorphism $F^\star : H \longrightarrow \bigotimes^n H$.)

Proof. Let $z_i = F^\star(g_i)$ $(i = 1, 2)$. Obviously $F^\star(\Omega) = z_1^2 = z_2^2$. If either $z_1 = 0$ or $z_2 = 0$, then $F^\star(\Omega) = 0$, and we are done. Therefore we will assume $z_1 \neq 0 \neq z_2$.

If the space generated by z_1 and z_2 is one-dimensional, then in fact (with this coefficient field) we have $z_1 = z_2$, and hence $F^\star(\Omega) = z_1^2 = z_1 z_2 = 0$, and we are done, as before.

It is not hard to see from Lemma 17 that the only remaining possibilities are that

$$z_k \;=\; 1 \otimes \cdots \otimes 1 \otimes g_k \otimes 1 \otimes \cdots \otimes 1$$

for $(k = 1, 2)$ — see Equation (45) – or the same values with z_1 and z_2 interchanged. Hence

$$F^\star(\Omega) \;=\; z_k^2 \;=\; 1 \otimes \cdots \otimes 1 \otimes g_k^2 \otimes 1 \otimes \cdots \otimes 1 \;=\; 1 \otimes \cdots \otimes 1 \otimes \Omega \otimes 1 \otimes \cdots \otimes 1.$$

This completes the proof of the Lemma. ∎

7.5 Real projective space. (Part 6.)

Again we can use the proof that has already appeared in §7.2, §7.3 and §7.4, with only minimal changes, and so we will not repeat it in detail. The only needed modifications are, first, that we need a specific presentation of the cohomology ring of projective space (Lemma 29 below), and, second, that we need a lemma to replace Lemmas 25, 28 and 30. The necessary replacement is Lemma 32. Where the previous lemmas considered $F^\star(a)$ and $F^\star(\Omega)$, this one considers $F^\star(g)$, where g is the unique generator appearing in the presentation below. Again, the definition of the operations F_t' should reflect this change.

For the real projective space P^n, we will again use cohomology with coefficients from $Z/2$. The following presentation of the cohomology of projective space can be found on page 403 of Munkres [26]. It defines what has been called a *truncated polynomial algebra over* $Z/2$.

Lemma 31 *(Coefficient group = Z_2, the field of integers modulo 2.) If P^n is the real projective space of dimension n, then the Z_2-CGR $H^\star(P^n) = H$ has the presentation*

$$\langle g \mid g^{n+1} = 0 \rangle,$$

with $g \in H_1$.

The following lemma is the counterpart for projective space to Lemma 25 for the 2-sphere, to Lemma 28 for the surface of genus 2, and to Lemma 30 for the Klein bottle.

Lemma 32 *Suppose that R is the ring $Z/2$ of integers modulo 2, and suppose that A is a space agreeing in cohomology with projective space P^n, for some n such that $n+1$ is not a power of 2. Then for any continuous $\overline{F}: A^m \longrightarrow A$, either $F^\star(g) = 0$ or there exists $i \in \{1, \ldots, m\}$ such that*

$$F^\star(g) \;=\; 1 \otimes \cdots \otimes 1 \otimes g \otimes 1 \otimes \cdots \otimes 1 \;=\; \eta_i(g), \tag{65}$$

with the g in the i^{th} position. (In fact, the conclusion holds for any CGR-homomorphism $F^\star: H \longrightarrow \bigotimes^n H$.)

Proof. We present the proof first for binary operations, i.e., $\overline{F}$ with $m = 2$. By way of contradiction, let us assume that the conclusion of the lemma is false, i.e. that

$$F^\star(g) \;=\; g \otimes 1 \;+\; 1 \otimes g.$$

Then the following equations hold modulo 2:

$$\begin{aligned} 0 \;=\; F^\star(0) \;=\; F^\star(g^{n+1}) \quad &= \quad (F^\star(g))^{n+1} \\ &= \quad (g \otimes 1 \;+\; 1 \otimes g)^{n+1} \\ &= \quad \sum_{j=0}^{n+1} \binom{n+1}{j} g^j \otimes g^{n+1-j}. \end{aligned}$$

The summands for $j = 0$ and $j = n+1$ are automatically zero (since $g^{n+1} = 0$). The other summands involve the linearly independent ring elements $g^j \otimes g^{n+1-j}$, and hence their coefficients must all be zero. In other words, the binomial coefficients

$$\binom{n+1}{j}$$

$(j = 1, \cdots, n)$ must all be zero (modulo 2). As is well known (and not hard to prove), this condition implies that $n+1$ is a power of 2. The resulting contradiction to our hypotheses completes the proof of the lemma for $m = 2$.

Now for arbitrary $m \geq 2$, we must have

$$F^\star(g) \;=\; \sum_{k \in K} \underbrace{1 \otimes \cdots \otimes 1}_{k} \otimes g \otimes \underbrace{1 \otimes \cdots \otimes 1}_{m-k-1},$$

for some $K \subseteq \{1, \cdots, m-1\}$. If the conclusion of the theorem were false, then we would have this representation with $|K| \geq 2$; without loss of generality, $1, 2 \in K$, and hence

$$F^\star(g) \;=\; g \otimes 1 \otimes \cdots \otimes 1 \;+\; 1 \otimes g \otimes 1 \otimes \cdots \otimes 1 \;+\; \cdots$$

Now select an arbitrary element $a \in P^n$, and define the binary section $\overline{F_2}: (P^n)^2 \longrightarrow P^n$ by

$$\overline{F_2}(x_1, x_2) \;=\; \overline{F}(x_1, x_2, a, \cdots, a).$$

It is not hard to check that

$$F_2^\star(g) \;=\; g \otimes 1 \;+\; 1 \otimes g,$$

and so the proof can be completed by reference to the case $m = 2$. ∎

7.6 The figure-eight. (Part 7.)

Let A be a space satisfying Hypothesis 7 of Theorem 1, for example, the figure-eight space (wedge of two circles). The following description of $H^\star(A)$ can be inferred from various items in Munkres [26].

Lemma 33 *If A is the figure-eight space, then the R-CGR $H^\star(A) = H$ has the presentation*

$$\langle g_1, g_2 \mid g_1 g_2 = 0;\; g_1^2 = g_2^2 = 0 \rangle,$$

with $g_i \in H_1$ (for $1 \leq i \leq 2$).

The following lemma is a rough cohomology counterpart to Lemma 13 (which dealt instead with the homotopy group). In the cohomology context, it plays the same role as Lemmas 25, 28, 30 and 32 above.

Lemma 34 *For any continuous $\overline{F} : A^n \longrightarrow A$, either $F^\star(H_1)$ is at most one-dimensional, or there exist $i \in \{1, \ldots, n\}$ and a linear automorphism ϕ of H_1, such that*

$$F^\star(v) \;=\; 1 \otimes \cdots \otimes \phi(v) \otimes \cdots \otimes 1 \;=\; \eta_i(\phi(v)) \tag{66}$$

for all $v \in H_1$ (with $\phi(v)$ in the i^{th} position and all other entries 1). (In fact, the conclusion holds for any CGR-homomorphism $F^\star : H \longrightarrow \bigotimes^n H$.)

Proof. Let the generators of H_1 be g_1, g_2 (as above), with $g_1 g_2 = 0$. Let $z_k = F^\star(g_k)$ ($k = 1, 2$). We first suppose that the space generated by $z_1 = F^\star(g_1)$ and $z_2 = F^\star(g_2)$ is at most one-dimensional. Since $\{g_1, g_2\}$ spans H_1, it follows readily that $F^\star(H_1)$ is at most one-dimensional, and the proof of the lemma is complete.

On the other hand, if this space is two-dimensional, then $\{z_1, z_2\}$ is linearly independent. It is obvious that $z_1 z_2 = 0$ (since we are given that $g_1 g_2 = 0$). Therefore we know from Lemma 17 that there exists i, with $1 \leq i \leq n$ such that

$$F^\star(g_k) \;=\; z_k \;=\; 1 \otimes \cdots \otimes 1 \otimes a_k^i \otimes 1 \otimes \cdots \otimes 1$$

for $k = 1, 2$. Since g_1, g_2 generate H_1, and since $F^\star$ is linear, we have

$$F^\star(v) \;=\; 1 \otimes \cdots \otimes 1 \otimes \phi(v) \otimes 1 \otimes \cdots \otimes 1$$

for some endomorphism ϕ of H_1. Since $F^\star(H_1)$ has dimension at least two, ϕ must be an automorphism. ∎

Definition of the operations F'_t: We now define the set B and the operations F'_t on B. In fact B can be taken as any set with more than one element. We then let c be any element of B, and define the operations F'_t as follows:

(A) If $F^\star_t(H_1)$ is at most one-dimensional, then

$$F'_t(x_1, \cdots, x_{n(t)}) \;=\; c.$$

(B) If there exist $i \leq N$ and a linear automorphism ϕ such that $\tau^\star(v) = \eta_i(\phi(v))$ for all v (this is Equation (66)), then

$$F'_t(x_1, \cdots, x_{n(t)}) \;=\; x_i.$$

It follows from Lemma 23 that condition (A) is independent of N, and from Lemma 24 that condition (B) is independent of N.

We now prove an analog of Lemma 26, namely

Lemma 35 *For any term τ, if $\tau^\star(H_1)$ is at most one-dimensional, then $\tau'(x) = c$ for any $x \in B^\omega$. If $\tau^\star(v) = \eta_i(\phi(v))$ for some i and some automorphism ϕ, then $\tau'(x) = x_i$ for any $x \in B^\omega$.*

Proof. The proof will be by induction on $|\tau|$.

Case 1: $\tau = x_i$. Then $\tau^\star = \eta_i$, by part (i) of the definition of $\tau^\star$ (found between Equations (56) and (57)). In other words, $\tau^\star(v) = \eta_i(\phi(v))$, with ϕ the identity automorphism. In this case the lemma asserts that $\tau'(x) = x_i$, and indeed this holds by Equation (6).

Case 2: $\tau = F_t(\tau_1, \cdots, \tau_{n(t)})$. Then

$$\tau^\star(v) \;=\; \tau^\bullet \circ F_t^\star(v) \qquad \text{where} \qquad \tau^\bullet \circ \eta_i \;=\; \eta_i^\star \tag{67}$$

by Equations (57) and (58). Lemma 34 obviously divides Case 2 into Cases 2A and 2B that follow.

Case 2A: $F_t^\star(H_1)$ is at most one-dimensional. In this case $F_t'(x) = c$ for all x, by clause (i) of the definition of F_t'. It follows immediately from Equation (67) that $\tau^\star(H_1)$ is at most one-dimensional, and from Equation (6) that $\tau'(x) = c$ for all x. This proves the Lemma in Case 2A.

Case 2B: There exist $i \le n(t)$ and a linear automorphism ϕ of H_1 such that $F_t^\star(v) = \eta_i(\phi(v))$ for all v. It follows from Equation (67) that

$$\tau^\star \;=\; \tau^\bullet \circ \eta_i \circ \phi \;=\; \tau_i^\star \circ \phi. \tag{68}$$

Moreover, by clause (ii) of the definition, we have $F_t'(x) \;=\; x_i$ for all x, and hence

$$\tau'(x) \;=\; \tau_i'(x) \tag{69}$$

for all x, by Equation (7). Lemma 34 again divides Case 2B into the cases 2B-I and 2B-II that follow:

Case 2B-I: $\tau_i^\star(H_1)$ is at most one-dimensional. By induction, $\tau_i'(x) = c$ for all x. Hence $\tau'(x) = c$ for all x, by (69). It is clear from Equation (68) that $\tau^\star(H_1)$ is at most one-dimensional. This proves the lemma in Case 2B-I.

Case 2B-II: There exist $j \le N$ and a linear automorphism ψ such that $\tau_i^\star(v) = \eta_j(\psi(v))$ for all v. By induction, $\tau_i'(x) = x_j$ for all x. Hence $\tau'(x) = x_j$ for all x, by Equation (69). Moreover, $\tau^\star = \eta_j \circ \psi \circ \phi$, by Equation (68). This proves the lemma for Case 2B-II, and hence for all cases. ∎

Completion of the proof of Part 7. We begin by establishing the hypothesis of Lemma 3 (from §2). As ever, Lemma 22 defines $\tau^\star$ from the homotopy type of the operation $\overline{\tau}^N$, and clearly Lemma 35 defines τ' from the operation $\tau^\star$. All in all, we have τ' defined from the operation $\overline{\tau}^N$, and so the hypothesis of Lemma 3 is satisfied. Thus the operations F_t' satisfy Σ, and hence Σ is undemanding. This completes the proof of Part 7.

8 Degrees and the Hopf invariant.

For the proof, in §9, of Part 1 of Theorem 1, we will need the notion of the degree of a continuous map

$$\overline{F} : (S^n)^k \longrightarrow S^n. \tag{70}$$

For that proof, it is enough to know that the degree is a vector of integers $(d_1, \cdots, d_k)$ for which the following five facts are true:

- The degree is an invariant of the homotopy class of $\overline{F}$.
- The degree of a constant map is $(0, \cdots, 0)$.
- The i^{th} projection map has degree $(0, \cdots, 0, 1, 0, \ldots, 0)$, with 1 in the i^{th} co-ordinate.
- The degree of a composite map may be calculated from a simple bilinear formula (see (76) below).
- If $n \neq 1, 3, 7$, then at most one component of the degree is odd (see Corollary 39 below).

The first four of these points are relatively straightforward. As we shall explain more fully, the last of them essentially contains the deep facts of algebraic topology that Adams developed in [1] for his celebrated result on H-spaces. We will say a few words about the definition of the degree and the verification of these properties, but the reader who wishes may proceed directly to the proof in §9.

Let us use the ring of integers for R. For any continuous map

$$f : S^n \longrightarrow S^n, \tag{71}$$

the functor $H^\star$ yields a homomorphism of R-CGR's

$$H^\star(f) : H^\star(S^n, R) \longrightarrow H^\star(S^n, R), \tag{72}$$

Recalling from (48) of §7 that $H^\star(S^n, R)$ has the presentation

$$\langle\, a \;:\; a \in H_n,\ a^2 \;=\; 0 \,\rangle, \tag{73}$$

we see that

$$[H^\star(f)](a) \;=\; da$$

for some integer d. This integer d is defined to be the *degree of* f. Every integer d is the degree of some map.

Now, given an operation (70) and i with $1 \leq i \leq k$, we define an *i-section* of $\overline{F}$ to be any function

$$f_i : S^n \longrightarrow S^n$$

that is defined by

$$f_i(x) \;=\; \overline{F}(a_1, \cdots, a_{i-1}, x, a_{i+1}, \cdots, a_n), \tag{74}$$

for some choice of $a_j \in S^n$ ($j \neq i$). There are many different i-sections, depending on the choice of the a_j, but all are homotopic to one another (since S^n is path-connected), and hence the degree

$$d_i \;=\; \text{degree}(f_i) \tag{75}$$

is a well-defined integer. The we then say that the degree of $\overline{F}$ is the vector $(d_1, \cdots, d_n)$.

We state without proof the following lemma for computing degrees of composite operations. The proof requires a small amount of homological algebra, including Equation (54).

Lemma 36 *Suppose that $\overline{F} : (S^n)^k \longrightarrow S^n$ has degree $(e_1, \cdots, e_k)$, and that $\overline{G_j} : (S^n)^N \longrightarrow S^n$ has degree $(g_1^j, \cdots, g_N^j)$ $(1 \leq j \leq k)$. Then the composite map $\overline{H} : (S^n)^N \longrightarrow S^n$, defined by*

$$\overline{H}(x) \;=\; \overline{F}(\overline{G_1}(x), \cdots, \overline{G_k}(x))$$

(for $x \in (S^n)^N$), has degree $(d_1, \cdots, d_N)$, where

$$d_i \;=\; \sum_{j=1}^{k} e_j g_i^j. \tag{76}$$

Incidentally, the categorical import of Lemma 36 is as follows. Let $\mathcal{C}_n$ denote the full subcategory of topological spaces, whose objects are the powers $(S^n)^k$ for $k \in \omega$. For each map $\overline{G} : (S^n)^N \longrightarrow (S^n)^k$, let $D(\overline{G})$ denote the $N \times k$ matrix whose i^{th} column is the degree of $\Pi_i \circ \overline{G}$ $(1 \leq i \leq k)$. The lemma may then be interpreted as asserting that D is a functor from $\mathcal{C}_n$ to the category of all rectangular matrices of integers. This functor makes a second appearance in Lemma 41 below, where the image category appears as the abstract clone (algebraic theory) of Abelian groups.

We have now covered the first four of our essential points about the degree. We go on to look into odd components of the degree (the fifth point). We first note that, for $n = 1, 3$ or 7, the multiplication of unimodular complex numbers, quaternions or Cayley numbers, respectively,

$$(x_1, \cdots, x_k) \longmapsto (\cdots((x_1 x_2) x_3) \cdots) x_k \tag{77}$$

has degree $(1, \cdots, 1)$. We will very briefly sketch the reasons why, for other values of n, there can be at most one odd component.

Continuous maps

$$G \colon S^{2n+1} \longrightarrow S^{n+1} \tag{78}$$

are classified according to an integral invariant $\Gamma(G)$, which is known as the *Hopf invariant* of G (introduced by H. Hopf in 1935 [16]). The original definition — recapitulated in Dieudonné [9, pp. 314–317] — involved the *linking number* of $G^{-1}[u]$ and $G^{-1}[v]$, for $u, v \in S^{n+1}$. An alternate definition may be found in Steenrod [34, p. 12]. See also, e.g., Novikov [30, p. 207], Hu [17, p. 326], Whitehead [45, p. 494]. For a comparison of the various available definitions, see Hu [17, pp. 334–335]. The two significant facts relating this invariant to our work are stated in the next lemma and theorem.

Lemma 37 *For any binary operation $\overline{F} \colon S^n \times S^n \longrightarrow S^n$, there is a mapping G as in (78) such that*

$$\Gamma(G) \;=\; \pm\, d_1 d_2, \tag{79}$$

where (d_1, d_2) is the degree of $\overline{F}$.

Theorem 38 *If G as in (78) has odd Hopf invariant, then $n = 1, 3$ or 7.*

It should be apparent to the reader that this Lemma and Theorem immediately entail the result of Adams mentioned at the outset, that if $n \neq 1, 3, 7$, there is no Hopf-algebra structure (Equation (1)) definable on S^n. We shall soon see that they are also adequate to entail Part 1.

Lemma 37 is due originally to H. Hopf [16]; his proof is recapitulated on page 319 of Dieudonné [9]. The result is stated without proof in Novikov [30, p. 194]. For a short proof see also page 13 of Steenrod [34].

As for Theorem 38, its hypotheses cannot hold for even n (those Hopf invariants are zero), and so the theorem is trivial for even n. In 1950, G.

W. Whitehead proved it for all $n \equiv 1 \pmod 4$. Using relations between Steenrod squares, J. Adem proved it around 1956 [3] for all n not equal to $2^k - 1$ for some k . H. Toda proved $n = 15$ as a special case. The final step was the hardest: in 1958, using secondary cohomology operations, J. F. Adams proved it for all remaining $2^k - 1$. A nice recapitulation of the proofs is in Dieudonné [9, pp.549–551]. Adams' proof was greatly simplified using K-theory by Adams and Atiyah [2] in 1966.

Corollary 39 *If* $n \neq 1, 3, 7$, *and*

$$\overline{F} : (S^n)^k \longrightarrow S^n, \tag{80}$$

is a continuous k-ary operation of degree $(d_1, \cdots, d_n)$, *then at most one* d_i *is odd.*

Proof. Contrapositively, we will assume that two d_i are odd, and work to prove that $n = 1, 3$ or 7. Without loss of generality, let us suppose that d_1 and d_2 are odd. Choose $a_2, a_3, \cdots, a_{n-1} \in S^n$ and define $\overline{H} : S^n \times S^n \longrightarrow S^n$ via

$$\overline{H}(x_1, x_2) \; = \; \overline{F}(x_1, x_2, a_3, \ldots, a_{n-1}).$$

Clearly $\overline{H}$ has degree (d_1, d_2), and hence by Lemma 37, there exists $G : S^{2n+1} \longrightarrow S^{n+1}$ of Hopf invariant $\pm d_1 d_2$, which is odd. Theorem 38 tells us immediately that $n = 1, 3$ or 7. ∎

9 The proof of Part 1 of Theorem 1.

As in previous proofs, we are given continuous operations $\overline{F_t} : (S^n)^{n(t)} \longrightarrow S^n$ such that

$$(S^n, \overline{F_t})_{t \in T} \models \Sigma$$

up to homotopy, and we need to construct operations F'_t modeling Σ, each a constant or a projection.

Definition of the operations F'_t**:** We now define the set B and the operations F'_t on B. In fact B can be taken as any set with more than one element. We then let c be any element of B, and define the operations F'_t as follows:

(A) If the degree of $\overline{F_t}$ is $(d_1, \cdots, d_{n(t)})$, with each d_i even, then

$$F'_t(x_1, \cdots, x_{n(t)}) \; = \; c.$$

(B) If d_i is odd, then

$$F'_t(x_1, \cdots, x_{n(t)}) \; = \; x_i.$$

(Of course, in clause (B), only one d_i can be odd, by Corollary 39 above.)

As in §5, we use Equations (6) and (7) to create an associated interpretation τ' for any term τ. As we remarked before, to complete the proof of Part 1. we need only establish that

$$(B, F'_t)_{t \in T} \models \Sigma.$$

In the next lemma, τ is any term in the language of Σ, and N is a positive integer chosen large enough that $i < n$ for all variables x_i appearing in τ. As was said in remarks following Equation (8), $\overline{\tau}^N$ is the continuous N-ary operation on S^n that is defined by Equations (6–8). The proof is like proofs that have come before, and hence left to the reader. (It is an inductive argument that seems to require a straightforward division into cases. The first four points at the start of §8 will be useful for some of the cases. For example, for τ a composite term, one needs Lemma 36 to calculate the degree of $\overline{\tau}^N$.)

Lemma 40 *For any term τ, if the degree of $\overline{\tau}^N$ is $(d_1, \cdots, d_N)$ with each d_i even, then $\tau'(x) = c$ for any $x \in B^\omega$. If some d_i is odd, then $\tau'(x) = x_i$ for any $x \in B^\omega$.*

Completion of the proof of Part 1. Lemma 40 immediately yields a construction of τ' from the homotopy class of the operation $\overline{\tau}^N$. Hence, by Lemma 3, the operations F'_t satisfy Σ, and the theorem is proved.

10 A sketch of the proof of Theorem 2.

We consider only the case where A is B^q, and the fundamental group

$$G \; = \; \Pi_b(B)$$

is free on k generators ($2 \leq k < \omega$). Then the fundamental group

$$\Pi_a(A) \;\cong\; G^q$$

a q^{th} power of the free group G.

Thus for each $t \in T$, the homomorphism

$$F_t^\star\colon \Pi_a(A)^{n(t)} \longrightarrow \Pi_{\overline{F_t(a,\cdots,a)}}(A)$$

takes the form of a homomorphism

$$F_t^\star\colon G^{qn} \longrightarrow G^q.$$

Now for $1 \leq j \leq q$, the composite homomorphism

$${}_jF_t^\star \;=\; G^{qn} \xrightarrow{F_t^\star} G^q \xrightarrow{\pi_j} G$$

is a homomorphism to which we can apply Lemma 13. Either

(i) ${}_jF_t^\star$ fails to be onto, or

(ii) ${}_jF_t^\star(x_1,\ldots,x_{qn}) = \phi(x_i)$ for some i with $1 \leq i \leq qn$, and for some automorphism ϕ of G.

We now define operations F_t' on $C = D^q$, of the type required for q-undemanding, as follows. First choose an element $d \in D$. Then define

$$F_t'\colon D^{qn} \longrightarrow D$$

in terms of its component mappings

$${}_jF_t' \;=\; D^{qn} \xrightarrow{F_t'} D^q \xrightarrow{\pi_j} D,$$

namely, if (i) holds, then ${}_jF_t'$ should be a constant map with value d; if (ii) holds, then ${}_jF_t'$ should be the i^{th} projection function.

The proof now proceeds in the manner of the proof of Part 2 (in §5). One needs to prove inductively, for every component of every term, that a failure to be onto ((i) above) corresponds to a constant value of that component, and that condition (ii) above corresponds in a similar way to an i^{th} coordinate projection. We omit the details.

11 Final remarks and problems.

One overall difficulty with the subject of topological algebra is the scarcity of examples. From this perspective, Theorem 1 is somewhat discouraging, making it seem harder than ever for us to discover new and interesting examples. Some possibilities of examples can be seen in §11.4 below.

11.1 Generalizations and extensions.

We have omitted some obvious generalizations. E.g. the surface of genus k for $k > 2$, various one-point joins of three or more spheres (of varying dimensions), and so on. In many cases such generalizations should be pretty straightforward. More generally, it would be an interesting study in CGR's to find the best possible common generalization of Lemmas 25, 28, 30, 32 and 34 (and of the technical Lemmas 17 and 19 on which they rely). Such endeavors might yield the best possible extension of our cohomological method.

Likewise, it would be a valuable project to find extensions of the group-theoretic results in §4, in order to extend the homotopy-theoretic proofs of §5 in the best possible way.

11.2 Our results, interpreted in clone theory.

For some more general remarks, it will be useful to speak in the context of the *clone* $\mathbf{C}(A)$ which is described in [39] and [40]. $\mathbf{C}(A)$ is the clone whose elements are the continuous operations on A, and whose operations are formed in the usual way by substituting n continuous m-ary operations $\overline{F_1}, \cdots, \overline{F_n}$ into a single n-ary operation $\overline{G}$ to form a single m-ary operation $\overline{H}$. In other words

$$\overline{H}(x_1, \cdots, x_m) \;=\; \overline{F}(\overline{G_1}(x_1, \cdots, x_m), \cdots, \overline{G_n}(x_1, \cdots, x_m)). \qquad (81)$$

(Another designation of this object — in the school of F. W. Lawvere [19] [20]; see also [46] or the appendix to [36] — is the *algebraic theory* of A in the category of spaces and continuous maps.)

We have avoided speaking of $\mathbf{C}(A)$, since the results proved here can in fact be stated and proved without any very complex categorical machinery. Nevertheless, those familiar with $\mathbf{C}(A)$ will recognize that each part of Theorem 1 is tantamount to the existence of a *clone homomorphism* $\overline{F} \longmapsto F'$

from $\mathbf{C}(A)$ to the clone[13] of a trite algebra.

To see this, we let A be any of the spaces described in Theorem 1. We let $\overline{F}_t$ $(t \in T)$ be an indexed collection of *all* continuous operations on A. Every time Equation (81) holds for some operations $\overline{H}$, $\overline{F}$, $\overline{G_1}$, $\ldots$, $\overline{G_n}$ among the collection of operations F_t, we include the formal equation

$$H(x_1, \cdots, x_m) \approx F(G_1(x_1, \cdots, x_m), \cdots, G_n(x_1, \cdots, x_m)) \tag{82}$$

in Σ, and we let Σ consist precisely of all Equations (82). Clearly Equations (81) tell us that the continuous operations $\overline{F}_t$ model Σ, and hence A is compatible with Σ. Hence any part of Theorem 1 will tell us that Σ is undemanding. In other words, there are operations F'_t $(t \in T)$ satisfying Σ such that

$$H'(x_1, \cdots, x_m) \;=\; F'(G'_1(x_1, \cdots, x_m), \cdots, G'_n(x_1, \cdots, x_m)) \tag{83}$$

holds whenever (82) holds; which is to say, whenever (81) holds. In other words, (81) implies (83), and so $\overline{F} \longmapsto F'$ is a clone homomorphism.

In fact, since Theorem 1 assumes only satisfaction up to homotopy, we easily see that we really have a homomorphism from the clone of homotopy classes of continuous operations on A to the clone of a trite algebra.

As an aside, we mention that an *operad* (see [21]) is like a clone in comprising maps $A^n \longrightarrow A$ and operators that act on tuples of maps. (In this case, one has all permutations of variables and a sort of distinct-variable composition of operations.) For discriminating classes defined by equations Σ, the operad is a coarser instrument than the clone (see [21, page 16]). Nevertheless, there is a large body of results linking operads to algebraic topology, and so there might be some payoff to investigation of operads in the context of this paper. (Thanks to the referee for pointing to operads.)

11.3 The lattice of interpretability.

The existence of the homomorphism $\overline{F} \longmapsto F'$ (in §11.2) tells us that A lies as low as possible in the *lattice of varietal interpretation* $\mathbf{L}$, which was introduced by W. D. Neumann in [28], and later described in more detail by

[13] Among clones with a constant, this one is the smallest (in the sense of §11.3 below). S. Świerczkowski showed that it is not completely meet-irreducible in the lattice of §11.3 (reported in [24]).

García and Taylor in [12]. (There is also a brief description in [40]. For some sidelights on the theory of this lattice, see Mycielski and Taylor [27].)

Now obviously, there exist spaces A such that no retraction $\overline{F} \longmapsto F'$ exists. This is of course the case when any demanding Σ is satisfiable on A, for instance, if A is the underlying space of a topological group. In this case, we wonder if, in some cases, we could find a concrete interpretation of Σ onto which one can retract $\mathbf{C}(A)$. In other words, we would like to know theorems of the following type for a specific space A, and a specific demanding Σ: Σ can be continuously modeled on A, and moreover the clone of all operations on A retracts onto the operations that model Σ. Another way to say this would be that Σ and $\mathbf{C}(A)$ have exactly the same location in the lattice $\mathbf{L}$.

Actually we know three kinds of space A for which we can prove a result of the desired kind: the q^{th}-power spaces of Theorem 2, the circle S^1 (see Theorems 41 and 42 below), and the solenoid (see Theorems 45 and 46 below).

As for the q^{th}-power spaces of Theorem 2, we leave it to the reader that the corresponding Σ is the set of all operations of the type required for q-undemanding, together with the equations that hold among them. For S^1 and the solenoid, we prove some theorems that are relatively easy extensions of the material that we already have. The first result is true for any n, but of course we already knew something stronger (Part 1) for $n \neq 1, 3, 7$.

Theorem 41 *The abstract clone $\mathbf{C}(S^n)$ lies below the theory of Abelian groups in the interpretability lattice. The same is true of the homotopy quotient of $\mathbf{C}(S^n)$.*

Proof. Suppose $\overline{F} : (S^n)^k \longrightarrow S^n$ is a k-ary operation. Let its degree (as defined in §8) be $(e_1, \cdots, e_k)$. Now define

$$\Phi(\overline{F}) \ = \ e_1 x_1 + \cdots + e_k x_k; \tag{84}$$

in other words, $\Phi(\overline{F})$ is the Abelian-group term appearing on the right-hand side of (84). (More properly, one should restore the parentheses that are obviously missing from (84), and then consider the equivalence class of this term, modulo the theory of Abelian groups.)

Now to see that Φ respects the formation of composite operations, we suppose that $\overline{G_j} : (S^n)^N \longrightarrow S^n$ has degree $(g_1^j, \cdots, g_N^j)$ $(1 \leq j \leq k)$. Appropriate applications of Equation (84) tell us that

$$\Phi(\overline{G}_j) \ = \ g_1^j x_1 + \cdots + g_N^j x_N$$

$(1 \leq j \leq k)$. Substituting these equations into (84) yields

$$\Phi(\overline{F})(\Phi(\overline{G}_1), \cdots, \Phi(\overline{G}_k)) \ = \ \sum_{i=1}^{N} \left(\sum_{j=1}^{k} g_i^j e_j \right) x_i \tag{85}$$

According to Lemma 36, the coefficients of the variables x_i appearing on the right-hand side of Equation (85) are precisely the degrees of the composite map $\overline{H} : (S^n)^N \longrightarrow S^n$, defined by

$$\overline{H}(x) \ = \ \overline{F}(\overline{G_1}(x), \cdots, \overline{G_k}(x)).$$

Therefore, the term appearing on the right-hand side of Equation (85) is $\Phi(\overline{H})$. Therefore Φ respects the formation of composite operations, and the lemma is proved. ∎

On the other hand, when $n = 1$, the reverse inequality is immediate since (by multiplication of complex numbers of unit modulus) S^1 is an Abelian group:

Theorem 42 *In the interpretability lattice, the abstract clone* $\mathbf{C}(S^1)$ *lies above the theory of Abelian groups .*

Now Theorems 41 and 42 tell us that the abstract clone $\mathbf{C}(S^1)$, and its homotopy quotient, are both precisely the same as the theory of Abelian groups, as far as the interpretability lattice is concerned.

As for the spheres S^3 and S^7, we do not know their position in the lattice, except that they lie below Abelian groups, by Theorem 41. By James [18], these two spheres cannot be Abelian groups, even up to homotopy, and hence they lie strictly below Abelian groups.

For another example of this kind of equality, we consider the case of solenoids. For the sake of definiteness, we will consider the so-called *dyadic solenoid*. This space is most simply defined to be the closed, hence compact, subset S_2 of the infinite product

$$(S^1)^\omega \ = \ S^1 \times S^1 \times S^1 \times \cdots$$

defined by

$$S_2 \ = \ \Big\{ \ x \in (S^1)^\omega \mid x_0 = 2x_1; \ x_1 = 2x_2; \ x_2 = 2x_3; \ \cdots \Big\}. \tag{86}$$

(Obviously, the definition is easily modified to form other solenoids by changing the multipliers: $x_0 = n_1 x_1$, $x_1 = n_2 x_2$, and so on.)

We will analyze this space with the Čech cohomology $\check{H}^\star(S_2)$. Much of what we said about ordinary cohomology in §6.3 holds true for Čech cohomology — $\check{H}^\star$ is a functor from spaces into graded R-monoids. For subtle spaces like the dyadic solenoid, Čech cohomology can differ from the ordinary cohomologies (such as the singular cohomology) that are described in §6.3. In particular $H^\star(S_2)$ is zero, while $\check{H}^\star(S_2)$ is not (see Lemma 43 below). A version of the Künneth formula (Theorem 21) holds for Čech cohomology with integer coefficients — see Spanier [33, Chapter 6, Exercise E5, page 360].

Obviously we need integer coefficients for the following lemma: a field would yield too much divisibility.

Lemma 43 *In dimension* 1*, the Čech cohomology of the dyadic solenoid, $\check{H}^\star(S_2)$, is isomorphic (as an Abelian group) to the group of rational numbers (under addition) with universe*

$$Z_{(2)} \;=\; \{\frac{m}{2^k} \;:\; m, k \in Z, k \geq 0\}.$$

Proof. This is essentially Exercise 5 on page 444 of Munkres [26]. That exercise contains an interesting alternate description of the dyadic solenoid as the intersection of a nested sequence of polyhedra. Then Theorem 73.4 on page 440 of [26] yields the Čech cohomology as the direct limit of the sequence

$$Z \longrightarrow Z \longrightarrow Z \longrightarrow \;\cdots$$

with each arrow indicating multiplication by 2. Up to isomorphism, this limit is clearly the group $Z_{(2)}$ described above. ∎

We omit the proof of the following easy lemma.

Lemma 44 *If $\phi\colon Z_{(2)} \longrightarrow Z_{(2)}$ is a group homomorphism, then there exists a rational α, with denominator a power of* 2*, such that $\phi(x) = \alpha x$ for all $x \in Z_{(2)}$*

Let us note that, as a set of rational numbers, $Z_{(2)}$ is closed under multiplication as well as addition, and hence is a ring. Hence we may speak of $Z_{(2)}$-modules. Here is a result analogous to Theorem 41 above.

Theorem 45 *The abstract clone* $\mathbf{C}(S_2)$ *lies below the theory of* $Z_{(2)}$*-modules in the interpretability lattice. The same is true of the homotopy quotient of* $\mathbf{C}(S_2)$*.*

Proof. Suppose $\overline{F}\colon (S_2)^k \longrightarrow S_2$ is a k-ary operation. In the usual way, it defines a group homomorphism

$$\check{F}^\star\colon \check{H}^\star(S_2) \longrightarrow \check{H}^\star(S_2) \otimes \cdots \otimes \check{H}^\star(S_2),$$

which in dimension 1 is a group homomorphism

$$\check{F}^\star\colon Z_{(2)} \longrightarrow Z_{(2)} \otimes \cdots \otimes Z_{(2)}$$

As usual, we let

$$e_i \;=\; \eta_i(e) = 1 \otimes \cdots \otimes e \otimes \cdots \otimes 1,$$

where 1 is the generator of $\check{H}^0(S_2)$, e denotes the unit element of $Z_{(2)} = \check{H}^1(S_2)$, and the e appears in the i^{th} of k tensor-factors. From Lemma 44 it is not hard to determine that

$$\check{F}^\star(e) \;=\; \alpha_1 e_1 + \cdots + \alpha_k e_k,$$

for scalars $\alpha_1, \cdots, \alpha_k \in Z_{(2)}$ Now as in the proof of Theorem 41 above, we define

$$\Phi(\overline{F}) \;=\; \alpha_1 x_1 + \cdots + \alpha_k x_k,$$

which may be regarded as a k-ary operation in the theory of $Z_{(2)}$-modules. We leave to the reader the job of proving that Φ respects the formation of composite operations. ∎

Theorem 46 *The abstract clone* $\mathbf{C}(S_2)$ *lies above the theory of* $Z_{(2)}$*-modules in the interpretability lattice.*

Proof. By its very definition (86), S_2 is a topological subgroup of $(S^1)^\omega$. To complete the proof, we need, in essence, to define division by 2 in a continuous way. Taking the representation of S_2 given in (86), we simply define

$$\theta(x_0, x_1, x_2, \cdots) \;=\; (x_1, x_2, x_3, \cdots).$$

This θ is clearly continuous, and moreover, it is apparent from the defining equations appearing in (86) that $2\theta(x) = x$ for all $x \in S_2$. ∎

Now Theorems 45 and 46 tell us that the abstract clone $\mathbf{C}(S_2)$, and its homotopy quotient, are precisely the same as the theory of $Z_{(2)}$-modules, as far as the interpretability lattice is concerned.

Of course, we already represented the theory of Abelian groups with $\mathbf{C}(S^1)$ in Theorems 41 and 42, and so Theorems 45 and 46 would be redundant if Abelian group theory and the theory of $Z_{(2)}$-modules happened to occupy the same spot in the interpretability lattice. In fact, as is pointed out in [25], this would be true only if there were a unit-preserving ring-homomorphism from $Z_{(2)}$ to Z. Obviously there is no such homomorphism, and so in fact $Z_{(2)}$-modules are strictly higher in the lattice.

11.4 A construction for topological algebras.

Suppose that A is a space that can continuously satisfy only undemanding Σ (such as the spaces of Theorem 1, and that B is the Σ-free algebra on A, with the Świerczkowski topology [35] (or a related topolgy — see [37], [4] or [7]). By construction, B is compatible with Σ; but one might suppose that there are no further compatibilities beyond those that follow from this one. More formally, the author asked in 1986 [39, page 38] whether the operations of $\mathbf{C}(B)$ will retract onto operations satisfying Σ.

In fact, even at that time, a negative answer was available in the literature. For n a fixed natural number ≥ 2, let F be an n-ary operation symbol. The equations

$$F(x_1, \cdots, x_n) \approx F(x_{\sigma(1)}, \cdots, x_{\sigma(n)}) \qquad (87)$$

$$F(x, \cdots, x) \approx x \qquad (88)$$

(with σ ranging over all permutations of $\{1, \cdots, n\}$) define the variety of n-*means*. In 1963, Eckmann, Ganea and Hilton [11] proved that if the space B is a CW-complex that admits a continuous n-mean (i.e., if B is compatible with Equations (87–88)), then B is an H-space up to homotopy, i.e. satisfies Equations (1) with equality replaced by homotopy. We now let B be the Σ-free algebra over A, for Σ the theory of n-means, and let A be one of the spaces of Theorem 1. According to Bateson [4], B is a CW-complex, and hence the Eckmann-Ganea-Hilton result applies, and so B admits a continuous n-mean. But H-space operations do not retract onto n-mean operations (as the reader may check), and hence the answer to the original question is negative in general. (At least from the homotopy point of view.)

Nevertheless, the material of this paper gives some hope for obtaining a positive answer in some special cases: at least there are now simple spaces A for which one can begin to consider the free Σ-algebra on A. This makes it possible to begin some further research in this direction.

11.5 Compatibility and H-spaces.

Problem. Does condition (i) imply condition (ii) for every topological space A?

(i) A is compatible with some Σ that is not q-undemanding for any q. (For q-undemanding, see before the statement of Theorem 2 in §2.)

(ii) A admits the structure of an H-space (Equations (1)) up to homotopy.

Certainly the weakened form of (i)

(i′) A is compatible with some demanding Σ.

is not strong enough in general to imply (ii). For example, Equation (17) of §2 is demanding, but is compatible with spaces (such as the square of the figure-eight) that cannot be H-spaces, not even up to homotopy. On the other hand, there are several special cases where (i′) is known to imply (ii):

- If A is a suspension and Σ is idempotent [38, Theorem 3.1].
- If A is a CW-complex and Σ defines n-means (as in Equations (87–88)).
- If A is a CW-complex and Σ is either congruence-modular or congruence-k-permutable for some k [38, Theorem 6.1].

Theorem 1 supplies a certain evidence for the supposition that (i) might imply (ii); or at least, it eliminates some possible counterexamples. The spaces A studied in those theorems were known to satisfy the negation of (ii). (That, in fact, was our motivation for studying them in this context.) Our results say, of course, that for each such A we also have a strong negation of (i), namely that every Σ compatible with A is undemanding.

If there is any hope of creating a space satisfying (i) but failing to satisfy (ii), then the method of the Świerczkowski topology, outlined in §11.4 above, might be one place to look. If Σ is demanding and A is not an H-space, why should we expect F in general to be an H-space, where F is the (suitably topologized) Σ-free algebra over A?

11.6 Compatibility and commutativity of homotopy.

Problem. Does Condition (i) of §11.5 imply

(iii) A has commutative fundamental group.

As is well known, Condition (ii) of §11.5 implies Condition (iii), and hence an affirmative solution to the Problem in §11.5 would imply a positive solution to the problem stated here. Moreover, much as before, Condition (i′) of §11.5 does not imply (iii). (Equation (17) of §2 is demanding, and also compatible with spaces (such as the square of the figure-eight) that have non-commutative fundamental group.)

We know that (i′) implies (iii) when Σ is idempotent. [37, Theorem 5.1], but such a result is unknown for (i′) implies (ii).

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