3.1 MOTIVATION

3.2 BELIEFS, SEQUENTIAL RATIONALITY, AND (WEAK) PERFECT BAYESIAN EQUILIBRIUM

In dynamic games of imperfect information, the notion of subgame perfection may not be much helpful. Consider the following example:

**Example 3-1.** Suppose an entrant \(E\) can choose among Out, In\(_1\), and In\(_2\). The incumbent \(I\) chooses between Fight and Accommodate if entry occurs, but \(I\) cannot distinguish between In\(_1\) and In\(_2\).

The game has two pure strategy Nash equilibria: (Out, Fight if entry occurs), and (In\(_1\), Accommodate if entry occurs). The first one again seems to be supported by empty threats: if entry indeed were to occur, it is always optimal for \(I\) to accommodate. But here subgame perfection would not eliminate this equilibrium. In fact, both equilibria are SPNE since the only subgame in the game is the game itself! In such situations, additional refinement on Nash equilibrium is needed. The basic idea is to require that a player’s strategy should be sequentially rational under some belief about what has happened in the game, and the belief should be reasonable in certain sense.
In the literature, there are different suggestions about what reasonable beliefs should be in different applications. It is generally agreed, however, that at the minimum beliefs should be consistent with strategies being played. When only this is required on beliefs, the solution concept is often called weak perfect Bayesian equilibrium (weak PBE). The term “weak” is dropped when some additional restrictions are imposed on beliefs, but sometimes people do not make such distinctions in the literature. That is, sometimes weak PBE is simply called PBE. We now formally define the notions of beliefs and sequential rationality under certain beliefs.

**Definition.** A system of beliefs $\mu$ in extensive form game $\Gamma_E$ is a specification of a probability $\mu(x) \in [0, 1]$ for each decision node $x$ in $\Gamma_E$ such that $\sum_{x \in H} \mu(x) = 1$ for all information sets $H$.

In Example 3-1, under one system of beliefs $I$ assigns equal probabilities to being at the node following $I_{n_1}$ and being at the node following $I_{n_2}$ if entry has occurred. Another belief could be to assign probability 1 and 0 to these two nodes respectively by $I$ should she is called to move. When an information set contains only one node, the player obviously should assigns probability one to being at this node if the information set is reached, and we usually omit this in describing the beliefs.

Now let $E[u_i | H, \mu, \sigma_i, \sigma_{-i}]$ denote player $i$’s expected payoff starting at her information set $H$ if her beliefs regarding the conditional probabilities of being at the various nodes in $H$ are given by $\mu$, if she follows strategy $\sigma_i$, and if her opponents use strategy $\sigma_{-i}$.

**Definition.** A strategy profile $\sigma = (\sigma_1, ... , \sigma_I)$ in $\Gamma_E$ is sequentially rational at information set $H$ given a system of beliefs $\mu$ if, denoting the player who moves at $H$ by $i$, we have

$$E[u_i | H, \mu, \sigma_i, \sigma_{-i}] \geq E[u_i | H, \mu, \sigma_i', \sigma_{-i}]$$

for all $\sigma_i' \in \Delta(S_i)$. If $\sigma$ satisfies this condition for all information sets $H$, then we say that $\sigma$ is sequentially rational given belief $\mu$. 

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Thus, a strategy profile is sequentially rational if each player is choosing a strategy that is optimal for her at each of her information sets, given her belief and the strategies of other players.

In Example 3-1, suppose that, when $I$ is called to move, $I$ believes that Firm $E$ has chosen $I_{n1}$ and $I_{n2}$ each with 50% chance, then strategy profile $(I_{n1}, \text{Accommodate if entry has occurred})$ is sequentially rational.

We are now ready to define weak perfect Bayesian equilibrium (WPBE).

Definition. A profile of strategies and system of beliefs $(\sigma, \mu)$ is a weak perfect Bayesian equilibrium (weak PBE or WPBE) in extensive form game $\Gamma_E$ if

(i) $\sigma$ is sequentially rational given belief $\mu$; and

(ii) $\mu$ is derived from $\sigma$ through Bayes’ rule whenever possible.

Condition (ii) says that for any information set $H$ such that $\text{Prob}(H | \sigma) > 0$, then

$$\mu(x) = \frac{\text{Prob}(x | \sigma)}{\text{Prob}(H | \sigma)} \quad \text{for all } x \in H.$$  

On the other hand, there is no restriction on beliefs in information sets that are not reached under the strategy profile. Additional restrictions on these beliefs would lead to notions of PBE and other equilibrium refinements.

In Example 3-1, there is a unique weak PBE, where the strategy profile is $(I_{n2}, \text{Accommodate if entry occurs})$ with $I$’s belief being that $E$ has chosen $I_{n1}$ with prob. 1 if entry occurs. To see that there is no other WPBE, notice that under any belief by $I$ sequential rationality must require $I$ to choose Accommodate if entry occurs. Therefore at any WPBE $E$ must choose $I_{n2}$ with probability 1. But then $I$ must assigns prob. 1 to $I_{n1}$ if entry occurs at any WPBE.

Notice the difference between Nash equilibrium and weak PBE:

A strategy profile $\sigma$ is a NE iff there exists a system of beliefs $\mu$ such that (i) $\sigma$ is sequentially rational given $\mu$ at all information sets $H$ such that $\text{Prob}(H) > 0$; and (ii) $\mu$ is derived from $\sigma$ through Bayes’ rule whenever possible.
While for weak PBE, \( \sigma \) needs to be sequentially rational at all information sets given \( \mu \). In other words, NE only requires that strategies be sequentially rational along the equilibrium path, while weak PBE requires that strategies be sequentially rational even at information sets that are not reached under the strategy profile.

The game in Example 3-1 is special in that player \( I \) has an optimal strategy independent of her beliefs. In general, this is not the case and the trick to find a weak PBE is to find a fixed point such that strategies are sequentially rational given the beliefs and beliefs are consistent with the strategies according to Bayes’ rule.

**Example 3-2.** In this game, it is optimal for \( I \) to choose Fight if she believes that \( \text{In}_1 \) has occurred and to choose Accommodate if she believes \( \text{In}_2 \) has occurred. Assume \( \gamma > 0 \).

It is easy to show that the game has no pure strategy weak PBE. But the game does have a mixed strategy weak PBE. To see this, suppose \( E \) chooses Out, \( \text{In}_1 \), and \( \text{In}_2 \) with probabilities \( \sigma_0 \), \( \sigma_1 \), and \( \sigma_2 \) respectively; \( I \) chooses Fight with prob. \( \sigma_F \) if entry occurs; \( I \)'s belief that \( \text{In}_1 \) has been played when entry occurs is \( \mu_1 \).

First, \( I \) is willing to choose Fight if and only if

\[
-\mu_1 - (1 - \mu_1) \geq -2\mu_1 + (1 - \mu_1),
\]
or $\mu_1 \geq \frac{2}{3}$.

Next, if $\mu_1 > \frac{2}{3}$ in any weak PBE, then $I$ must choose Fight with prob. 1. But then $E$ must play $\text{In}_2$ with prob. 1, and the weak PBE would then require $\mu_1 = 0$, a contradiction.

Next, if $\mu_1 < \frac{2}{3}$ in any weak PBE, then $I$ must choose Accommodate with prob. 1. But then $E$ must play $\text{In}_1$ with prob. 1, and the weak PBE would then require $\mu_1 = 1$, again a contradiction.

Thus at any weak PBE $\mu_1 = \frac{2}{3}$. $E$’s strategy at the weak PBE must then randomize between $\text{In}_1$ and $\text{In}_2$ when choosing to enter the market, with $\sigma_1 = 2\sigma_2$. But then $I$ must randomize between Fight and Accommodate if entry occurs so that $E$ would be indifferent between $\text{In}_1$ and $\text{In}_2$. Thus

$$-\sigma_F + 3(1 - \sigma_F) = \gamma\sigma_F + 2(1 - \sigma_F),$$

or $\sigma_F = \frac{1}{2 + \gamma}$. This implies that $E$’s payoff from playing either $\text{In}_1$ or $\text{In}_2$, given $I$’s strategy, is $3 - \frac{4}{2 + \gamma} > 1$, and thus $\sigma_0 = 0$. Therefore the unique weak PBE is $(\sigma_0, \sigma_1, \sigma_2) = (0, \frac{2}{3}, \frac{1}{3})$, $\sigma_F = \frac{1}{2 + \gamma}$, and $\mu_1 = \frac{2}{3}$.

Let us now look at an example where a player may have infinitely many strategies.

**Example 3-3.** A bargaining model. A seller owns an object that has zero value to him but has value $v$ to a buyer. $v$ can take two possible values: $v_H$ and $v_L$, with $v_H > v_L > 0$. The value of $v$ is known only to the buyer. The probability of $v = v_H$ is $\lambda$, which is common knowledge. In period 1, the seller offers a price to the buyer, and the buyer can either accept or reject the seller’s offer. If the buyer accepts, the object is transferred at the price and the game ends. If the buyer rejects the seller’s offer, then at period 2 the seller can make another offer and the game ends either the offer is accepted or rejected. The discount factor for both the seller and the buyer is $\delta$. Assume that the buyer always accept the seller’s offer if the buyer is indifferent. What are the pure strategy WPBE in this game?
Let $S$’s strategy be $(p_1, p_2)$, its belief of $B$ having $v_H$ be $\mu_1$ and $\mu_2$ in the beginning of the first and second periods. First notice that at any WPBE $v_H \geq p_2 \geq v_L$. Given $(p_1, p_2)$ and $v_H \geq p_2 \geq v_L$, the best response for the $v_L$ type is $\sigma_L = \text{accept } p_1$ iff $p_1 \leq v_l$ and accept $p_2$ iff $p_1 > v_l$ but $p_2 \leq v_l$. The best response for $v_H$ is $\sigma_H = \text{accept } p_1$ iff $p_1 \leq \min\{v_H, v_H - \delta(v_H - p_2)\}$ and accept $p_2$ iff $p_1 > v_H - \delta(v_H - p_2)$ and $p_2 \leq v_H$. Thus, there can be no WPBE at which both $v_H$ and $v_L$ buy at the second period, because that would imply $p_2 = v_L$ and the seller could be better off offering $p_1 = v_L$. There can also be no WPBE at which only $v_H$ buys at period 2, because that would imply $p_1 > v_H - \delta(v_H - p_2) > v_L$ and $p_2 = v_H$, but the seller would then be better off offering $p_1 = v_H$. Thus the WPBE is

$$p_1 = v_L \text{ and } p_2 = v_L \text{ or } v_H; (\sigma_L, \sigma_H); \text{ with } \mu_1 = \lambda \text{ and } \mu_2 = 0 \text{ or } 1$$

if

$$v_l \geq \lambda[v_H - \delta(v_H - v_l)] + (1 - \lambda)\delta v_l,$$

or

$$v_l \geq \lambda v_H.$$  

The WPBE is

$$p_1 = v_H - \delta(v_H - v_l) \text{ and } p_2 = v_L; (\sigma_L, \sigma_H); \text{ with } \mu_1 = \lambda \text{ and } \mu_2 = 0$$

if

$$v_l \leq \lambda v_H.$$  

The first (second) WPBE above, when it exists, is often called a pooling (separating) equilibrium in the literature. We can also simply say what the WPBE strategies are, when we can specify the equilibrium beliefs.

The concept of weak PBE does not impose enough requirement on beliefs in some situations. It is possible that a weak PBE may even not be a subgame perfect Nash
equilibrium. Thus the concept of weak PBE may be too weak. Stronger concepts
such as perfect Bayesian equilibrium (PBE) and sequential equilibrium have been
proposed in the literature. I will spare you from the details of these concepts at this
point.