Online Supplement to

"Identification of Peer Effects via a Root Estimator"

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A Proof of Consistency and Asymptotic Normality

Proof of Proposition 1. Let $\boldsymbol{\delta}_0 = (\boldsymbol{\beta}_0', \boldsymbol{\gamma}_0')'$, $\boldsymbol{\Sigma} = \text{diag}_{g=1}^n\{\boldsymbol{\Sigma}_g\}$, $\mathbf{S} = \mathbf{I}_{mn} - \lambda_0 \mathbf{A}$, and $\mathbf{G} = \mathbf{A}\mathbf{S}^{-1}$. Note that $\mathbf{A}$, $\mathbf{S}$ and $\mathbf{G}$ are symmetric. The root estimator $\widehat{\lambda}$ is given by

$$\widehat{\lambda} = \frac{b_n - \sqrt{b_n^2 - a_n c_n}}{a_n} \tag{A.1}$$

where

$$\begin{aligned}
a_n &= n^{-1}\mathbf{y}'\mathbf{AMAMAy} = n^{-1}(\mathbf{Z}\boldsymbol{\delta}_0 + \mathbf{u})'\mathbf{GMAMG}(\mathbf{Z}\boldsymbol{\delta}_0 + \mathbf{u}) \\
b_n &= n^{-1}\mathbf{y}'\mathbf{AMAMy} = n^{-1}(\mathbf{Z}\boldsymbol{\delta}_0 + \mathbf{u})'\mathbf{GMAMS}^{-1}(\mathbf{Z}\boldsymbol{\delta}_0 + \mathbf{u}) \\
c_n &= n^{-1}\mathbf{y}'\mathbf{MAMy} = n^{-1}(\mathbf{Z}\boldsymbol{\delta}_0 + \mathbf{u})'\mathbf{S}^{-1}\mathbf{MAMS}^{-1}(\mathbf{Z}\boldsymbol{\delta}_0 + \mathbf{u}).
\end{aligned}$$

As $\mathbf{A}$, $\mathbf{G}$, and $\mathbf{M}$ are bounded in both row and column sum norms by Lemma 1 in Jin and Lee (2012), it follows by Lemmata 3 and 4 in Jin and Lee (2012) that $a_n = \text{E}(a_n) + o_p(1)$, $b_n = \text{E}(b_n) + o_p(1)$, and $c_n = \mathrm{E}(c_n) + o_p(1)$, where

$$\begin{aligned}
\mathrm{E}(a_n) &= n^{-1}\boldsymbol{\delta}_0'\mathbf{Z}'\mathbf{GMAMGZ}\boldsymbol{\delta}_0 + n^{-1}\mathrm{tr}(\mathbf{GMAMG\Sigma}) \\
\mathrm{E}(b_n) &= n^{-1}\boldsymbol{\delta}_0'\mathbf{Z}'\mathbf{GMAMS}^{-1}\mathbf{Z}\boldsymbol{\delta}_0 + n^{-1}\mathrm{tr}(\mathbf{GMAMS}^{-1}\mathbf{\Sigma}) \\
\mathrm{E}(c_n) &= n^{-1}\boldsymbol{\delta}_0'\mathbf{Z}'\mathbf{S}^{-1}\mathbf{MAMS}^{-1}\mathbf{Z}\boldsymbol{\delta}_0 + n^{-1}\mathrm{tr}(\mathbf{S}^{-1}\mathbf{MAMS}^{-1}\mathbf{\Sigma}).
\end{aligned}$$

As $\mathbf{S}^{-1} = \mathbf{I}_{mn} + \lambda_0\mathbf{G}$, $\mathbf{MZ} = \mathbf{0}$, and $\mathrm{tr}(\mathbf{BM}) = \mathrm{tr}(\mathbf{B}) + O(1)$ for any $n \times n$ matrix $\mathbf{B}$ that is bounded in both row and column sum norms (see Lemma 1 in Jin and Lee, 2012), we have

$$\begin{aligned}
\mathrm{E}(a_n) &= d_n + n^{-1}\mathrm{tr}(\mathbf{GAG\Sigma}) + o(1), \\
\mathrm{E}(b_n) &= \lambda_0[d_n + n^{-1}\mathrm{tr}(\mathbf{GAG\Sigma})] + n^{-1}\mathrm{tr}(\mathbf{GA\Sigma}) + o(1), \\
\mathrm{E}(c_n) &= \lambda_0^2[d_n + n^{-1}\mathrm{tr}(\mathbf{GAG\Sigma})] + 2\lambda_0 n^{-1}\mathrm{tr}(\mathbf{GA\Sigma}) + o(1),
\end{aligned}$$

where $d_n = n^{-1}\boldsymbol{\delta}_0'\mathbf{Z}'\mathbf{GMAMGZ}\boldsymbol{\delta}_0$. By the continuous mapping theorem,

$$\begin{aligned}
& b_n^2 - a_n c_n \\
= & \{\lambda_0[d_n + n^{-1}\mathrm{tr}(\mathbf{GAG\Sigma})] + n^{-1}\mathrm{tr}(\mathbf{GA\Sigma})\}^2 \\
& -[d_n + n^{-1}\mathrm{tr}(\mathbf{GAG\Sigma})]\{\lambda_0^2[d_n + n^{-1}\mathrm{tr}(\mathbf{GAG\Sigma})] + 2\lambda_0 n^{-1}\mathrm{tr}(\mathbf{GA\Sigma})\} + o_p(1) \\
= & n^{-2}[\mathrm{tr}(\mathbf{GA\Sigma})]^2 + o_p(1).
\end{aligned}$$

As $\mathbf{A}^k = (1-m)^{-k}\mathbf{I}_{mn} - [\sum_{j=1}^{k}(1-m)^{-j}](\mathbf{I}_n \otimes \boldsymbol{\iota}_m\boldsymbol{\iota}_m')$, we have

$$\mathbf{G} = \textstyle\sum_{k=1}^{\infty} \lambda_0^{k-1}\mathbf{A}^k = (1-\lambda_0)^{-1}(1-m-\lambda_0)^{-1}[(1-\lambda_0)\mathbf{I}_{mn} - (\mathbf{I}_n \otimes \boldsymbol{\iota}_m\boldsymbol{\iota}_m')].$$

Thus, under Assumption A1, $\mathrm{tr}(\mathbf{GA\Sigma}) = (1-\lambda_0)^{-1}(m+\lambda_0-1)^{-1}\sum_{g=1}^{n}\sum_{i=1}^{m}\sigma_{i,g}^2 > 0$, and $\mathrm{plim}_{n\to\infty} a_n = d_n + [m - 2(1-\lambda_0)](1-\lambda_0)^{-2}(1-m-\lambda_0)^{-2}n^{-1}\sum_{g=1}^{n}\sum_{i=1}^{m}\sigma_{i,g}^2 \neq 0$. Therefore,

$$\widehat{\lambda} = \frac{\lambda_0[d_n + n^{-1}\mathrm{tr}(\mathbf{GAG\Sigma})] + n^{-1}\mathrm{tr}(\mathbf{GA\Sigma}) - \sqrt{n^{-2}[\mathrm{tr}(\mathbf{GA\Sigma})]^2}}{d_n + n^{-1}\mathrm{tr}(\mathbf{GAG\Sigma})} + o_p(1) = \lambda_0 + o_p(1).$$

Let $\widehat{\boldsymbol{\delta}} = (\widehat{\boldsymbol{\beta}}', \widehat{\boldsymbol{\gamma}}')'$. Then,

$$\begin{aligned}
\widehat{\boldsymbol{\delta}} &= (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'(\mathbf{y} - \widehat{\lambda}\mathbf{A}\mathbf{y}) \\
&= (\mathbf{Z}'\mathbf{Z})^{-1}\mathbf{Z}'[\mathbf{Z}\boldsymbol{\delta}_0 + \mathbf{u} - (\widehat{\lambda} - \lambda_0)\mathbf{A}\mathbf{y}] \\
&= \boldsymbol{\delta}_0 + (n^{-1}\mathbf{Z}'\mathbf{Z})^{-1}n^{-1}\mathbf{Z}'\mathbf{u} - (\widehat{\lambda} - \lambda_0)(n^{-1}\mathbf{Z}'\mathbf{Z})^{-1}n^{-1}\mathbf{Z}'\mathbf{G}(\mathbf{Z}\boldsymbol{\delta}_0 + \mathbf{u}) \\
&= \boldsymbol{\delta}_0 + o_p(1),
\end{aligned}$$

where the last equality follows by Lemma 4 in Jin and Lee (2012). Therefore, the root estimator $\widehat{\boldsymbol{\theta}} = (\widehat{\lambda}, \widehat{\boldsymbol{\delta}}')'$ is consistent.

Let

$$\mathbf{f}(\boldsymbol{\theta}) = \begin{bmatrix} \mathbf{Z}'\mathbf{u}(\boldsymbol{\theta}) \\ \mathbf{u}(\boldsymbol{\theta})'\mathbf{A}\mathbf{u}(\boldsymbol{\theta}) \end{bmatrix},$$

where $\mathbf{u}(\boldsymbol{\theta}) = \mathbf{y} - \lambda\mathbf{A}\mathbf{y} - \mathbf{X}\boldsymbol{\beta} - \mathbf{A}\mathbf{X}\boldsymbol{\gamma}$. The root estimator $\widehat{\boldsymbol{\theta}}$ satisfies $n^{-1}\mathbf{f}(\widehat{\boldsymbol{\theta}}) = \mathbf{0}$. By the mean value theorem,

$$\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) = -\left[n^{-1}\frac{\partial\mathbf{f}(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'}\right]^{-1} n^{-1/2}\mathbf{f}(\boldsymbol{\theta}_0) + o_p(1).$$

It follows by Lemmata 4 and 5 in Jin and Lee (2012) that $\sqrt{n}(\widehat{\boldsymbol{\theta}} - \boldsymbol{\theta}_0) \xrightarrow{d} N(\mathbf{0}, \text{plim}_{n\to\infty}\mathbf{D}^{-1}\boldsymbol{\Omega}\mathbf{D}'^{-1})$ where

$$\boldsymbol{\Omega} = \text{Var}(n^{-1/2}\mathbf{f}(\boldsymbol{\theta}_0)) = n^{-1}\begin{bmatrix} \mathbf{Z}'\boldsymbol{\Sigma}\mathbf{Z} & \mathbf{0} \\ \mathbf{0} & 2\text{tr}(\boldsymbol{\Sigma}\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}) \end{bmatrix}$$

and

$$\mathbf{D} = -\text{E}(n^{-1}\frac{\partial\mathbf{f}(\boldsymbol{\theta}_0)}{\partial\boldsymbol{\theta}'}) = n^{-1}\begin{bmatrix} \mathbf{Z}'\mathbf{G}(\mathbf{X}\boldsymbol{\beta}_0 + \mathbf{A}\mathbf{X}\boldsymbol{\gamma}_0) & \mathbf{Z}'\mathbf{Z} \\ 2\text{tr}(\boldsymbol{\Sigma}\mathbf{A}\mathbf{G}) & \mathbf{0} \end{bmatrix}.$$

□

References

Jin, F. and Lee, L. F. (2012). Approximated likelihood and root estimators for spatial interaction in spatial autoregressive models, *Regional Science and Urban Economics* **42**: 446–458.