Appendices to "Two Stage Least Squares Estimation of Spatial Autoregressive Models with Endogenous Regressors and Many Instruments"

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A Some Useful Lemmas

To simplify notations, we drop the K subscript on Q_K, P_K and M_K. Let MI refer to the Markov inequality, and CSI to the Cauchy-Schwarz inequality.

Lemma A.1 *For an estimator given by* $\sqrt{n}(\hat{\delta}_n - \delta_0) = \hat{H}_n^{-1}\hat{h}_n$, *suppose there is a decomposition,* $\hat{h}_n = h_n + T^h + Z^h$, $\hat{H}_n = H_n + T^H + Z^H$,

$$(h_n + T^h)(h_n + T^h)' - h_n h_n' H_n^{-1} T^{H\prime} - T^H H_n^{-1} h_n h_n' = \hat{A}(K) + Z^A(K),$$

such that $T^h = o_p(1)$, $h_n = O_p(1)$, $H_n = O(1)$, *the determinant of* H_n *is bounded away from zero,* $\rho_{K,n} = \mathrm{tr}(S(K)) = o(1)$, $\left\|T^H\right\|^2 = o_p(\rho_{K,n})$, $\left\|T^h\right\|\left\|T^H\right\| = o_p(\rho_{K,n})$, $\left\|Z^h\right\| = o_p(\rho_{K,n})$, $\left\|Z^H\right\| = o_p(\rho_{K,n})$, $Z^A(K) = o_p(\rho_{K,n})$, $\mathrm{E}[\hat{A}(K)] = \sigma_\epsilon^2 H_n + H_n S(K) H_n + o(\rho_{K,n})$. *Then (2) is satisfied.*

Proof. This is Lemma A.1 in Donald and Newey (2001). ■

Lemma A.2 (i) $\mathrm{tr}(P) = K$. (ii) *Suppose that* $\{A_n\}$ *is UB. For* $B_n = PA_n$, $\mathrm{tr}(B_n) = O(K)$, $\mathrm{tr}(B_n^2) = O(K)$, *and* $\sum_i (B_{ii})^2 = O(K)$, *where* B_{ii}*'s are diagonal elements of* B_n.

Proof. See Liu and Lee (2010). ■

Lemma A.3 *Under Assumption 4* (iii), *we have* (i) $\sum_i P_{ii}^2 = o(K)$, $\sum_{i\neq j} P_{ii}P_{jj} = K^2 + o(K)$, $\sum_{i\neq j} P_{ij}P_{ij} = \sum_{i\neq j} P_{ij}P_{ji} = K + o(K)$; (ii) $\sum_i M_{ii}P_{ii} = o(K)$, $\sum_{i\neq j} M_{ii}P_{jj} = K\mathrm{tr}(M) + o(K) = O(K^2)$, $\sum_{i\neq j} M_{ij}P_{ij} = \sum_{i\neq j} M_{ij}P_{ji} = \mathrm{tr}(M) + o(K) = O(K)$; (iii) $\sum_i M_{ii}^2 = O(K)$, $\sum_{i\neq j} M_{ii}M_{jj} = \mathrm{tr}^2(M) - \sum_i M_{ii}^2 = O(K^2)$, $\sum_{i\neq j} M_{ij}M_{ij} = \mathrm{tr}(MM') - \sum_i M_{ii}^2 = O(K)$, $\sum_{i\neq j} M_{ij}M_{ji} = \mathrm{tr}(M^2) - \sum_i M_{ii}^2 = O(K)$; *and* (iv) *if Assumption 5* (ii) *also holds,* $\sum_i M_{ii}^2 = o(K)$.

Proof. (i) is Lemma A.2 in Donald and Newey (2001). For (ii), as $\mathrm{tr}(MP) = \mathrm{tr}(M) = O(K)$, it follows that $|\sum_i M_{ii}P_{ii}| \leq \sum_i |M_{ii}P_{ii}| \leq (\max_i |M_{ii}|)\mathrm{tr}(P) = o(K)$, $\sum_{i\neq j} M_{ii}P_{jj} = \sum_i M_{ii} \sum_j P_{jj} - \sum_i M_{ii}P_{ii} = \mathrm{tr}(M)\mathrm{tr}(P) + o(K) = O(K^2)$, and $\sum_{i\neq j} M_{ij}P_{ij} = \sum_{i\neq j} M_{ij}P_{ji} = \sum_{i,j} M_{ij}P_{ji} - \sum_i M_{ii}P_{ii} = \mathrm{tr}(M) + o(K) = O(K)$. For (iii), $\sum_i M_{ii}^2 = O(K)$ by Lemma A.2 (ii). As $\mathrm{tr}(M) = O(K)$, $\mathrm{tr}(MM') = O(K)$, and $\mathrm{tr}(M^2) = O(K)$ by Lemma A.2, it follows that $\sum_{i\neq j} M_{ii}M_{jj} = \sum_{i,j} M_{ii}M_{jj} - \sum_i M_{ii}^2 = \mathrm{tr}^2(M) - \sum_i M_{ii}^2 = O(K^2)$, $\sum_{i\neq j} M_{ij}M_{ij} = \sum_{i,j} M_{ij}M_{ij} - \sum_i M_{ii}^2 = \mathrm{tr}(MM') - \sum_i M_{ii}^2 = O(K)$, and $\sum_{i\neq j} M_{ij}M_{ji} = \sum_{i,j} M_{ij}M_{ji} - \sum_i M_{ii}^2 = \mathrm{tr}(M^2) - \sum_i M_{ii}^2 = O(K)$. For (iv), $\sum_i M_{ii}^2 \leq (\max_i |M_{ii}|) \sum_i |M_{ii}| = o(K)$. ■

Lemma A.4 *Under Assumptions 1-4,* (i) $\Delta_K = o(1)$. (ii) $F_n'(I_n - P)\epsilon_n/\sqrt{n} = O_p(\Delta_K^{1/2})$ *and* $F_n'(I_n - P)V_n/\sqrt{n} = O_p(\Delta_K^{1/2})$. (iii)

$$\mathrm{E}(V_n'PV_n) = \begin{bmatrix} \mathrm{tr}(M'M)(\gamma_0'\Sigma_u\gamma_0 + 2\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2) & * \\ \mathrm{tr}(M)(\Sigma_u\gamma_0 + \sigma_{u\epsilon}') & K\Sigma_u \end{bmatrix} = O(K), \tag{1}$$

and $\mathrm{E}(V_n'P\epsilon_n) = [\mathrm{tr}(M)(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2), K\sigma_{u\epsilon}]' = O(K)$. (iv) $\mathrm{E}(V_n'P\epsilon_n\epsilon_n'PV_n) - \mathrm{E}(V_n'P\epsilon_n)\mathrm{E}(\epsilon_n'PV_n) = O(K)$, $\mathrm{E}[(V_n'PV_n)^2] - \mathrm{E}^2(V_n'PV_n) = O(K)$, *and, if Assumption 5* (ii) *also holds,*

$$\begin{aligned} &\mathrm{E}(V_n'P\epsilon_n\epsilon_n'PV_n) - \mathrm{E}(V_n'P\epsilon_n)\mathrm{E}(\epsilon_n'PV_n) \qquad (2)\\ = &\begin{bmatrix} (\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)^2\mathrm{tr}(MM') + \sigma_\epsilon^2(\gamma_0'\Sigma_u\gamma_0 + 2\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)\mathrm{tr}(M^2) & * \\ [(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)\sigma_{u\epsilon}' + \sigma_\epsilon^2(\Sigma_u\gamma_0 + \sigma_{u\epsilon}')]\mathrm{tr}(M) & (\sigma_{u\epsilon}'\sigma_{u\epsilon} + \sigma_\epsilon^2\Sigma_u)K \end{bmatrix} + o(K). \end{aligned}$$

(v) $\sqrt{\Delta_K/n} = o(K/n + \Delta_K)$ *and* $\sqrt{K\Delta_K/n} = o(K^2/n + \Delta_K)$. (vi) *Let* Γ_n *be either* F_n *or* $(I_n - P)F_n$,

$$\mathrm{E}(\frac{1}{n}\Gamma_n'\epsilon_n\epsilon_n'PV_n) = [\frac{1}{n}\Gamma_n' vec_D(M)E(\epsilon_{ni}^2u_{ni})\gamma_0 + \frac{1}{n}\Gamma_n' vec_D(M)\mu_3, \frac{1}{n}\Gamma_n' vec_D(P)E(\epsilon_{ni}^2u_{ni})],$$

$\mathrm{E}(\frac{1}{n}F_n'\epsilon_n\epsilon_n'PV_n) = O(\sqrt{K/n})$, $\mathrm{E}[\frac{1}{n}F_n'(I_n - P)\epsilon_n\epsilon_n'PV_n] = O(\sqrt{K\Delta_K/n})$, *and* $E(\frac{1}{n}h_nh_n'H_n^{-1}V_n'F_n) = O(1/n)$.

Proof. (i) and the first part of (ii) is in Lemma A.3 of Donald and Newey (2001). For the second part of (ii), let $\bar{U}_n = (\bar{u}_{n1}', \cdots, \bar{u}_{nn}')'$, where $\bar{u}_{ni} = u_{ni}\gamma_0 + \epsilon_{ni}$. $\mathrm{E}(V_nV_n') = \mathrm{E}(G_n\bar{U}_n\bar{U}_n'G_n' + U_nU_n') = \mathrm{E}(\bar{u}_{ni}^2)G_nG_n' + \mathrm{E}(u_{ni}u_{ni}')I_n = (\gamma_0'\Sigma_u\gamma_0 + 2\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)G_nG_n' + \mathrm{tr}(\Sigma_u)I_n$. Let $\lambda_{\max}$ be the largest eigenvalue of $\mathrm{E}(V_nV_n')$. $Var[\frac{1}{\sqrt{n}}F_n'(I_n-P)V_n] = \frac{1}{n}F_n'(I_n-P)\mathrm{E}(V_nV_n')(I_n-P)F_n \leq \lambda_{\max}\frac{1}{n}F_n'(I_n-P)F_n = O(\Delta_K)$. So $\frac{1}{\sqrt{n}}F_n'(I_n-P)V_n = O_p(\Delta_K^{1/2})$.

For (iii),

$$\mathrm{E}(V_n'PV_n) = \begin{bmatrix} \mathrm{E}(\bar{U}_n'M'M\bar{U}_n) & \mathrm{E}(\bar{U}_n'M'U_n) \\ * & \mathrm{E}(U_n'PU_n) \end{bmatrix},$$

where $\bar{U}_n = U_n\gamma_0 + \epsilon_n$ and $M = PG_n$. $\mathrm{E}(\bar{U}_n'M'M\bar{U}_n) = \mathrm{tr}(M'M)\mathrm{E}(\bar{u}_{ni}^2) = \mathrm{tr}(M'M)(\gamma_0'\Sigma_u\gamma_0 + 2\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)$, $\mathrm{E}(\bar{U}_n'M'U_n) = \sum_i M_{ii}\mathrm{E}(\bar{u}_{ni}u_{ni}) = \mathrm{tr}(M)(\gamma_0'\Sigma_u + \sigma_{u\epsilon})$ and $\mathrm{E}(U_n'PU_n) = \sum_i P_{ii}\mathrm{E}(u_{ni}'u_{ni}) = \mathrm{tr}(P)\Sigma_u = K\Sigma_u$. As $\mathrm{tr}(M'M) = O(K)$ and $\mathrm{tr}(M) = O(K)$ by Lemma A.2, $\mathrm{E}(V_n'PV_n)$ is given by (1). Next, $\mathrm{E}(V_n'P\epsilon_n) = \mathrm{E}[\epsilon_n'M'\bar{U}_n, \epsilon_n'PU_n]'$, where $\mathrm{E}(\epsilon_n'M'\bar{U}_n) = \sum_i M_{ii}\mathrm{E}(\bar{u}_{ni}\epsilon_{ni}) = \mathrm{tr}(M)(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)$ and $\mathrm{E}(\epsilon_n'PU_n) = \sum_i P_{ii}\mathrm{E}(\epsilon_{ni}u_{ni}) = \mathrm{tr}(P)\sigma_{u\epsilon} = K\sigma_{u\epsilon}$. Hence $\mathrm{E}(V_n'P\epsilon_n) = [\mathrm{tr}(M)(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2), K\sigma_{u\epsilon}]' = O(K)$ by Lemma A.2.

For (iv),

$$\mathrm{E}(V_n'P\epsilon_n\epsilon_n'PV_n) = \begin{bmatrix} \mathrm{E}(\bar{U}_n'M'\epsilon_n\epsilon_n'M\bar{U}_n) & \mathrm{E}(\bar{U}_n'M'\epsilon_n\epsilon_n'PU_n) \\ * & \mathrm{E}(U_n'P\epsilon_n\epsilon_n'PU_n) \end{bmatrix}.$$

It follows by Lemma A.3 that

$$\begin{aligned} & \mathrm{E}(\bar{U}_n'M'\epsilon_n\epsilon_n'M\bar{U}_n) \\ = \ & \textstyle\sum_i M_{ii}^2\mathrm{E}(\epsilon_{ni}^2\bar{u}_{ni}^2) + \sum_{i\neq j}(M_{ii}M_{jj} + M_{ji}M_{ij})\mathrm{E}(\bar{u}_{ni}\epsilon_{ni}\epsilon_{nj}\bar{u}_{nj}) + \sum_{i\neq j}M_{ij}M_{ij}\mathrm{E}(\epsilon_{ni}^2\bar{u}_{nj}^2) \\ = \ & [\mathrm{E}(\epsilon_{ni}^2\bar{u}_{ni}^2) - 2(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)^2 - \sigma_\epsilon^2(\gamma_0'\Sigma_u\gamma_0 + 2\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)]\textstyle\sum_i M_{ii}^2 \\ & +(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)^2[\mathrm{tr}^2(M) + \mathrm{tr}(M^2)] + \sigma_\epsilon^2(\gamma_0'\Sigma_u\gamma_0 + 2\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)\mathrm{tr}(M'M). \end{aligned}$$

Similarly, $\mathrm{E}(\bar{U}_n'M'\epsilon_n\epsilon_n'PU_n) = (\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)\sigma_{u\epsilon}K\mathrm{tr}(M) + [(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)\sigma_{u\epsilon} + \sigma_\epsilon^2(\gamma_0'\Sigma_u + \sigma_{u\epsilon})]\mathrm{tr}(M) + o(K)$ and $\mathrm{E}(U_n'P\epsilon_n\epsilon_n'PU_n) = \sigma_{u\epsilon}'\sigma_{u\epsilon}K^2 + (\sigma_{u\epsilon}'\sigma_{u\epsilon} + \sigma_\epsilon^2\Sigma_u)K + o(K)$. Hence, $\mathrm{E}(V_n'P\epsilon_n\epsilon_n'PV_n) - \mathrm{E}(V_n'P\epsilon_n)\mathrm{E}(\epsilon_n'PV_n) = O(K)$ by (iii). Furthermore, under Assumption 5 (ii), $\sum_i M_{ii}^2 = o(K)$ and (2) follows. On the other hand, $\mathrm{E}[(V_n'PV_n)^2] = \mathrm{E}(V_n'M\bar{U}_n\bar{U}_n'M'V_n) + \mathrm{E}(V_n'PU_nU_n'PV_n)$. As $V_n = [G_n\bar{U}_n, U_n]$, by similar arguments as above, $\mathrm{E}(\bar{U}_n'M'M\bar{U}_n\bar{U}_n'M'M\bar{U}_n) = (\gamma_0'\Sigma_u\gamma_0 + 2\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)^2\mathrm{tr}^2(M'M) + O(K)$, $\mathrm{E}(\bar{U}_n'M'M\bar{U}_n\bar{U}_n'M'U_n) = (\gamma_0'\Sigma_u\gamma_0 + 2\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)(\gamma_0'\Sigma_u + \sigma_{u\epsilon})\mathrm{tr}(M'M)\mathrm{tr}(M) + O(K)$, $\mathrm{E}(U_n'M\bar{U}_n\bar{U}_n'M'U_n) = (\Sigma_u\gamma_0 + \sigma_{u\epsilon}')(\gamma_0'\Sigma_u + \sigma_{u\epsilon})\mathrm{tr}^2(M) + O(K)$, $\mathrm{E}(\bar{U}_n'M'U_nU_n'M\bar{U}_n) = (\gamma_0'\Sigma_u + \sigma_{u\epsilon})(\Sigma_u\gamma_0 + \sigma_{u\epsilon}')\mathrm{tr}^2(M) + O(K)$, $\mathrm{E}(\bar{U}_n'M'U_nU_n'PU_n) = (\gamma_0'\Sigma_u + \sigma_{u\epsilon})\Sigma_uK\mathrm{tr}(M) + O(K)$, and $\mathrm{E}(U_n'PU_nU_n'PU_n) = \Sigma_u^2K^2 + O(K)$. Hence, $\mathrm{E}[(V_n'PV_n)^2] - \mathrm{E}^2(V_n'PV_n) = O(K)$ by (iii).

The first part of (v) is Lemma A.3 (vi) in Donald and Newey (2001). The second part of (v) is in Liu and Lee (2010). For (vi), $\mathrm{E}(\frac{1}{n}\Gamma_n'\epsilon_n\epsilon_n'PV_n) = [\mathrm{E}(\frac{1}{n}\Gamma_n'\epsilon_n\epsilon_n'M\bar{U}_n), \mathrm{E}(\frac{1}{n}\Gamma_n'\epsilon_n\epsilon_n'PU_n)]$, where $\mathrm{E}(\frac{1}{n}\Gamma_n'\epsilon_n\epsilon_n'M\bar{U}_n) = \frac{1}{n}\Gamma_n' vec_D(M)E(\epsilon_{ni}^2 u_{ni})\gamma_0 + \frac{1}{n}\Gamma_n' vec_D(M)\mu_3$ and $\mathrm{E}(\frac{1}{n}\Gamma_n'\epsilon_n\epsilon_n'PU_n) = \frac{1}{n}\Gamma_n' vec_D(P)E(\epsilon_{ni}^2 u_{ni})$. On the other hand, let $T_1 = F_n'(I_n - P)\epsilon_n/\sqrt{n}$ and $T_2 = V_n'P\epsilon_n/\sqrt{n}$. By the covariance inequality, $|\mathrm{E}(e_j'h_nT_2'e_k)| \leq \sqrt{\mathrm{Var}(e_j'h_n)}\sqrt{\mathrm{Var}(e_k'T_2)}$. As $\mathrm{Var}(h_n) = O(1)$ and $\mathrm{Var}(T_2) = \mathrm{E}(T_2T_2') - \mathrm{E}(T_2)\mathrm{E}(T_2') = O(K/n)$ by (iv), it follows that $\mathrm{E}(h_nT_2') = \mathrm{E}(\frac{1}{n}F_n'\epsilon_n\epsilon_n'PV_n) = O(\sqrt{K/n})$. Similarly, $|\mathrm{E}(e_j'T_1T_2'e_k)| \leq \sqrt{\mathrm{Var}(e_j'T_1)}\sqrt{\mathrm{Var}(e_k'T_2)}$, where e_j and e_k are, respectively, the jth and kth unit vectors. As $\mathrm{Var}(T_1) = \sigma_\epsilon^2 F_n'(I_n - P)F_n/n = O(\Delta_K)$ and $\mathrm{Var}(T_2) = O(K/n)$, we have $\mathrm{E}(T_1T_2') = \mathrm{E}[\frac{1}{n}F_n'(I_n - P)\epsilon_n\epsilon_n'PV_n] = O(\sqrt{K\Delta_K/n})$. Let v_{ni} be the ith row of V_n. $\mathrm{E}(\frac{1}{n}h_nh_n'H_n^{-1}V_n'F_n) = \frac{1}{n^2}\sum_{i,j}\mathrm{E}(F_n'e_i\epsilon_{n,i}\epsilon_{n,j}e_j'F_nH_n^{-1}V_n'F_n) = \frac{1}{n^2}\sum_i F_n'e_ie_i'F_nH_n^{-1}\mathrm{E}(\epsilon_{n,i}^2 v_{ni}')e_i'F_n = O(\frac{1}{n})$, because elements of F_n are uniformly bounded (implied by elements of $\bar{Z}_{2n}$ being uniformly bounded in Assumption 4 (ii)). ■

Lemma A.5 *Suppose that $\{A_n\}$ is a sequence of $n \times n$ UBC matrices, the elements of the $n \times k$ matrix C_n are uniformly bounded, and $\epsilon_{n1}, \cdots, \epsilon_{nn}$ are i.i.d. with zero mean and finite variance σ^2. Then, $\frac{1}{\sqrt{n}}C_n'A_n\epsilon_n = O_p(1)$ and $\frac{1}{n}C_n'A_n\epsilon_n = o_p(1)$. Furthermore, if the limit of $\frac{1}{n}C_n'A_nA_n'C_n$ exists and is positive definite, then $\frac{1}{\sqrt{n}}C_n'A_n\epsilon_n \xrightarrow{D} N(0, \sigma^2 \lim_{n\to\infty}\frac{1}{n}C_n'A_nA_n'C_n)$.*

Proof. See Lee (2004). ■

Lemma A.6 (i) *$\frac{1}{n}Z_n'PZ_n = H_n + \sum_{i=1}^4 R_i^H$, where $R_1^H = -e_F(K) = O(\Delta_K)$, $R_2^H = \frac{1}{n}(F_n'V_n + V_n'F_n) = O(1/\sqrt{n})$, $R_3^H = \frac{1}{n}V_n'PV_n = O(K/n)$, and $R_4^H = -\frac{1}{n}[F_n'(I_n - P)V_n + V_n'(I_n - P)F_n] = O(\sqrt{\Delta_K/n}) = o(K/n + \Delta_K)$;*

(ii) *$[Z_n'P\epsilon_n - \mathrm{E}(V_n'P\epsilon_n)]/\sqrt{n} = h_n + \sum_{i=1}^2 R_i^h$, where $R_1^h = -F_n'(I_n - P)\epsilon_n/\sqrt{n} = O(\Delta_K^{1/2})$ and $R_2^h = [V_n'P\epsilon_n - \mathrm{E}(V_n'P\epsilon_n)]/\sqrt{n} = O(\sqrt{K/n})$;*

(iii) *$\mathrm{E}(R_1^hR_1^{h\prime}) = -\mathrm{E}(R_1^hh_n') = -\mathrm{E}(h_nh_n'H_n^{-1}R_1^{H\prime}) = \sigma_\epsilon^2 e_F(K)$ and $\mathrm{E}(h_nh_n'H_n^{-1}R_2^{H\prime}) = O(1/n)$.*

Proof. $\frac{1}{n}Z_n'PZ_n = \frac{1}{n}(F_n + V_n)'P(F_n + V_n) = \frac{1}{n}F_n'PF_n + \frac{1}{n}(F_n'PV_n + V_n'PF_n) + \frac{1}{n}V_n'PV_n = H_n + \sum_{i=1}^4 R_i^H$, where $R_1^H = -\frac{1}{n}F_n'(I_n - P)F_n = -e_F(K) = O(\Delta_K)$, $R_2^H = \frac{1}{n}(F_n'V_n + V_n'F_n) = O(1/\sqrt{n})$ by Lemma A.5, $R_3^H = \frac{1}{n}V_n'PV_n$, and $R_4^H = -\frac{1}{n}[F_n'(I_n - P)V_n + V_n'(I_n - P)F_n] = O(\sqrt{\Delta_K/n}) = o(K/n + \Delta_K)$ by Lemma A.4 (ii) and (v). As $\frac{1}{n}\mathrm{E}(V_n'PV_n) = O(K/n)$ by Lemma A.4 (iii), $R_3^H = \frac{1}{n}V_n'PV_n = O(K/n)$ by MI.

For (ii), $[Z_n' P\epsilon_n - \mathrm{E}(V_n' P\epsilon_n)]/\sqrt{n} = h_n - F_n'(I_n - P)\epsilon_n/\sqrt{n} + [V_n' P\epsilon_n - \mathrm{E}(V_n' P\epsilon_n)]/\sqrt{n} = h_n + R_1^h + R_2^h$, where $R_1^h = -F_n'(I_n - P)\epsilon_n/\sqrt{n} = O(\Delta_K^{1/2})$ by Lemma A.4 (ii), and $R_2^h = [V_n' P\epsilon_n - \mathrm{E}(V_n' P\epsilon_n)]/\sqrt{n}$. As $E(V_n' P\epsilon_n \epsilon_n' P V_n) - \mathrm{E}(V_n' P\epsilon_n)\mathrm{E}(\epsilon_n' P V_n) = O(K)$ by Lemma A.4 (iv), it follows that $R_2^h = O(\sqrt{K/n})$.

For (iii), $\mathrm{E}(R_1^h R_1^{h\prime}) = -\mathrm{E}(R_1^h h_n') = -\mathrm{E}(h_n h_n' H_n^{-1} R_1^{H\prime}) = \sigma_\epsilon^2 e_F(K)$ is trivial. By Lemma A.4 (vi), $\mathrm{E}(h_n h_n' H_n^{-1} R_2^{H\prime}) = O(1/n)$. ■

Lemma A.7 *If* $\sup_K \frac{|\hat{S}_\xi(K) - S_\xi(K)|}{S_\xi(K)} \xrightarrow{p} 0$, *then* $\frac{S_\xi(\hat{K})}{\inf_K S_\xi(K)} \xrightarrow{p} 1$.

Proof. Donald and Newey (2001) Lemma A.9. ■

Lemma A.8 *Suppose* $\sqrt{n}(\tilde{\delta}_n - \delta_0) = O(1)$, $\tilde{\sigma}_{u\epsilon}' = \frac{1}{n} Z_{2n}'(y_n - Z_n\tilde{\delta}_n)$, *and* $\tilde{\sigma}_\epsilon^2 = \frac{1}{n}(y_n - Z_n\tilde{\delta}_n)'(y_n - Z_n\tilde{\delta}_n)$. *Then,* (i)

$$\sqrt{n}(\tilde{\sigma}_{u\epsilon} - \sigma_{u\epsilon})' = \frac{1}{\sqrt{n}}\bar{Z}_{2n}'\epsilon_n + \sqrt{n}(\frac{1}{n}U_n'\epsilon_n - \sigma_{u\epsilon}) - \frac{1}{n}[\bar{Z}_{2n}'F_n + \mathrm{E}(U_n'V_n)]\sqrt{n}(\tilde{\delta}_n - \delta_0) + o(1) = O(1),$$

and (ii) $\sqrt{n}(\tilde{\sigma}_\epsilon^2 - \sigma_\epsilon^2) = \sqrt{n}(\frac{1}{n}\epsilon_n'\epsilon_n - \sigma_\epsilon^2) - \frac{2}{n}\mathrm{E}(\epsilon_n' V_n)\sqrt{n}(\tilde{\delta}_n - \delta_0) + o(1) = O(1)$.

Proof. For (i), $\tilde{\sigma}_{u\epsilon}' = \frac{1}{n}Z_{2n}'\epsilon_n - \frac{1}{n}Z_{2n}'Z_n(\tilde{\delta}_n - \delta_0)$. As $\frac{1}{n}\bar{Z}_{2n}'V_n = o(1)$, $\frac{1}{n}U_n'F_n = o(1)$, and $\frac{1}{n}[U_n'V_n - \mathrm{E}(U_n'V_n)] = o(1)$, we have $\frac{1}{n}Z_{2n}'Z_n = \frac{1}{n}(\bar{Z}_{2n} + U_n)'(F_n + V_n) = \frac{1}{n}[\bar{Z}_{2n}'F_n + \mathrm{E}(U_n'V_n)] + o(1)$. Hence, $\sqrt{n}(\tilde{\sigma}_{u\epsilon} - \sigma_{u\epsilon})' = \sqrt{n}(\frac{1}{n}Z_{2n}'\epsilon_n - \sigma_{u\epsilon}) - (\frac{1}{n}Z_{2n}'Z_n)\sqrt{n}(\tilde{\delta}_n - \delta_0) = \frac{1}{\sqrt{n}}\bar{Z}_{2n}'\epsilon_n + \sqrt{n}(\frac{1}{n}U_n'\epsilon_n - \sigma_{u\epsilon}) - \frac{1}{n}[\bar{Z}_{2n}'F_n + \mathrm{E}(U_n'V_n)]\sqrt{n}(\tilde{\delta}_n - \delta_0) + o(1)$, where $\frac{1}{\sqrt{n}}\bar{Z}_{2n}'\epsilon_n = O(1)$, $\sqrt{n}(\frac{1}{n}U_n'\epsilon_n - \sigma_{u\epsilon}) = O(1)$, and $\frac{1}{n}[\bar{Z}_{2n}'F_n + \mathrm{E}(U_n'V_n)] = O(1)$. For (ii), $\tilde{\sigma}_\epsilon^2 = \frac{1}{n}[\epsilon_n - Z_n(\tilde{\delta}_n - \delta_0)]'[\epsilon_n - Z_n(\tilde{\delta}_n - \delta_0)] = \frac{1}{n}\epsilon_n'\epsilon_n - \frac{2}{n}\epsilon_n'Z_n(\tilde{\delta}_n - \delta_0) + \frac{1}{n}(\tilde{\delta}_n - \delta_0)'Z_n'Z_n(\tilde{\delta}_n - \delta_0)$. As $\frac{1}{n}\epsilon_n'F_n = o(1)$ and $\frac{1}{n}[\epsilon_n'V_n - \mathrm{E}(\epsilon_n'V_n)] = o(1)$, it follows that $\frac{1}{n}\epsilon_n'Z_n = \frac{1}{n}\epsilon_n'(F_n + V_n) = \frac{1}{n}\epsilon_n'F_n + \frac{1}{n}[\epsilon_n'V_n - \mathrm{E}(\epsilon_n'V_n)] + \frac{1}{n}\mathrm{E}(\epsilon_n'V_n) = \frac{1}{n}\mathrm{E}(\epsilon_n'V_n) + o(1)$. Hence, $\sqrt{n}(\tilde{\sigma}_\epsilon^2 - \sigma_\epsilon^2) = \sqrt{n}(\frac{1}{n}\epsilon_n'\epsilon_n - \sigma_\epsilon^2) - \frac{2}{n}\epsilon_n'Z_n\sqrt{n}(\tilde{\delta}_n - \delta_0) + \frac{1}{n}(\tilde{\delta}_n - \delta_0)'Z_n'Z_n\sqrt{n}(\tilde{\delta}_n - \delta_0) = \sqrt{n}(\frac{1}{n}\epsilon_n'\epsilon_n - \sigma_\epsilon^2) - \frac{2}{n}\mathrm{E}(\epsilon_n'V_n)\sqrt{n}(\tilde{\delta}_n - \delta_0) + o(1)$, where $\sqrt{n}(\frac{1}{n}\epsilon_n'\epsilon_n - \sigma_\epsilon^2) = O(1)$ and $\frac{1}{n}\mathrm{E}(\epsilon_n'V_n) = O(1)$.
■

B Proofs

Proof of Lemma 2.1. For

$$\pi_K = \begin{pmatrix} \gamma_0'\pi_q^{0\prime} & \lambda_0\gamma_0'\pi_q^{0\prime} & \cdots & \lambda_0^p\gamma_0'\pi_q^{0\prime} & 0 \\ 0 & 0 & \cdots & 0 & \pi_q^{0\prime} \end{pmatrix}',$$

$Q_K\pi_K = [(\sum_{j=0}^{p}\lambda_0^j W_n^{j+1})\Psi_q(X_n)\pi_q^0\gamma_0, \Psi_q(X_n)\pi_q^0]$. Hence, when $\sup_n ||\lambda_0 W_n||_\infty < 1$, we have $||F_n - Q_K\pi_K||_\infty = ||[G_n\bar{Z}_{2n}\gamma_0 - (\sum_{j=0}^{p}\lambda_0^j W_n^{j+1})\Psi_q(X_n)\pi_q^0\gamma_0, \bar{Z}_{2n} - \Psi_q(X_n)\pi_q^0]||_\infty = ||[G_n(\bar{Z}_{2n} - \Psi_q(X_n)\pi_q^0)\gamma_0 + (G_n - \sum_{j=0}^{p}\lambda_0^j W_n^{j+1})\Psi_q(X_n)\pi_q^0\gamma_0, \bar{Z}_{2n} - \Psi_q(X_n)\pi_q^0]||_\infty$, which is $o(1)$ as $n, K \to \infty$, since $||\bar{Z}_{2n} - \Psi_q(X_n)\pi_q^0||_\infty \to 0$ and $||G_n - \sum_{j=0}^{p}\lambda_0^j W_n^{j+1}||_\infty \leq ||\lambda_0 W_n||_\infty^{p+1}||G_n||_\infty \to 0$. Finally, as $\frac{1}{n}||\cdot||^2 \leq (||\cdot||_\infty)^2$, the desired result follows. ■

Proof of Proposition 1. The 2SLS estimator satisfies $\sqrt{n}(\hat{\delta}_{2sls,n} - \delta_0 - b_{2sls,n}) = (\frac{1}{n}Z_n'PZ_n)^{-1}[Z_n'P\epsilon_n - \mathrm{E}(V_n'P\epsilon_n)]/\sqrt{n}$, where $\mathrm{E}(V_n'P\epsilon_n) = [\mathrm{tr}(M)(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2), K\sigma_{u\epsilon}]'$ by Lemma A.4 (iii). As $K/n \to 0$ and $\Delta_K = o(1)$ by Lemma A.4 (i), we have $\frac{1}{n}Z_n'PZ_n = H_n + o(1) \xrightarrow{p} \bar{H}$ and $[Z_n'P\epsilon_n - \mathrm{E}(V_n'P\epsilon_n)]/\sqrt{n} = h_n + o(1) \xrightarrow{d} N(0, \sigma_\epsilon^2\bar{H})$ by Lemma A.6. The conclusion then follows by the Slutzky theorem. ■

Proof of Corollary 2. (i) and (ii) are trivial. For (iii), because $0 < \eta < 1$ implies $K^{2\eta} < K^{1+\eta}$, therefore $K^\eta/\sqrt{n} < \sqrt{K^{1+\eta}/n} \to 0$. As $\sqrt{n}(\hat{\delta}_{2sls,n} - \delta_0 - b_{2sls,n}) = O(1)$, $(K^\eta/\sqrt{n})\sqrt{n}(\hat{\delta}_{2sls,n} - \delta_0 - b_{2sls,n}) = o(1)$. Note that $b_{2sls,n} = O(K/n)$ and $K^\eta b_{2sls,n} = O(K^{1+\eta}/n) = o(1)$. It follows that $K^\eta(\hat{\delta}_{2sls,n} - \delta_0) = K^\eta(\hat{\delta}_{2sls,n} - \delta_0 - b_{2sls,n}) + K^\eta b_{2sls,n} = o(1)$. ■

Proof of Proposition 3. The 2SLS estimator satisfies $\hat{\delta}_{2sls,n} - \delta_0 - b_{2sls,n} = (\frac{1}{n}Z_n'PZ_n)^{-1}\frac{1}{n}[Z_n'P\epsilon_n - \mathrm{E}(V_n'P\epsilon_n)]$. By Lemma A.6, $(\frac{1}{n}Z_n'PZ_n)^{-1} = O(1)$ and $\frac{1}{n}[Z_n'P\epsilon_n - \mathrm{E}(V_n'P\epsilon_n)] = O(1/\sqrt{n})$. By Lemma A.4 (iii), $\frac{1}{n}\mathrm{E}(V_n'P\epsilon_n) = O(K/n) = O(1)$. The conclusion then follows by the Slutzky theorem. ■

Proof of Proposition 4. It is sufficient to show that $\frac{1}{\sqrt{n}}(\hat{b}_{2sls,n} - b_{2sls,n}) = o(1)$. As $\tilde{G}_n - G_n = (\tilde{\lambda}_n - \lambda_0)\tilde{G}_n G_n$, $\sqrt{n}(\tilde{\delta}_n - \delta_0) = O(1)$, $\sqrt{n}(\tilde{\sigma}_{u\epsilon} - \sigma_{u\epsilon}) = O(1)$, and $\sqrt{n}(\tilde{\sigma}_\epsilon^2 - \sigma_\epsilon^2) = O(1)$ by Lemma A.8, $\frac{1}{\sqrt{n}}[\mathrm{tr}(P\tilde{G}_n)(\tilde{\sigma}_{u\epsilon}\tilde{\gamma}_n + \tilde{\sigma}_\epsilon^2) - \mathrm{tr}(M)(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)] = \frac{1}{\sqrt{n}}(\tilde{\lambda}_n - \lambda_0)\mathrm{tr}(P\tilde{G}_n G_n)(\tilde{\sigma}_{u\epsilon}\tilde{\gamma}_n + \tilde{\sigma}_\epsilon^2) + \frac{1}{\sqrt{n}}\mathrm{tr}(M)[\tilde{\sigma}_{u\epsilon}(\tilde{\gamma}_n - \gamma_0) + (\tilde{\sigma}_{u\epsilon} - \sigma_{u\epsilon})\gamma_0 + (\tilde{\sigma}_\epsilon^2 - \sigma_\epsilon^2)] = O(K/n) = o(1)$. Similarly, $K(\tilde{\sigma}_{u\epsilon} - \sigma_{u\epsilon})/\sqrt{n} = O(K/n) = o(1)$. On the other hand, as $\frac{1}{n}Z_n'P_K Z_n = O(1)$, the desired result follows. ■

Proof of Proposition 5. The 2SLS estimator satisfies $\sqrt{n}(\hat{\delta}_{2sls,n} - \delta_0) = \hat{H}_n^{-1}\hat{h}_n$, with $\hat{H}_n = \frac{1}{n}Z_n'PZ_n$ and $\hat{h}_n = Z_n'P\epsilon_n/\sqrt{n}$. By Lemma A.6 (i), $\hat{H}_n = H_n + T_1^H + T_2^H + Z^H$, where $T_1^H = R_1^H = O(\Delta_K)$, $T_2^H = R_2^H = O(1/\sqrt{n})$, $Z^H = R_3^H + R_4^H = O(K/n) + o(K/n + \Delta_K)$. On the other hand, $\hat{h}_n = h_n + T_1^h + T_2^h$, where $T_1^h = R_1^h = O(\Delta_K^{1/2})$ by Lemma A.6 (ii) and $T_2^h = V_n'P\epsilon_n/\sqrt{n}$. As $E(V_n'PV_n) = O(K)$ by Lemma A.4 (iii) and $E(\epsilon_n'P\epsilon_n) = \sigma_\epsilon^2\mathrm{tr}(P) = \sigma_\epsilon^2 K$, it follows by MI that $E|V_n'P\epsilon_n| \leq [E(V_n'PV_n)]^{1/2}[E(\epsilon_n'P\epsilon_n)]^{1/2} = O(K)$. Hence, $T_2^h = V_n'P\epsilon_n/\sqrt{n} = O(K/\sqrt{n})$. For

$\rho_{K,n} = \mathrm{tr}(S(K))$, $\rho_{K,n} = O(K^2/n + \Delta_K)$ by Assumption 5 (i). It follows that $\|Z^H\| = o(\rho_{K,n})$. For $T^H = T_1^H + T_2^H$ and $T^h = T_1^h + T_2^h$, $\|T^H\|^2 = o(\rho_{K,n})$, $\|T^h\| \|T^H\| = o(\rho_{K,n})$.

Next, by Lemma A.6 (iii), $\mathrm{E}(T_1^h T_1^{h\prime}) = -\mathrm{E}(T_1^h h_n') = -\mathrm{E}(h_n h_n' H_n^{-1} T_1^{H\prime}) = \sigma_\epsilon^2 e_F(K)$. By Lemma A.4 (iv), $\mathrm{E}(T_2^h T_2^{h\prime}) = \frac{1}{n}\mathrm{E}(V_n' P \epsilon_n)\mathrm{E}(\epsilon_n' P V_n) + O(K/n)$. When $\mu_3 = 0$ and $E(\epsilon_{ni}^2 u_{ni}) = 0$, $\mathrm{E}(h_n T_2^{h\prime}) = \frac{1}{n}\mathrm{E}(F_n' \epsilon_n \epsilon_n' P V_n) = 0$. By Lemma A.4 (v), $\mathrm{E}(T_1^h T_2^{h\prime}) = -\frac{1}{n}\mathrm{E}[F_n'(I_n - P)\epsilon_n \epsilon_n' P V_n] = O(\sqrt{K\Delta_K/n}) = o(\rho_{K,n})$, and, by Lemma A.6 (iii), $\mathrm{E}(h_n h_n' H_n^{-1} T_2^{H\prime}) = O(1/n) = o(\rho_{K,n})$. Then, for $Z^A(K) = 0$ and $\hat{A}(K)$ defined in Lemma A.1, $\mathrm{E}[\hat{A}(K)] = \mathrm{E}[(h_n + T_1^h + T_2^h)(h_n + T_1^h + T_2^h)' - h_n h_n' H_n^{-1}(T_1^H + T_2^H)' - (T_1^H + T_2^H) H_n^{-1} h_n h_n'] = \sigma_\epsilon^2 H_n + \sigma_\epsilon^2 e_F(K) + \frac{1}{n}\mathrm{E}(V_n' P \epsilon_n)\mathrm{E}(\epsilon_n' P V_n) + o(\rho_{K,n}) = \sigma_\epsilon^2 H_n + H_n S(K) H_n + o(\rho_{K,n})$. So all conditions of Lemma A.1 are satisfied. The desired result follows as $\mathrm{E}(V_n' P \epsilon_n) = [\mathrm{tr}(M)(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2), K\sigma_{u\epsilon}]'$ by Lemma A.4 (iii). ■

Proof of Proposition 6. For $\rho_{K,n} = \mathrm{tr}(S(K))$, $\rho_{K,n} = O(K/n + \Delta_K)$ by Assumption 5 (i). The C2SLS estimator satisfies $\sqrt{n}(\hat{\delta}_{c2sls,n} - \delta_0) = \hat{H}_n^{-1}\hat{h}_n$, with $\hat{H}_n = \frac{1}{n} Z_n' P Z_n$ and $\hat{h}_n = \frac{1}{\sqrt{n}} Z_n' P \epsilon_n - \frac{1}{\sqrt{n}}[\mathrm{tr}(P\tilde{G}_n)(\tilde{\sigma}_{u\epsilon}\tilde{\gamma}_n + \tilde{\sigma}_\epsilon^2), K\tilde{\sigma}_{u\epsilon}]'$. By Lemma A.6 (i), $\hat{H}_n = H_n + \sum_{j=1}^3 T_j^H + Z^H$, where $T_1^H = R_1^H = O(\Delta_K)$, $T_2^H = R_2^H = O(1/\sqrt{n})$, $T_3^H = \frac{1}{n}\mathrm{E}(V_n' P V_n) = O(K/n)$, and $Z^H = \frac{1}{n} V_n' P V_n - \frac{1}{n}\mathrm{E}(V_n' P V_n) + R_4^H$. By Lemma A.4 (iv), we have $\mathrm{E}[(V_n' P V_n)^2] = \mathrm{E}^2(V_n' P V_n) + O(K)$. It follows that $\mathrm{E}[\frac{1}{n} V_n' P V_n - \frac{1}{n}\mathrm{E}(V_n' P V_n)]^2 = O(K/n^2) = o(K^2/n^2)$. Hence, $\frac{1}{n} V_n' P V_n - \frac{1}{n}\mathrm{E}(V_n' P V_n) = o(K/n)$. As $R_4^H = o(K/n + \Delta_K)$ by Lemma A.6, $Z^H = o(\rho_{K,n})$.

On the other hand, $\hat{h}_n = \frac{1}{\sqrt{n}}[Z_n' P \epsilon_n - \mathrm{E}(V_n' P \epsilon_n)] - \frac{1}{\sqrt{n}}\{[\mathrm{tr}(P\tilde{G}_n)(\tilde{\sigma}_{u\epsilon}\tilde{\gamma}_n + \tilde{\sigma}_\epsilon^2), K\tilde{\sigma}_{u\epsilon}]' - \mathrm{E}(V_n' P \epsilon_n)\}$. $\frac{1}{\sqrt{n}}[Z_n' P \epsilon_n - \mathrm{E}(V_n' P \epsilon_n)] = h_n + T_1^h + T_2^h$, where $T_1^h = R_1^h = O(\Delta_K^{1/2})$ and $T_2^h = R_2^h = O(\sqrt{K/n})$ by Lemma A.6 (ii). As $\tilde{G}_n - G_n = (\tilde{\lambda}_n - \lambda_0)\tilde{G}_n G_n$, $\frac{1}{\sqrt{n}}[\mathrm{tr}(P\tilde{G}_n)(\tilde{\sigma}_{u\epsilon}\tilde{\gamma}_n + \tilde{\sigma}_\epsilon^2) - \mathrm{tr}(M)(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)] = \frac{1}{\sqrt{n}}(\tilde{\lambda}_n - \lambda_0)\mathrm{tr}(P\tilde{G}_n G_n)(\tilde{\sigma}_{u\epsilon}\tilde{\gamma}_n + \tilde{\sigma}_\epsilon^2) + \frac{1}{\sqrt{n}}\mathrm{tr}(M)[\tilde{\sigma}_{u\epsilon}(\tilde{\gamma}_n - \gamma_0) + (\tilde{\sigma}_{u\epsilon} - \sigma_{u\epsilon})\gamma_0 + (\tilde{\sigma}_\epsilon^2 - \sigma_\epsilon^2)] = \frac{1}{\sqrt{n}}(\tilde{\lambda}_n - \lambda_0) tr(PG_n^2)(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2) + \frac{1}{\sqrt{n}} tr(M)[\sigma_{u\epsilon}(\tilde{\gamma}_n - \gamma_0) + \gamma_0'(\tilde{\sigma}_{u\epsilon} - \sigma_{u\epsilon})' + (\tilde{\sigma}_\epsilon^2 - \sigma_\epsilon^2)] + o(K/n)$. Let $P_0 = Q_0(Q_0'Q_0)^{-1}Q_0'$. The preliminary estimator is given by $\tilde{\delta}_n = (Z_n' P_0 Z_n)^{-1} Z_n' P_0 y_n$. As $\frac{1}{n} Q_0' Z_n = \frac{1}{n} Q_0'(F_n + V_n) = \frac{1}{n} Q_0' F_n + o(1)$, $\sqrt{n}(\tilde{\delta}_n - \delta_0) = L_0 + o(1)$, where $L_0 = \sqrt{n}(F_n' P_0 F_n)^{-1} F_n' P_0 \epsilon_n$. By Lemma A.8, $\sqrt{n}(\tilde{\sigma}_{u\epsilon} - \sigma_{u\epsilon})' = L_1 + o(1)$ and $\sqrt{n}(\tilde{\sigma}_\epsilon^2 - \sigma_\epsilon^2) = L_2 + o(1)$, where $L_1 = \frac{1}{\sqrt{n}}\bar{Z}_{2n}'\epsilon_n + \sqrt{n}(\frac{1}{n} U_n' \epsilon_n - \sigma_{u\epsilon}) - \frac{1}{n}[\bar{Z}_{2n}' F_n + \mathrm{E}(U_n' V_n)] L_0$ and $L_2 = \sqrt{n}(\frac{1}{n}\epsilon_n'\epsilon_n - \sigma_\epsilon^2) - \frac{2}{n}\mathrm{E}(\epsilon_n' V_n) L_0$. Let $T_3^h = -\frac{1}{n}[a_1, a_2]'$, where $a_1 = \mathrm{tr}(PG_n^2)(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2) e_1' L_0 + \mathrm{tr}(M)[\sigma_{u\epsilon}(0_{m\times 1}, I_m) L_0 + \gamma_0' L_1 + L_2]$ and $a_2 = K L_1'$. Let $Z^h = \frac{1}{\sqrt{n}}[\mathrm{tr}(P\tilde{G}_n)(\tilde{\sigma}_{u\epsilon}\tilde{\gamma}_n + \tilde{\sigma}_\epsilon^2) - \mathrm{tr}(M)(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2), K(\tilde{\sigma}_{u\epsilon} - \sigma_{u\epsilon})] - T_3^h = o(K/n)$. Hence, $\hat{h}_n = h_n + \sum_{j=1}^3 T_j^h + Z^h$. It follows that, for $T^H = \sum_{j=1}^3 T_j^H$ and $T^h = \sum_{j=1}^3 T_j^h$, we have $\|T^H\|^2 = o(\rho_{K,n})$, $\|T^h\| \|T^H\| = o(\rho_{K,n})$, $\|Z^h\| = o(\rho_{K,n})$ and $\|Z^H\| = o(\rho_{K,n})$.

Next, by Lemma A.6 (iii), $\mathrm{E}(T_1^h T_1^{h\prime}) = -\mathrm{E}(T_1^h h_n') = -\mathrm{E}(h_n h_n' H_n^{-1} T_1^{H\prime}) = \sigma_\epsilon^2 e_F(K)$. $\mathrm{E}(T_2^h T_2^{h\prime}) = \frac{1}{n}[\mathrm{E}(V_n' P\epsilon_n \epsilon_n' P V_n) - \mathrm{E}(V_n' P \epsilon_n)\mathrm{E}(\epsilon_n' P V_n)] = \frac{1}{n}\Pi_1(K) + o(K/n)$, where $\Pi_1(K)$ denotes the leading-order matrix given in Lemma A.4 (iv). As $\mu_3 = 0$ and $\mathrm{E}(\epsilon_{ni}^2 u_{ni}) = 0$, we have $\mathrm{E}(h_n T_2^{h\prime}) = 0$, and $\mathrm{E}(T_1^h T_2^{h\prime}) = 0$. By Lemma A.6 (iii), $\mathrm{E}(h_n h_n' H_n^{-1} T_2^{H\prime}) = O(1/n) = o(\rho_{K,n})$. $\mathrm{E}(h_n h_n' H_n^{-1} T_3^{H\prime}) = \frac{1}{n}\sigma_\epsilon^2 \mathrm{E}(V_n' P V_n)$.

$\mathrm{E}(L_0 h_n') = (F_n' P_0 F_n)^{-1} F_n' P_0 \mathrm{E}(\epsilon_n \epsilon_n') F_n = \sigma_\epsilon^2 I_{m+1}$. As $\mathrm{E}(\epsilon_{ni}^2 u_{ni}) = 0$, $\mathrm{E}(L_1 h_n') = \frac{1}{n}\bar{Z}_{2n}' \mathrm{E}(\epsilon_n \epsilon_n') F_n - \frac{1}{n}[\bar{Z}_{2n}' F_n + \mathrm{E}(U_n' V_n)]\mathrm{E}(L_0 h_n') = \frac{1}{n}\sigma_\epsilon^2 \bar{Z}_{2n}' F_n - \frac{1}{n}\sigma_\epsilon^2[\bar{Z}_{2n}' F_n + \mathrm{E}(U_n' V_n)] = -\frac{1}{n}\sigma_\epsilon^2 \mathrm{E}(U_n' V_n)$, where $\mathrm{E}(U_n' V_n) = \mathrm{E}[U_n' G_n (U_n \gamma_0 + \epsilon_n), U_n' U_n] = [\mathrm{tr}(G_n)(\Sigma_u \gamma_0 + \sigma_{u\epsilon}'), n\Sigma_u]$. As $\mu_3 = 0$, $\mathrm{E}(L_2 h_n') = -\frac{2}{n}\mathrm{E}(\epsilon_n' V_n)\mathrm{E}(L_0 h_n') = -\frac{2}{n}\sigma_\epsilon^2 \mathrm{E}(\epsilon_n' V_n)$, where $\mathrm{E}(\epsilon_n' V_n) = \mathrm{E}[\epsilon_n' G_n (U_n \gamma_0 + \epsilon_n), \epsilon_n' U_n] = [\mathrm{tr}(G_n)(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2), n\sigma_{u\epsilon}]$. It follows that $\mathrm{E}(a_1 h_n') = \mathrm{tr}(PG_n^2)(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2) e_1' \mathrm{E}(L_0 h_n') + \mathrm{tr}(M)[\sigma_{u\epsilon}(0_{m\times 1}, I_m)\mathrm{E}(L_0 h_n') + \gamma_0' \mathrm{E}(L_1 h_n') + \mathrm{E}(L_2 h_n')] = \mathrm{tr}(PG_n^2)(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)\sigma_\epsilon^2 e_1' + \mathrm{tr}(M)\sigma_\epsilon^2[\sigma_{u\epsilon}(0_{m\times 1}, I_m) - \frac{1}{n}\gamma_0' \mathrm{E}(U_n' V_n) - \frac{2}{n}\mathrm{E}(\epsilon_n' V_n)]$.$\mathrm{E}(a_2' h_n') = K\mathrm{E}(L_1 h_n') = -\frac{K}{n}\sigma_\epsilon^2 \mathrm{E}(U_n' V_n)$. Hence,

$$\mathrm{E}(T_3^h h_n') = -\frac{1}{n}\mathrm{E}([a_1, a_2]' h_n')$$
$$= \frac{1}{n}\begin{bmatrix} \frac{1}{n}\mathrm{tr}(M)\mathrm{tr}(G_n)\sigma_\epsilon^2(\gamma_0'\Sigma_u\gamma_0 + 3\sigma_{u\epsilon}\gamma_0 + 2\sigma_\epsilon^2) - \mathrm{tr}(PG_n^2)\sigma_\epsilon^2(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2) & \mathrm{tr}(M)\sigma_\epsilon^2(\gamma_0'\Sigma_u + \sigma_{u\epsilon}) \\ \frac{K}{n}\mathrm{tr}(G_n)\sigma_\epsilon^2(\Sigma_u\gamma_0 + \sigma_{u\epsilon}') & K\sigma_\epsilon^2\Sigma_u \end{bmatrix}.$$

Then for $Z^A(K) = [(T_1^h + T_2^h)T_3^{h\prime}]^s + T_3^h T_3^{h\prime}$ in Lemma A.1,

$$\begin{aligned} \mathrm{E}[\hat{A}(K)] &= \mathrm{E}[(h_n + T_1^h + T_2^h)(h_n + T_1^h + T_2^h)' + (T_3^h h_n')^s - (h_n h_n' H_n^{-1} T^{H\prime})^s] \\ &= \mathrm{E}[h_n h_n' + T_1^h T_1^{h\prime} + T_2^h T_2^{h\prime} + (T_1^h h_n')^s + (T_3^h h_n')^s - (h_n h_n' H_n^{-1} T^{H\prime})^s] \\ &= \sigma_\epsilon^2 H_n + \sigma_\epsilon^2 e_F(K) + \frac{1}{n}\Pi_1(K) + \mathrm{E}(T_3^h h_n')^s - \frac{2}{n}\sigma_\epsilon^2 \mathrm{E}(V_n' P V_n) + o(\rho_{K,n}) \\ &= \sigma_\epsilon^2 H_n + H_n S(K) H_n + o(\rho_{K,n}). \end{aligned}$$

So all conditions of Lemma A.1 are satisfied. Let $\Pi_2(K) = n\mathrm{E}(T_3^h h_n')^s - 2\sigma_\epsilon^2 \mathrm{E}(V_n' P V_n)$.

$$\Pi_2(K) = \begin{bmatrix} \Pi_{2,11} & * \\ [\frac{K}{n}\mathrm{tr}(G_n) - \mathrm{tr}(M)]\sigma_\epsilon^2(\Sigma_u\gamma_0 + \sigma_{u\epsilon}') & 0 \end{bmatrix},$$

where $\Pi_{2,11} = 2[\frac{1}{n}\mathrm{tr}(M)\mathrm{tr}(G_n) - \mathrm{tr}(M'M)]\sigma_\epsilon^2(\gamma_0'\Sigma_u\gamma_0 + 2\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2) + 2[\frac{1}{n}\mathrm{tr}(M)\mathrm{tr}(G_n) - \mathrm{tr}(PG_n^2)]\sigma_\epsilon^2(\sigma_{u\epsilon}\gamma_0 + \sigma_\epsilon^2)$. ■

Proof of Proposition 7. Let $\Omega_{1\xi} = \xi' H_n^{-1}\Omega_1 H_n^{-1}\xi$, $\Omega_{2\xi} = \xi' H_n^{-1}\Omega_2 H_n^{-1}\xi$, $\Pi = \Pi_1(K) + \Pi_2(K)$ and $\Pi_\xi = \xi' H_n^{-1}\Pi H_n^{-1}\xi$. Let $\hat{\Omega}_{1\xi} = \xi'\hat{H}_n^{-1}\hat{\Omega}_1\hat{H}_n^{-1}\xi$, $\hat{\Omega}_{2\xi} = \xi'\hat{H}_n^{-1}\hat{\Omega}_2\hat{H}_n^{-1}\xi$ and $\hat{\Pi}_\xi = \xi'\hat{H}_n^{-1}\hat{\Pi}\hat{H}_n^{-1}\xi$. Let $\tilde{R}(K) = \frac{1}{n}[Z_n'(I_n - P)Z_n + \hat{\Omega}_2 - V_n'V_n]$ and $\tilde{R}_\xi(K) = \xi'\hat{H}_n^{-1}\tilde{R}(K)\hat{H}_n^{-1}\xi$. Let $\tilde{S}_\xi(K) = \frac{1}{n}\hat{\Omega}_{1\xi} + \hat{\sigma}_\epsilon^2\tilde{R}_\xi(K)$ for 2SLS and $\tilde{S}_\xi(K) = \frac{1}{n}\hat{\Pi}_\xi + \hat{\sigma}_\epsilon^2\tilde{R}_\xi(K)$ for C2SLS. As $\frac{1}{n}V_n'V_n$ does not depend on K, $\hat{K} = \arg\min_K \hat{S}_\xi(K) = \arg\min_K \tilde{S}_\xi(K)$. Let $R(K) = \frac{1}{n}F_n'(I_n - P)F_n$, $R_\xi(K) = \xi' H_n^{-1}R(K)H_n^{-1}\xi$. We have $S_\xi(K) = \frac{1}{n}\Omega_{1\xi} + \sigma_\epsilon^2 R_\xi(K)$ for 2SLS and $S_\xi(K) = \frac{1}{n}\Pi_\xi + \sigma_\epsilon^2 R_\xi(K)$ for C2SLS. By Lemma A.7, we only need to show $\sup_K \frac{|\hat{S}_\xi(K) - S_\xi(K)|}{S_\xi(K)} \xrightarrow{p} 0$. $\frac{|\hat{S}_\xi(K) - S_\xi(K)|}{S_\xi(K)} \leq \frac{1}{n}\frac{A}{S_\xi(K)} + \hat{\sigma}_\epsilon^2\frac{|\tilde{R}_\xi(K) - R_\xi(K)|}{S_\xi(K)} + |\hat{\sigma}_\epsilon^2 - \sigma_\epsilon^2|\frac{R_\xi(K)}{S_\xi(K)}$, where $A = |\hat{\Omega}_{1\xi} - \Omega_{1\xi}|$ for 2SLS and $A = |\hat{\Pi}_\xi - \Pi_\xi|$ for C2SLS. Let $\tilde{G}_n = G_n(\tilde{\lambda}_n)$. As $\tilde{G}_n - G_n = (\tilde{\lambda}_n - \lambda_0)\tilde{G}_n G_n$, we have $\mathrm{tr}(\hat{M} - M) = (\tilde{\lambda}_n - \lambda_0)\mathrm{tr}(P\tilde{G}_n G_n) = o(K)$ and $\mathrm{tr}(P\tilde{G}_n^2 - PG_n^2) = \mathrm{tr}(P\tilde{G}_n^2 - P\tilde{G}_n G_n + P\tilde{G}_n G_n - PG_n^2) = (\tilde{\lambda}_n - \lambda_0)\mathrm{tr}(P\tilde{G}_n^2 G_n) + (\tilde{\lambda}_n - \lambda_0)\mathrm{tr}(P\tilde{G}_n G_n^2) = o(K)$. Similarly, $\mathrm{tr}(\hat{M}^2 - M^2) = o(K)$ and $\mathrm{tr}(\hat{M}'\hat{M} - M'M) = o(K)$. It follows that $|\hat{\Omega}_{1\xi} - \Omega_{1\xi}| = o(K^2)$ and $|\hat{\Pi}_\xi - \Pi_\xi| = o(K)$. For 2SLS, $|nS_\xi(K)| \geq c \cdot K^2$ by Assumption 6 (ii), which implies $\frac{1}{n}|\hat{\Omega}_{1\xi} - \Omega_{1\xi}|/S_\xi(K) = o(1)$, uniformly in K. On the other hand, for C2SLS, $|nS_\xi(K)| \geq c \cdot K$ by Assumption 6 (iii), which implies $\frac{1}{n}|\hat{\Pi}_\xi - \Pi_\xi|/S_\xi(K) = o(1)$. As $R_\xi(K)/S_\xi(K) = O(1)$ uniformly in K for both 2SLS and C2SLS because $0 < R_\xi(K)/S_\xi(K) < 1$ for all K, it remains to show that $\sup_K \frac{|\tilde{R}_\xi(K) - R_\xi(K)|}{S_\xi(K)} \xrightarrow{p} 0$. As $|nS_\xi(K)| \geq c(K + n\Delta_K)$ for 2SLS and C2SLS, $\sup_K \frac{|\tilde{R}_\xi(K) - R_\xi(K)|}{S_\xi(K)} \xrightarrow{p} 0$ can be proved by the same argument for both estimators as follows.

For subsequent analysis, it is useful to consider a more general form, $\sup_K \frac{|\tilde{R}_*(K) - R_*(K)|}{S_\xi(K)}$, where $\tilde{R}_*(K) = \xi_{1n}'\tilde{R}(K)\xi_{2n}$ and $R_*(K) = \xi_{1n}'R(K)\xi_{2n}$ for any arbitrary (random) vectors ξ_{1n} and ξ_{2n} which converge (in probability) to well defined limits. As $\tilde{R}(K) - R(K) = \frac{1}{n}[F_n'(I_n - P)V_n]^s - \frac{1}{n}(V_n'PV_n - \Omega_2) + \frac{1}{n}(\hat{\Omega}_2 - \Omega_2)$, we have $\tilde{R}_*(K) - R_*(K) = \frac{1}{n}\xi_{1n}'[F_n'(I_n - P)V_n]^s\xi_{2n} - \frac{1}{n}\xi_{1n}'(V_n'PV_n - \Omega_2)\xi_{2n} + \frac{1}{n}\xi_{1n}'(\hat{\Omega}_2 - \Omega_2)\xi_{2n}$.

First, we show $\sup_K \frac{|\frac{1}{n}\xi_{1n}'F_n'(I_n - P)V_n\xi_{2n}|}{S_\xi(K)} \xrightarrow{p} 0$, where $V_n = G_n\bar{U}_n e_1' + U_n(0_{m\times 1}, I_m)$. Given any $\alpha > 0$ and positive integer r, by Chebyshev's inequality, we have $\Pr(\sup_K \frac{|\frac{1}{n}\xi_{1n}'F_n'(I_n - P)G_n\bar{U}_n e_1'\xi_{2n}|}{S_\xi(K)} > \alpha) \leq \sum_K \frac{(e_1'\xi_{2n})^{2r}\mathrm{E}[\xi_{1n}'F_n'(I_n - P)G_n\bar{U}_n]^{2r}}{\alpha^{2r}[nS_\xi(K)]^{2r}}$. Let $\lambda_{G,\max}$ be the largest eigenvalue of $G_n G_n'$. By Theorem 2 of Whittle (1960), for positive constants c_1, c_2 and c, $\mathrm{E}[\xi_{1n}'F_n'(I_n - P)G_n\bar{U}_n]^{2r} \leq c_1[\xi_{1n}'F_n'(I_n - P)G_n G_n'(I_n - P)F_n\xi_{1n}]^r \leq c_1[\lambda_{G,\max}\xi_{1n}'F_n'(I_n - P)F_n\xi_{1n}]^r = O(n\Delta_K)^r \leq c_2[nS_\xi(K)]^r$. Hence, for $r = 1$, $\Pr(\sup_K \frac{|\xi_{1n}'F_n'(I_n - P)G_n\bar{U}_n e_1'\xi_{2n}|}{nS_\xi(K)} > \alpha) \leq c \cdot \alpha^{-2r}\sum_K [nS_\xi(K)]^{-r} \to 0$ by Assumption 7. Similarly, we have $\sup_K \frac{|\xi_{1n}'F_n'(I_n - P)U_n(0_{m\times 1}, I_m)\xi_{2n}|}{nS_\xi(K)} \xrightarrow{p} 0$. Thus, $\sup_K \frac{|\xi_{1n}'F_n'(I_n - P)V_n\xi_{2n}|}{nS_\xi(K)} \xrightarrow{p} 0$.

Next, we show $\sup_K \frac{|\xi_{1n}'(V_n'PV_n - \Omega_2)\xi_{2n}|}{nS_\xi(K)} \xrightarrow{p} 0$. Given any $\alpha > 0$ and positive integer r, by Chebyshev's inequality, we have

$$\Pr(\sup_K \frac{|\frac{1}{n}\xi_{1n}' e_1[\bar{U}_n' G_n' P G_n \bar{U}_n - \mathrm{E}(\bar{U}_n' G_n' P G_n \bar{U}_n)] e_1' \xi_{2n}|}{S_\xi(K)} > \alpha)$$
$$\leq \sum_K \frac{(\xi_{1n}' e_1)^{2r} (e_1' \xi_{2n})^{2r} \mathrm{E}[\bar{U}_n' G_n' P G_n \bar{U}_n - \mathrm{E}(\bar{U}_n' G_n' P G_n \bar{U}_n)]^{2r}}{\alpha^{2r}[n S_\xi(K)]^{2r}}.$$

For some positive constants c_3, c_4, $\mathrm{E}[\bar{U}_n' G_n' P G_n \bar{U}_n - \mathrm{E}(\bar{U}_n' G_n' P G_n \bar{U}_n)]^{2r} \leq c_3 \mathrm{tr}[(G_n' P G_n)^2]^r = O(K^r) \leq c_4[n S_\xi(K)]^r$, where the first inequality follows by Theorem 2 of Whittle (1960), the equality follows by Lemma A.2 (ii), and the last inequality follows as $|n S_\xi(K)| \geq c \cdot K$. Hence, for $r = 1$, $\Pr(\sup_K \frac{|\frac{1}{n}\xi_{1n}' e_1[\bar{U}_n' G_n' P G_n \bar{U}_n - \mathrm{E}(\bar{U}_n' G_n' P G_n \bar{U}_n)] e_1' \xi_{2n}|}{S_\xi(K)} > \alpha) \leq c \cdot \alpha^{-2r} \sum_K [n S_\xi(K)]^{-r} \to 0$. On the other hand, let $U_{1n} = U_n(0_{m\times 1}, I_m)\xi_{1n}$ and $U_{2n} = U_n(0_{m\times 1}, I_m)\xi_{2n}$. Given any $\alpha > 0$, by Chebyshev's inequality, we have $\Pr(\sup_K \frac{|U_{1n}' P U_{2n} - \mathrm{E}(U_{1n}' P U_{2n})|}{n S_\xi(K)} > \alpha) \leq \sum_K \frac{\mathrm{E}[U_{1n}' P U_{2n} - \mathrm{E}(U_{1n}' P U_{2n})]^2}{\alpha^2 [n S_\xi(K)]^2}$. By an argument similar to that in the proof of Lemma A.4 (iv), $\mathrm{E}[U_{1n}' P U_{2n} - \mathrm{E}(U_{1n}' P U_{2n})]^2 = O(K) \leq c \cdot n S_\xi(K)$. Hence, $\Pr(\sup_K \frac{|U_{1n}' P U_{2n} - \mathrm{E}(U_{1n}' P U_{2n})|}{n S_\xi(K)} > \alpha) \leq c \cdot \alpha^{-2} \sum_K [n S_\xi(K)]^{-1} \to 0$. Similarly, $\Pr(\sup_K \frac{|\frac{1}{n}\xi_{1n}' e_1[\bar{U}_n' G_n' P U_{2n} - \mathrm{E}(\bar{U}_n' G_n' P U_{2n})]|}{S_\xi(K)} > \alpha) \to 0$.

Lastly, as $\frac{1}{n}|\xi_{1n}'(\hat{\Omega}_2 - \Omega_2)\xi_{2n}| = o(K/n)$ and $|n S_\xi(K)| \geq c \cdot K$, we have $\sup_K \frac{|\xi_{1n}'(\hat{\Omega}_2 - \Omega_2)\xi_{2n}|}{n S_\xi(K)} \xrightarrow{p} 0$. In summary, we have $\sup_K \frac{|\tilde{R}_*(K) - R_*(K)|}{S_\xi(K)} \xrightarrow{p} 0$. As ξ_{1n} and ξ_{2n} can be arbitrary vectors, we have shown, in particular, that each component of $\frac{\tilde{R}(K) - R(K)}{S_\xi(K)}$ converges to 0 uniformly in K.

Note that $\tilde{R}_\xi(K) - R_\xi(K) = \xi' \hat{H}_n^{-1} \tilde{R}(K)(\hat{H}_n^{-1} - H_n^{-1})\xi + \xi'(\hat{H}_n^{-1} - H_n^{-1})\tilde{R}(K) H_n^{-1}\xi + \frac{1}{n}\xi' H_n^{-1}[F_n'(I_n - P)V_n]^s H_n^{-1}\xi - \frac{1}{n}\xi' H_n^{-1}(V_n' P V_n - \Omega_2) H_n^{-1}\xi + \frac{1}{n}\xi' H_n^{-1}(\hat{\Omega}_2 - \Omega_2) H_n^{-1}\xi$, where $\frac{|\xi' \hat{H}_n^{-1} \tilde{R}(K)(\hat{H}_n^{-1} - H_n^{-1})\xi|}{S_\xi(K)} \leq \frac{|\xi' \hat{H}_n^{-1}[\tilde{R}(K) - R(K)](\hat{H}_n^{-1} - H_n^{-1})\xi|}{S_\xi(K)} + \frac{|\xi' \hat{H}_n^{-1} R(K)(\hat{H}_n^{-1} - H_n^{-1})\xi|}{S_\xi(K)}$. As $\hat{H}_n^{-1} - H_n^{-1} = o(1)$ and each component of $\frac{\tilde{R}(K) - R(K)}{S_\xi(K)}$ converges to 0 uniformly in K, we have $\sup_K \frac{|\xi' \hat{H}_n^{-1}[\tilde{R}(K) - R(K)](\hat{H}_n^{-1} - H_n^{-1})\xi|}{S_\xi(K)} \xrightarrow{p} 0$. On the other hand, as $\hat{H}_n^{-1} - H_n^{-1} = o(1)$ and $R(K)/S_\xi(K) = O(1)$ uniformly in K, we have $\sup_K \frac{|\xi' \hat{H}_n^{-1} R(K)(\hat{H}_n^{-1} - H_n^{-1})\xi|}{S_\xi(K)} \xrightarrow{p} 0$. Hence, $\sup_K \frac{|\xi' \hat{H}_n^{-1} \tilde{R}(K)(\hat{H}_n^{-1} - H_n^{-1})\xi|}{S_\xi(K)} \xrightarrow{p} 0$. Similarly, we have $\sup_K \frac{|\xi'(\hat{H}_n^{-1} - H_n^{-1})\tilde{R}(K) H_n^{-1}\xi|}{S_\xi(K)} \xrightarrow{p} 0$. Hence, $\sup_K \frac{|\tilde{R}_\xi(K) - R_\xi(K)|}{S_\xi(K)} \xrightarrow{p} 0$, and the desired result follows.

■

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