Estimation of a Local-Aggregate Network Model with Sampled Networks*

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Abstract

This paper considers the estimation of a network model with sampled networks. Chandrasekhar and Lewis (2011) show that the estimation with sampled networks could be biased due to measurement error induced by sampling and propose a bias correction by restricting the estimation to sampled nodes to avoid measurement error in the regressors. However, measurement error may still exist in the instruments and thus induce the weak instrument problem when the sampling rate is low. For a local-aggregate model, we show that the instrument based on the outdegrees of sampled nodes is free of measurement error and thus remains informative even if the sampling rate is low. Simulation studies suggest that the 2SLS estimator with the proposed instrument works well when the sampling rate is low and the other instruments are weak.

JEL classification: C13, C21

Key words: social networks, local-average models, local-aggregate models, sampling of networks, weak instruments

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1 Introduction

In recent years, the literature on identification and estimation of social network models has progressed significantly (see Blume et al., 2011, for a recent survey). In his seminal work, Manski (1993) introduces a linear-in-means social interaction model with endogenous effects, contextual effects, and correlated effects. Manski shows that this model suffers from the *reflection problem* and the above-mentioned effects cannot be separately identified. Bramoullé et al. (2009) generalize Manski's linear-in-means model to a *local-average* network model, where the endogenous effect is represented by the *average* outcome of an agent's friends. They provide identification conditions for the local-average model and suggest using the characteristics of indirect friends as an instrument for the endogenous effect. Liu and Lee (2010) consider a *local-aggregate* network model where the endogenous effect is given by the *aggregate* outcome of the friends. They show that in the local-aggregate model, the Bonacich centrality (Bonacich, 1987) can be used as an additional instrument to achieve identification and improve estimation efficiency.

The above-mentioned papers assume that the outcomes, covariates and connections of the agents in a network can be fully observed, which may be unrealistic in some practical applications. Sojourner (2011) considers a linear-in-means model with missing data on covariates. He shows that random assignment of agents to peer groups can help to overcome the missing data problem. On the other hand, Chandrasekhar and Lewis (2011) consider the missing data problem on network connections. They show that the estimation of sampled networks could be biased due to the measurement error induced by sampling. They propose a simple bias correction by restricting the estimation to the sampled agents, whose friends are observed, to avoid measurement error in the regressors. However, measurement errors may still exist in the instruments. For the local-average model, the instrument based on the characteristics of indirect friends is less informative when the sampling rate is low and thus may induce the weak instrument problem. In this paper, we show that, for the local-aggregate model with sampled network data, the instrument based on the number of direct connections, which is the leading-order term of the Bonacich centrality, has no measurement error and thus remains informative even if the sampling rate is low.

The rest of the paper is organized as follows. Section 2 introduces basic concepts and notations. Section 3 discusses identification and estimation of the network model with sampled networks. Section 4 provides simulation evidence for the finite sample performance of the estimator. Section 5 concludes and generalizes the proposed estimator to estimate a network model with network fixed effects.

2 Networks and Sampling

A *network* is represented by a *graph* $G = (V, E)$ where V is a set of n *nodes* and E is a set of ordered pairs of nodes called *arcs*. An arc $e_{ij} = (i, j)$ is considered to be directed from i to j where j is called the *head* and i is called the *tail* of the arc.[1] We assume there is no arc that points to itself so that $e_{ii} \notin E$ for all $i \in V$.

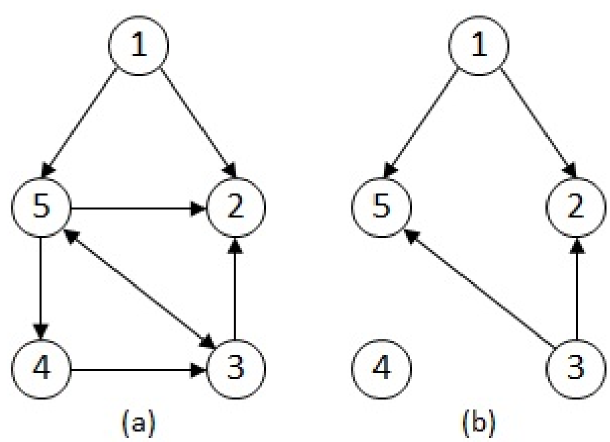

Figure 1: (a) the full network; (b) a star subgraph with $S = \{1, 2, 3\}$.

Denote the set of m randomly sampled nodes by S. Following Chandrasekhar and Lewis (2011), there are two different sampling schemes. In the first sampling scheme, the researcher surveys a set of m nodes and asks each node to nominate his/her friends among the other $m - 1$ nodes sampled. The sampled network $G^{|S} = (S, E^{|S})$, where $E^{|S} = \{e_{ij} | e_{ij} \in E, i \in$

[1] In this paper, we focus on the estimation of *directed* graphs. The estimators can be easily modified to estimate a *undirected* graph.

$S, j \in S\}$, is called the *induced subgraph*. In the second sampling scheme, the researcher has information of all the nodes in V and survey a set of m nodes and asks each node to nominate his/her friends from the list of the n nodes in V. The sampled network $G^S = (V, E^S)$, where $E^S = \{e_{ij} | e_{ij} \in E, i \in S, j \in V\}$, is called the *star subgraph*. In this paper, we focus on the estimation of a star subgraph. Figure 1 gives an example of a star subgraph.

3 Social Network Models with Sampled Networks

In a social network model, the connections in a network $G = (V, E)$ are represented by an $n \times n$ *adjacency matrix* $A = [a_{ij}]$ where $a_{ij} = 1$ if $e_{ij} \in E$ and $a_{ij} = 0$ otherwise. The model of the full network is given by

$$Y = \lambda_0 AY + \alpha_0 l_n + X\beta_0 + AX\gamma_0 + \epsilon. \tag{1}$$

Here, $Y = (y_1, \cdots, y_n)'$ where y_i is the observed outcome of the ith node. l_n is an $n \times 1$ vector of ones. $X = (x_1', \cdots, x_n')'$ where x_i is a $1 \times k$ vector of exogenous characteristics of the ith node. ϵ is an $n \times 1$ vector of i.i.d. innovations. According to Manski (1993), λ_0 captures the endogenous effect, where an agent's outcome may depend on the outcomes of his/her friends, and γ_0 captures the exogenous (contextual) effect, where an agent's outcome may depend on the exogenous characteristics of his/her friends. How to identify and estimate those two different effects has been a main interest for social interaction models.

If $(I_n - \lambda_0 A)$ is invertible, the reduced form equation of (1) is given by

$$Y = (I_n - \lambda_0 A)^{-1}(\alpha_0 l_n + X\beta_0 + AX\gamma_0 + \epsilon). \tag{2}$$

For identification and estimation of model (1), we need to find instruments for AY. As $(I_n - \lambda_0 A)^{-1} = I_n + \lambda_0 (I_n - \lambda_0 A)^{-1} A$, from (2), we have

$$\mathrm{E}(AY|A, X) = \alpha_0 A(I_n - \lambda_0 A)^{-1} l_n + AX\beta_0 + A^2(I_n - \lambda_0 A)^{-1} X(\lambda_0 \beta_0 + \gamma_0). \tag{3}$$

We will discuss the potential instruments implied by (3) in the following subsections.

Suppose we can observe (y_i, x_i) for all $i \in V$ and arcs e_{ij} if and only if $i \in S$ in the data. In other words, the sampled network can be represented by a star subgraph $G^S = (V, E^S)$. As argued by Chandrasekhar and Lewis (2011), this sampling scheme is quite common. For instance, consider the network data collected by Banerjee et al. (2011) from 43 villages in Karnataka, India, in order to study the diffusion of microfinance. The data includes a full census that collected demographic data on all households in the villages and a follow-up survey to a subsample of villagers asking them to nominate their social connections with other villagers. The resulting sampled network can be considered as a star subgraph.

Denote the corresponding adjacency matrix based on the sampled arcs by $A^* = [a^*_{ij}]$, where $a^*_{ij} = 1$ if $e_{ij} \in E^S$ and $a^*_{ij} = 0$ otherwise. The model with a sampled network is given by

$$Y = \lambda_0 A^* Y + \alpha_0 l_n + X\beta_0 + A^* X\gamma_0 + \epsilon. \tag{4}$$

For the estimation of model (4), we consider two different specifications of the network model, namely, the local-average model and the local-aggregate model.

3.1 The Local-Average Model

For network models, it is quite common to row-normalize the adjacency matrix A such that the sum of each row of A is unity. Let $d_i = \sum_{j=1}^n a_{ij}$ denote the *outdegree* of node i (i.e. the number of tails adjacent to a node). The row-normalized A is given by $\bar{A} = [\bar{a}_{ij}]$ where $\bar{a}_{ij} = a_{ij}/d_i$.[2] With a row-normalized adjacency matrix, the network model is

$$Y = \lambda_0 \bar{A} Y + \alpha_0 l_n + X\beta_0 + \bar{A} X\gamma_0 + \epsilon,$$

where $\bar{A}Y$ and $\bar{A}X$ represent the average outcome and average characteristics of the connections respectively. Therefore, we call this model the local-average model.

We assume that $|\lambda_0| < 1$ so that $(I_n - \lambda_0 \bar{A})^{-1} = \sum_{j=0}^{\infty} (\lambda_0 \bar{A})^j$. As $\bar{A} l_n = l_n$, we have

[2] For simplicity, we assume that $d_i > 0$ for all $i \in V$.

$\alpha_0\bar{A}(I_n - \lambda_0\bar{A})^{-1}l_n = \frac{\alpha_0}{1-\lambda_0}l_n$. Hence, it follows from (3) that

$$\mathrm{E}(\bar{A}Y|\bar{A}, X) = \frac{\alpha_0}{1-\lambda_0}l_n + \bar{A}X\beta_0 + (\bar{A}^2X + \lambda_0\bar{A}^3X + \cdots)(\lambda_0\beta_0 + \gamma_0).$$

If $\lambda_0\beta_0 + \gamma_0 = 0$, $\mathrm{E}(\bar{A}Y|\bar{A}, X)$ becomes a linear combination of l_n and $\bar{A}X$, and thus the local-average model cannot be identified. If $\lambda_0\beta_0 + \gamma_0 \neq 0$, then $\bar{A}^2X$ can be used as an instrument for $\bar{A}Y$ under the identification condition given by Bramoullé et al. (2009). Let $\bar{Z} = [\bar{A}Y, l_n, X, \bar{A}X]$, $Q_1 = [l_n, X, \bar{A}X, \bar{A}^2X]$, and $P_1 = Q_1(Q_1'Q_1)^{-1}Q_1'$. The 2SLS estimator of $\delta_0 = (\lambda_0, \alpha_0, \beta_0', \gamma_0')'$ is given by $\hat{\delta}_n = (\bar{Z}'P_1\bar{Z})^{-1}\bar{Z}'P_1Y$.

For a star subgraph, let $\bar{A}^*$ denote the row-normalized A^*. As $\bar{A}^*$ is misspecified, it introduces measurement errors to both regressors and instruments. Chandrasekhar and Lewis (2011) show that the 2SLS estimator for the local-average model with sampled networks is inconsistent because the measurement error in the instruments is correlated with that in the regressors. They propose a simple correction by estimating the model only with the sampled nodes. Let $\bar{a}_i$ ($\bar{a}_i^*$) denote the ith row of $\bar{A}$ ($\bar{A}^*$). As $\bar{a}_i^* = \bar{a}_i$ for $i \in S$, there is no measurement error in the regressors $[\bar{a}_i^*Y, 1, x_i, \bar{a}_i^*X]$ of a sampled node. Measurement error may still exist in the instruments, but they are uncorrelated with ϵ_i. Therefore, the 2SLS estimator is consistent.

Without loss of generality, suppose the first m nodes are sampled. The observed outcomes of the sampled nodes $Y^S = (y_1, \cdots, y_m)'$ are given by

$$Y^S = \lambda_0\bar{A}^SY + \alpha_0l_m + X^S\beta_0 + \bar{A}^SX\gamma_0 + \epsilon^S, \tag{5}$$

where $\bar{A}^S = (\bar{a}_1', \cdots, \bar{a}_m')$ is an $m \times n$ matrix consisting of the first m rows of $\bar{A}$, $X^S = (x_1', \cdots, x_m')'$ is an $m \times k$ matrix of exogenous characteristics of the sampled nodes, and ϵ^S is an $m \times 1$ vector of corresponding error terms. Let $\bar{Z}^S = [\bar{A}^SY, l_m, X^S, \bar{A}^SX]$, $Q_1^S = [l_m, X^S, \bar{A}^SX, \bar{A}^S\bar{A}^*X]$, and $P_1^S = Q_1^S(Q_1^{S\prime}Q_1^S)^{-1}Q_1^{S\prime}$. The bias-corrected 2SLS estimator proposed by Chandrasekhar and Lewis (2011) is given by $\hat{\delta}_n = (\bar{Z}^{S\prime}P_1^S\bar{Z}^S)^{-1}\bar{Z}^{S\prime}P_1^SY^S$.

However, the misspecified adjacency matrix $\bar{A}^*$ introduces measurement error to the instrument $\bar{A}^S\bar{A}^*X$ in Q_1^S. When the sampling rate m/n is low, the instrument is less informative. This can be seen from Figure 1. The instrument $\bar{A}^S\bar{A}^*X$ represents the characteristics of "second-order" connections. For the full network, following the arrows, nodes 3 and 4 are indirect connections of node 1 through node 5, and thus the characteristics of nodes 3 and 4 provide useful information for the identification of the endogenous effect. However, due to sampling, the arcs pointing from node 5 to nodes 3 and 4 are missing. Hence, the instrument based on the characteristics of friends' friends is not available in the sampled network.

3.2 The Local-Aggregate Model

Without row normalization of the adjacency matrix, AY and AX represent, respectively, the aggregate outcome and aggregate characteristics of the connections. Hence, we call model (1) the local-aggregate model. Let $||\cdot||_\infty$ denote the row sum matrix norm. If $\sup ||\lambda_0 A||_\infty < 1$, then $(I_n - \lambda_0 A)^{-1} = \sum_{j=0}^{\infty}(\lambda_0 A)^j$. It follows from (3) that

$$\mathrm{E}(AY|A,X) = \alpha_0(Al_n + \lambda_0 A^2 l_n + \cdots) + AX\beta_0 + (A^2X + \lambda_0 A^3 X + \cdots)(\lambda_0\beta_0 + \gamma_0).$$

The term $Al_n + \lambda_0 A^2 l_n + \cdots$ represents the Bonacich centrality (Bonacich, 1987), which is a sum of a node's direct and indirect connections discounted by λ_0. As pointed out by Liu and Lee (2010), the Bonacich centrality of a network in general is not proportional to l_n and thus provides information for identification.[3] Hence, unlike the local-average model, the local-aggregate model can be identified even if $\lambda_0\beta_0 + \gamma_0 = 0$. For the estimation, we use the vector of outdegrees Al_n, which is the leading order component of the Bonacich centrality, as an additional instrument. Let $Z = [AY, l_n, X, AX]$, $Q_2 = [l_n, X, AX, A^2X, Al_n]$, and $P_2 = Q_2(Q_2'Q_2)^{-1}Q_2'$. The 2SLS estimator of the local-aggregate model is given by $\hat{\delta}_n = (Z'P_2Z)^{-1}Z'P_2Y$.

For a star subgraph, the observed outcomes of the sampled nodes $Y^S = (y_1, \cdots, y_m)'$ are

[3] An exception would be the complete network where $A = l_n l_n' - I_n$. In this case, the Bonacich centrality of each node is the same.

given by

$$Y^S = \lambda_0 A^S Y + \alpha_0 l_m + X^S \beta_0 + A^S X \gamma_0 + \epsilon^S, \tag{6}$$

where A^S is an $m \times n$ matrix consisting of the first m rows of A. Let $Z^S = [A^S Y, l_m, X^S, A^S X]$, $Q_2^S = [l_m, X^S, A^S X, A^S A^* X, A^S l_n]$, and $P_2^S = Q_2^S (Q_2^{S\prime} Q_2^S)^{-1} Q_2^{S\prime}$. The bias-corrected 2SLS estimator for the sampled network is given by $\hat{\delta}_n = (Z^{S\prime} P_2^S Z^S)^{-1} Z^{S\prime} P_2^S Y^S$. For the sampled nodes, their outdegrees do not have measurement error and thus the instrument $A^S l_n$ is informative even if the sampling rate is low.

The usefulness of the additional instrument $A^S l_n$ can be seen in Figure 1. Although the star subgraph in Figure 1 (b) cannot be identified for a local-average model, it is identifiable for a local-aggregate model. For the sampled nodes, the outdegrees of nodes 1 and 3 are two and the outdegree of node 2 is zero. The variation in outdegrees provides an additional channel for identification.

4 Simulation Evidence

To investigate the finite sample performance of the estimators, we conduct a limited simulation experiment for the local-aggregate model (1). We consider a network with $n = 500$ nodes. The sampled network is a star subgraph with m sampled nodes, $m \in \{100, 200, 300, 400, 500\}$, with the corresponding sampling rates $\psi \in \{20\%, 40\%, 60\%, 80\%, 100\%\}$. The number of simulation repetitions is 50000. For each repetition, node i, $i = 1, \cdots, n$, is assigned an outdegree $d_i \in \{1, 2, 3\}$ with equal probability and randomly nominates d_i friends. The (i, j)th element of adjacency matrix A is one if node i nominates j as a friend and zero otherwise. X and ϵ are generated from $N(0, I_n)$ and $N(0, \sigma^2 I_n)$ respectively. The data are generated with $\lambda_0 = 0.1$, $\alpha_0 = 1$, $\beta_0 = 0.5$, $\gamma_0 = 0.1$, and $\sigma^2 = 2$.

We consider three estimators, (i) the IV estimator based on the n observations and the misspecified adjacency matrix, $\hat{\delta}_{IV-1} = (Q_1^{*\prime} Z^*)^{-1} Q_1^{*\prime} Y$, where $Z^* = [A^* Y, l_n, X, A^* X]$ and $Q_1^* = [l_n, X, A^* X, (A^*)^2 X]$; (ii) the IV estimator based on the m sampled nodes, $\hat{\delta}_{IV-2} = (Q_1^{S\prime} Z^S)^{-1} Q_1^{S\prime} Y^S$, where $Z^S = [A^S Y, l_m, X^S, A^S X]$ and $Q_1^S = [l_m, X^S, A^S X, A^S A^* X]$; and

(iii) the 2SLS estimator based on the m sampled nodes and an additional instrument on outdegrees, $\hat{\delta}_{2SLS} = (Z^{S\prime}P_2^S Z^S)^{-1}Z^{S\prime}P_2^S Y^S$, where $Q_2^S = [Q_1^S, A^S l_n]$ and $P_2^S = Q_2^S(Q_2^{S\prime}Q_2^S)^{-1}Q_2^{S\prime}$. We report the mean, standard deviation (SD), and root mean square error (RMSE) of the empirical distribution of the estimates. The first two estimators are unstable with extremely large SD when the sampling rate is low. To reduce the impact of outliers, we drop the estimates below the 1st percentile and above the 99th percentile of the empirical distribution when calculating the mean, SD and RMSE. As a measure for the quality of the instruments, we also report the first-stage partial F-statistics of each estimator.

The simulations results are reported in Table 1. The IV estimator based on the misspecified adjacency matrix $\hat{\delta}_{IV-1}$ is severely biased for a moderate sampling rate ($\psi \leq 60\%$). When the sampling rate is low ($\psi = 20\%$), $\hat{\lambda}_{IV-1}$ is downwards biased by 58%, $\hat{\alpha}_{IV-1}$ is upwards biased by 22.8% and $\hat{\gamma}_{IV-1}$ is upwards biased by 28%. The IV estimator $\hat{\delta}_{IV-2}$ based on the sampled nodes and IV matrix Q_1^S is also biased when the sampling rate is low ($\psi = 20\%$), $\hat{\lambda}_{IV-2}$ is upwards biased by 26% and $\hat{\gamma}_{IV-2}$ is downwards biased by 12%. The bias reduces as the sampling rate increases to 40%. The IV estimators $\hat{\delta}_{IV-1}$ and $\hat{\delta}_{IV-2}$ are unstable with extremely large SDs when the sampling rate is low. The SDs remain quite large for a moderate sampling rate. As indicated by the first-stage partial F-statistics, the instrument $A^S A^* X$ in Q_1^S is not very informative in the simulation experiment. The 2SLS estimator $\hat{\delta}_{2SLS}$ with an additional instrument of outdegrees $A^S l_n$ is essentially unbiased for all sampling rates considered. It also has much smaller SDs than the other two estimators.

5 Concluding Remarks

In this paper, we consider the estimation of a network model with sampled networks. Chandrasekhar and Lewis (2011) show that the estimates with sampled networks could be biased due to measurement error induced by sampling and propose a simple bias correction by restricting the estimation to sampled nodes to avoid measurement error in the regressors. However, measurement error may still exist in the instruments and thus induce the weak instrument problem when the sampling rate is low. For the local-aggregate model, we show that the instrument based on outdegrees is free of measurement error and thus remains informative even if the sampling rate is low. Simulation studies show that the 2SLS estimator with the additional instrument based on outdegrees works well in finite samples when the sampling rate is low and other instruments are weak.

The 2SLS estimator can be easily generalized to estimate a model with R networks. Suppose the observed outcomes of the m_r sampled nodes in network r are given by (6) for $r = 1, \cdots, R$. Let $\text{diag}\{B_k\}$ denote a 'generalized' block diagonal matrix in which the diagonal blocks are $m_k \times n_k$ matrices B_k's. For the R sampled networks, we have

$$Y^S = \lambda_0 A^S Y + L^S \alpha + X^S \beta_0 + A^S X \gamma_0 + \epsilon^S,$$

where $Y^S = (Y_1^{S\prime}, \cdots, Y_R^{S\prime})'$, $X^S = (X_1^{S\prime}, \cdots, X_R^{S\prime})'$, $\epsilon^S = (\epsilon_1^{S\prime}, \cdots, \epsilon_R^{S\prime})'$, $A^S = \text{diag}\{A_r^S\}_{r=1}^R$ and $L^S = \text{diag}\{l_{m_r}\}_{r=1}^R$. The vector of unknown parameters $\alpha = (\alpha_1, \cdots, \alpha_R)'$ captures network-specific fixed effects. To avoid the incidental parameter problem induced by α, we conduct a *within* transformation with a projector $J^S = \text{diag}\{J_r^S\}_{r=1}^R$ where $J_r^S = I_{m_r} - \frac{1}{m_r} l_{m_r} l_{m_r}'$. As $J^S L^S = 0$, the transformed model is

$$J^S Y^S = \lambda_0 J^S A^S Y + J^S X^S \beta_0 + J^S A^S X \gamma_0 + J^S \epsilon^S.$$

Like for the case with a single sampled network, we can use the characteristics of indirect friends $A^S A^* X$ and the number of direct friends $A^S l_n$ as instruments. Let $Z^S = J^S[A^S Y, X^S, A^S X]$, $Q^S = J^S[X^S, A^S X, A^S A^* X, A^S l_n]$, and $P^S = Q^S(Q^{S\prime} Q^S)^{-1} Q^{S\prime}$. The 2SLS estimator of $\theta_0 = (\lambda_0, \beta_0', \gamma_0')'$ is given by $\hat{\theta}_{2sls} = (Z^{S\prime} P^S Z^S)^{-1} Z^{S\prime} P^S J^S Y^S$.[4]

[4] Note that the 2SLS estimator $\hat{\theta}_{2sls}$ is consistent but may not be efficient. To improve estimation efficiency, consider the many-instrument 2SLS estimator in Liu and Lee (2010).

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Table 1: Estimation of a Sampled Network

	$\lambda_0 = 0.1$	$\alpha_0 = 1$	$\beta_0 = 0.5$	$\gamma_0 = 0.1$	F-$stat$
	sampling rate: 20%				
$\hat{\delta}_{IV-1}$	0.042(1.504)[1.505]	1.228(0.738)[0.773]	0.499(0.098)[0.098]	0.128(0.714)[0.714]	8.708
$\hat{\delta}_{IV-2}$	0.126(2.189)[2.189]	0.933(5.440)[5.441]	0.499(0.427)[0.427]	0.088(1.079)[1.079]	8.608
$\hat{\delta}_{2SLS}$	0.099(0.134)[0.134]	1.002(0.357)[0.357]	0.500(0.138)[0.138]	0.100(0.119)[0.119]	73.312
	sampling rate: 40%				
$\hat{\delta}_{IV-1}$	0.077(1.288)[1.288]	1.172(1.270)[1.281]	0.500(0.105)[0.105]	0.113(0.585)[0.585]	6.622
$\hat{\delta}_{IV-2}$	0.095(1.472)[1.472]	1.014(3.670)[3.670]	0.501(0.215)[0.215]	0.103(0.718)[0.718]	7.279
$\hat{\delta}_{2SLS}$	0.100(0.092)[0.092]	1.000(0.246)[0.246]	0.500(0.095)[0.095]	0.100(0.082)[0.082]	70.086
	sampling rate: 60%				
$\hat{\delta}_{IV-1}$	0.085(0.959)[0.959]	1.123(1.415)[1.421]	0.500(0.097)[0.097]	0.108(0.428)[0.428]	6.963
$\hat{\delta}_{IV-2}$	0.107(0.893)[0.893]	0.982(2.205)[2.205]	0.499(0.123)[0.123]	0.097(0.431)[0.431]	8.103
$\hat{\delta}_{2SLS}$	0.100(0.074)[0.074]	0.999(0.197)[0.197]	0.500(0.077)[0.077]	0.100(0.066)[0.066]	70.023
	sampling rate: 80%				
$\hat{\delta}_{IV-1}$	0.101(0.627)[0.627]	1.047(1.234)[1.235]	0.500(0.081)[0.081]	0.100(0.289)[0.289]	8.283
$\hat{\delta}_{IV-2}$	0.104(0.521)[0.521]	0.989(1.294)[1.294]	0.500(0.085)[0.085]	0.098(0.254)[0.254]	9.407
$\hat{\delta}_{2SLS}$	0.100(0.063)[0.063]	0.999(0.170)[0.170]	0.500(0.067)[0.067]	0.100(0.057)[0.057]	70.646
	sampling rate: 100%				
$\hat{\delta}_{IV-1}$	0.102(0.333)[0.333]	0.995(0.827)[0.827]	0.500(0.069)[0.069]	0.100(0.166)[0.166]	10.964
$\hat{\delta}_{IV-2}$	0.102(0.333)[0.333]	0.995(0.827)[0.827]	0.500(0.069)[0.069]	0.100(0.166)[0.166]	10.964
$\hat{\delta}_{2SLS}$	0.100(0.056)[0.056]	0.999(0.151)[0.151]	0.500(0.060)[0.060]	0.100(0.051)[0.051]	71.597

Mean(SD)[RMSE]