Technical Appendix to "Nonparametric Estimation of Large Auctions with Risk Averse Bidders"

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Proof of Lemma A.1. Consider the risk neutral bidding function $s_{RN,n}(\cdot)$ given by the differential equation

$$v - s_{RN,n}(v) = \frac{1}{n-1}\frac{F(v)}{f(v)}s'_{RN,n}(v).$$

Let $G_{RN,n}(\cdot)$ be the distribution function of the equilibrium bids $b = s_{RN,n}(v)$ and $g_{RN,n}(\cdot)$ be its density. As $G_{RN,n}(b) = F(v)$ and $g_{RN,n}(b) = f(v)/s'_{RN,n}(v)$, the corresponding inverse bidding function is $s^{-1}_{RN,n}(b) = b + \frac{1}{n-1}\frac{G_{RN,n}(b)}{g_{RN,n}(b)}$. Guerre, Perrigne and Vuong (2000) have shown $g_{RN,n}(b) \geq c_g > 0$ for all n. As $\lim_{n\to\infty} s_{RN,n}(v) = v$ by Proposition 1 in Fibich and Gavious (2010), we have $\lim_{n\to\infty} g_{RN,n}(s_{RN,n}(v)) = f(v)$ where $f(v) \geq c_f > 0$ by the definition of $\mathcal{F}_R$. It follows that $\lim_{n\to\infty} \sup |s^{-1}_{RN,n}(b) - b| = 0$ or, equivalently, $\lim_{n\to\infty} \sup |v - s_{RN,n}(v)| = 0$. As $0 \leq v - s_n(v) \leq v - s_{RN,n}(v)$ for all $v \in S(F)$ (Riley and Samuelson, 1981), the uniform equicontinuity of $s_n(\cdot)$ follows by the uniform convergence of $s_n(v)$ and the compactness of $S(F)$ (Rudin, 1976, Theorem 7.24). ■

Proof of Lemma A.2. It follows from Proposition 3 that

$$\tilde{G}_n(b) = \frac{1}{nL}\textstyle\sum_{i,l}\mathbf{1}(B_{il} \leq b) = \frac{1}{nL}\sum_{i,l}\mathbf{1}(G_n(B_{il}) \leq G_n(b)) = \frac{1}{nL}\sum_{i,l}\mathbf{1}(u_{il} \leq G_n(b)),$$

where $u_{il} = G_n(B_{il})$ is uniformly distributed on $[0,1]$ since $B_{il} \sim G_n(\cdot)$. Hence,

$$\begin{aligned} |\tilde{G}_n(b) - G_n(b)|_{0,S(G_n)} &= |\frac{1}{nL}\sum_{i,l}\mathbf{1}(u_{il} \leq G_n(b)) - G_n(b)|_{0,S(G_n)} \\ &= |\frac{1}{nL}\sum_{i,l}\mathbf{1}(u_{il} \leq u) - u|_{0,[0,1]} = O(1/r_G), \end{aligned}$$

where the last step holds because the empirical distribution of uniform distribution on $[0,1]$ (which does not depend on n) converges uniformly to the true distribution at the rate of r_G by the Chung-Smirnov theorem (Chung, 1949). ■

Proof of Lemma A.3. Following the proof of Lemma B.2 in Guerre, Perrigne and Vuong (2000), the proof is divided into three steps. The first step studies the uniform bias of $\tilde{g}_n(\cdot)$, the second step studies its uniform variance bound, and the last step establishes the exponential-type inequality.

Step 1 (Uniform Bias). $\mathrm{E}[\tilde{g}_n(b)] = \mathrm{E}[\frac{1}{nLh}\sum_{i,l} K(\frac{B_{il}-b}{h})] = \int K(u)\, g_n(b+hu)\, du$. Without loss of generality, suppose $u \geq 0$. Then for $b \in C_n$ and L sufficiently large, $\tilde{b} \in [b, b+hu] \subset C_n'$, where C_n' is a closed inner subset of $S(G_n)$. Since $g_n(\cdot)$ admits R continuous derivatives, a Taylor expansion gives $g_n(b+hu) - g_n(b) \leq hu g_n^{(1)}(b) + \cdots + \frac{(hu)^{R-1}}{(R-1)!} g_n^{(R-1)}(b) + \frac{|hu|^R}{R!} |g_n|_{R,C_n'}$. As $K(\cdot)$ is of order R, moments of order strictly smaller than R vanish. We have

$$|\mathrm{E}[\tilde{g}_n(b)] - g_n(b)|_{0,C_n} = \sup_{b\in C_n} |\int K(u)\,(g_n(b+hu) - g_n(b))\, du| \leq h^R M^R |g_n|_{R,C_n'},$$

where $M^R = \frac{1}{R!}\int |u|^R K(u)\, du$. It follows from the definition of r_g and h that

$$r_g\, |\mathrm{E}[\tilde{g}_n(b)] - g_n(b)|_{0,C_n} \leq \phi^R M^R |g_n|_{R,C_n'}. \tag{1}$$

Step 2 (Uniform Variance). Consider a density $g_n^*(b)$, such that $g_n^*(b) = g_n(b)$ if $b \in S(G_n)$ and $g_n^*(b) = 0$ otherwise. For $b \in C_\infty$, where $C_\infty = \lim_{n\to\infty} C_n$, $\mathrm{Var}[\tilde{g}_n(b)] = \mathrm{Var}[\frac{1}{nLh}\sum_{i,l} K(\frac{B_{il}-b}{h})] = \frac{1}{nLh^2}\mathrm{Var}[K(\frac{B-b}{h})] \leq \frac{1}{nLh^2}\mathrm{E}[K(\frac{B-b}{h})]^2 = \frac{1}{nLh}\int K^2(u)\, g_n^*(b+hu)\, du$. Let $Q = \int K^2(u)\, du$, we have

$$|\mathrm{Var}[\tilde{g}_n(b)]|_{0,C_\infty} \leq \frac{1}{nLh} Q\, |g_n^*|_{0,C_\infty} \leq \frac{1}{\phi r_g^2 \log(nL)} Q\, |g_n|_0. \tag{2}$$

Step 3 (Exponential-type Inequality). In this step, we establish the exponential-type inequality for the probability of deviation of $\tilde{g}_n(b) - g_n(b)$ in sup-norm over C_n. Let $e(c_1, c_2) = c_1 + 2c_2 |K|_1 + \phi^R M^R |g_n|_{R,C_n'}$, where c_1, c_2 are strictly positive constants. From the triangular inequality and (1), we have

$$\Pr\left[r_g\, |\tilde{g}_n(b) - g_n(b)|_{0,C_n} > e(c_1,c_2)\right] \leq \Pr\left[r_g\, |\tilde{g}_n(b) - \mathrm{E}[\tilde{g}_n(b)]|_{0,C_n} > e(c_1,c_2) - \phi^R M^R |g_n|_{R,C_n'}\right]. \tag{3}$$

Let $\tilde{g}_n(b) - \mathrm{E}[\tilde{g}_n(b)] = (1/nL)\sum_{j=1}^{nL} \zeta_{j,nL}(b)$, where $\zeta_{j,nL}(b) = \frac{1}{h}K(\frac{B_j-b}{h}) - \frac{1}{h}\mathrm{E}[K(\frac{B_j-b}{h})]$. By the triangular inequality we have $\left|r_g \zeta_{j,nL}\right| \leq \frac{2r_g}{h}|K|_0 = \frac{2nL}{\phi r_g \log(nL)}|K|_0$. As the $\zeta_{j,nL}$'s are independent zero-mean random variables, it follows from (2) that $\mathrm{Var}\left(r_g \zeta_{j,nL}\right) = nL r_g^2 \mathrm{Var}(\tilde{g}_n) \leq$

$\frac{nL}{\phi \log(nL)} Q \left| g_n \right|_0$. Hence, the Bernstein inequality gives

$$\Pr\left[r_g \left| \tilde{g}_n(b) - \mathrm{E}[\tilde{g}_n(b)] \right| > c_1\right] = \Pr[|\sum_{j=1}^{nL} r_g \zeta_{j,nL}(b)| > nLc_1] \le 2\exp[-\frac{\phi c_1^2 \log(nL)}{2Q \left|g_n\right|_0 + 4c_1 \left|K\right|_0 / (3r_g)}], \tag{4}$$

for any $b \in C_n$, c_1, n and L.

Note that $C_n \subset C_\infty$ for all $n < \infty$ and $C_\infty = C(V)$. Suppose C_∞ is covered by T intervals of the form $B_t \equiv B(b_t, \Delta) = \{b \in S(F) : b \in [b_t - \Delta, b_t + \Delta]\}$, where $b_t \in C_\infty$ and $\Delta > 0$. Consider a minimal covering (i.e. a covering with the smallest T) for C_∞ with the covering number denoted by $T(\Delta)$. For any $b \in B_t$, by the triangular inequality,

$$r_g \left| \tilde{g}_n(b) - \mathrm{E}[\tilde{g}_n(b)] \right| \le \sup_{1 \le t \le T(\Delta)} \sup_{b \in B_t} \left| \frac{r_g}{nL} \sum_{j=1}^{nL} [\zeta_{j,nL}(b_t) - \zeta_{j,nL}(b)] \right| + \sup_{1 \le t \le T(\Delta)} \left| \frac{r_g}{nL} \sum_{j=1}^{nL} \zeta_{j,nL}(b_t) \right|,$$

which implies that

$$\begin{aligned} & \Pr\left[r_g \sup\nolimits_{b \in C_n} \left| \tilde{g}_n(b) - \mathrm{E}[\tilde{g}_n(b)] \right| > e(c_1, c_2) - \phi^R M^R \left|g_n\right|_{R, C_n'}\right] \\ \le\ & \Pr\left[\sup\nolimits_{1 \le t \le T(\Delta)} \sup\nolimits_{b \in B_t} \left| \frac{r_g}{nL} \sum\nolimits_{j=1}^{nL} \left(\zeta_{j,nL}(b_t) - \zeta_{j,nL}(b)\right) \right| > e(c_1, c_2) - c_1 - \phi^R M^R \left|g_n\right|_{R, C_n'}\right] \\ & + \Pr\left[\sup\nolimits_{1 \le t \le T(\Delta)} \left| \frac{r_g}{nL} \sum\nolimits_{j=1}^{nL} \zeta_{j,nL}(b_t) \right| > c_1\right]. \end{aligned} \tag{5}$$

By the mean value theorem, $\left|\frac{1}{h} K(\frac{B - b_t}{h}) - \frac{1}{h} K(\frac{B - b}{h})\right| \le \frac{\Delta |K|_1}{h^2}$. Therefore, by the triangular inequality, $\left|\zeta_{j,nL}(b_t) - \zeta_{j,nL}(b)\right| \le \frac{\Delta |K|_1}{h^2} + \mathrm{E}[\frac{\Delta |K|_1}{h^2}] = \frac{2\Delta |K|_1}{h^2}$. Let $\Delta = c_2 h^2 / r_g$. It follows that $\sup_{1 \le t \le T(\Delta)} \sup_{b \in B_t} \left| \frac{r_g}{nL} \sum_{j=1}^{nL} \left(\zeta_{j,nL}(b_t) - \zeta_{j,nL}(b)\right) \right| \le \frac{2 r_g \Delta |K|_1}{h^2} = 2c_2 \left|K\right|_1$. Hence, by the definition of $e(c_1, c_2)$,

$$\begin{aligned} & \Pr\left[\sup\nolimits_{1 \le t \le T(\Delta)} \sup\nolimits_{b \in B_t} \left| \frac{r_g}{nL} \sum\nolimits_{j=1}^{nL} \left(\zeta_{j,nL}(b_t) - \zeta_{j,nL}(b)\right) \right| > e(c_1, c_2) - c_1 - \phi^R M^R \left|g_n\right|_{R, C_n'}\right] \\ =\ & \Pr\left[\sup\nolimits_{1 \le t \le T(\Delta)} \sup\nolimits_{b \in B_t} \left| \frac{r_g}{nL} \sum\nolimits_{j=1}^{nL} \left(\zeta_{j,nL}(b_t) - \zeta_{j,nL}(b)\right) \right| > 2c_2 \left|K\right|_1\right] = 0. \end{aligned} \tag{6}$$

Let $P(c_1,c_2) = 2T\left(c_2 h^2/r_g\right)\exp[-\frac{\phi c_1^2 \log(nL)}{2Q|g_n|_0+4c_1|K|_0/(3r_g)}]$. It follows from (3), (5), (6) and (4) that

$$\begin{aligned}
\Pr[r_g\,|\tilde{g}_n(b)-g_n(b)|_{0,C_n}>e(c_1,c_2)] &\leq \Pr[r_g\,|\tilde{g}_n(b)-\mathrm{E}[\tilde{g}_n(b)]|_{0,C_n}>e(c_1,c_2)-\phi^R M^R\,|g_n|_{R,C_n'}] \\
&\leq \Pr[r_g \sup_{1\leq t\leq T(\Delta)}|\tilde{g}_n(b_t)-\mathrm{E}[\tilde{g}_n(b_t)]|>c_1] \\
&\leq \textstyle\sum_{t=1}^{T(\Delta)}\Pr[r_g\,|\tilde{g}_n(b_t)-\mathrm{E}[\tilde{g}_n(b_t)]|>c_1] \\
&\leq P(c_1,c_2).
\end{aligned}$$

The covering number $T(\Delta)$ is of order Δ^{-1}. Hence $T\left(c_2 h^2/r_g\right) = O([nL/\log(nL)]^{(R+2)/(2R+1)})$. By taking c_1 sufficiently large, $P(c_1,c_2)$ converges as $nL\to\infty$. By Proposition 3, $e(c_1,c_2)=O(1)$. The desired result follows from the Borel-Cantelli Lemma. ■

References

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