1. A pyramid of constant density $\delta$ gm/cm$^3$ has a square base of sidelength 40 cm, and a height of 10 cm.

(a) Find the mass of the pyramid.

Mass = volume $\times$ density $= \delta \times \int_0^{10} (4z)^2 \, dz = \frac{\delta}{3} \int_0^{10} 16z^3 \, dz = \frac{16000\delta}{3}$ gm.

(b) Find the center of mass of the pyramid.

Put the apex of the pyramid at the origin, with the base of the pyramid perpendicular to the $z$ axis. Then by symmetry, the center of mass has $x$-coordinate $\bar{x} = 0$ and $y$-coordinate $\bar{y} = 0$.

Furthermore,

$$\bar{z} = \frac{\delta}{16000\delta} \int_0^{10} z(4z)^2 \, dz = \frac{3}{16000} \cdot \frac{16z^4}{4} \bigg|_0^{10} = \frac{30}{4} = 15 \frac{2}{2} \text{ cm}.$$  

2. A metal plate of constant density 5 gm/cm$^2$ has a shape bounded by the curve $y = \sqrt{x}$, the $x$-axis, and the line $x = 1$.

(a) Find the mass of the plate.

Mass = area $\times$ density $= 5 \times \int_0^1 \sqrt{x} \, dx = \frac{10}{3}$ gm.

(b) Find the center of mass of the plate.

$$\bar{x} = \frac{5}{10} \int_0^1 x\sqrt{x} \, dx = \frac{3}{5} \text{ cm}, \quad \bar{y} = \frac{5}{10} \int_0^1 y(1-y^2) \, dy = \frac{3}{8} \text{ cm}.$$  

3. Find the mass of the plate in the previous problem if, instead of constant density, the plate has density:

(a) $\delta(x) = 1 + x$,

Mass $= \int_0^1 (1 + x)\sqrt{x} \, dx = \frac{16}{15}$ gm.

(b) $\delta(y) = 1 + y$.

Mass $= \int_0^1 (1 + y)(1 - y^2) \, dy = \frac{11}{12}$ gm.

4. Find the area of the region bounded by the curve $r = \sqrt{\theta}$, for $0 \leq \theta \leq \pi$. $\int_0^\pi [\sqrt{\theta}^2/2] \, d\theta = \pi^2/4.$
5. Find the area of the region that lies within the limaçon \( r = 3 + 2 \cos(\theta) \) and outside the circle \( r = 4 \).

\[
\frac{1}{2} \int_{-\pi/3}^{\pi/3} [(3 + 2 \cos \theta)^2 - 4^2] d\theta = \frac{13\sqrt{3}}{2} - \frac{5\pi}{3}.
\]

6. Find the area of the region common to the circles \( r = \cos(\theta) \) and \( r = \sqrt{3} \sin(\theta) \).

(Hint: \( \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6} \).)

\[
\frac{1}{2} \int_0^{\pi/6} (\sqrt{3} \sin \theta)^2 d\theta + \frac{1}{2} \int_{\pi/6}^{\pi/2} (\cos \theta)^2 d\theta = \frac{1}{24} \left(5\pi - 6\sqrt{3}\right) \approx 0.221486.
\]

7. Find the exact length of the polar curve \( r = e^\theta \), for \( 0 \leq \theta \leq \pi/2 \).

\[
\int_0^{\pi/2} \sqrt{x'(\theta)^2 + y'(\theta)^2} d\theta = \sqrt{2} \int_0^{\pi/2} e^\theta d\theta = \sqrt{2}(e^{\pi/2} - 1).
\]

8. Consider the parametric curve given by \( x(t) = e^t - t \) and \( y(t) = 4e^{t/2} \) for \( 0 \leq t \leq 1 \). Find the length of this curve.

\[
\int_0^1 \sqrt{(e^t - 1)^2 + (2e^{t/2})^2} dt = \int_0^1 \sqrt{e^{2t} - 2e^t + 1 + 4e^t} dt = \int_0^1 \sqrt{e^{2t} + 2e^t + 1} dt = \int_0^1 e^t + 1 dt = e.
\]

9. Derive the volume formulas for the following shapes by using an appropriate integral:

(a) a circular cylinder of height \( H \) whose radius is \( R \),

\[
\int_0^H \pi R^2 \, dh
\]

(b) a circular cone of height \( H \) whose radius at the base is \( R \), \( \int_0^H \pi \left(R \frac{h}{H}\right)^2 \, dh \)

(c) a square pyramid of height \( H \) whose base has side length \( S \), \( \int_0^H \frac{\sqrt{3}}{4} \left(S \frac{h}{H}\right)^2 \, dh \)

(d) a triangular pyramid of height \( H \) whose base is an equilateral triangle with side length \( S \),

\[
\int_0^H \frac{\sqrt{3}}{4} \left(S \frac{h}{H}\right)^2 \, dh
\]

(e) a sphere of radius \( R \), \( \int_{-R}^R \pi (R^2 - h^2) \, dh \)

10. Suppose that \( 0 \leq f(x) \leq g(x) \) for \( x \geq a \). If \( \int_a^\infty f(x)dx \) converges and \( \int_a^\infty g(x)dx \) diverges, then is the area between the curves \( f(x) \) and \( g(x) \) for \( x \geq a \) finite or infinite? Infinite: this area is \( \int_a^\infty g(x)dx - \int_a^\infty f(x)dx \), which is a finite area removed from an infinite area.

11. Determine whether the following integrals converge or diverge:

(a) \( \int_1^\infty \frac{5 - 2 \sin(e^x)}{x^2} \, dx \) converges
(b) \( \int_{1}^{\infty} \frac{1}{x + \ln x} \, dx \) diverges

(c) \( \int_{1}^{\infty} \frac{x^2 + x + 1}{x^5 + 3x^2 + 1} \, dx \) converges

(d) \( \int_{1}^{\infty} \frac{1}{e^x - x} \, dx \) converges

(e) \( \int_{0}^{\infty} \frac{e^x}{e^{2x} + 1} \, dx \) converges

12. Consider the region bounded by \( y = \sqrt{x} \), \( y = 0 \), \( x = 1 \).

(a) Sketch the solid obtained by rotating the above region around the \( x \)-axis.

(b) Using the sketch, write a Riemann sum approximating the volume of the solid.

\[
\sum \pi (\sqrt{x})^2 \triangle x = \sum \pi x \triangle x
\]

(c) Convert your sum into an integral and find the volume.

\[
\int_{0}^{1} \pi x \, dx = \frac{\pi}{2}
\]

(d) Repeat parts (a)-(c) with the same region rotated around the \( y \)-axis.

\[
\sum \pi (1 - (y^2)^2) \triangle y = \sum \pi (1 - y^4) \triangle y
\]

\[
\int_{0}^{1} \pi (1 - y^4) \, dy = \frac{4\pi}{5}
\]

13. Find the volume of the solid whose base is the region in the \( xy \)-plane bounded by the curves \( y = x \) and \( y = x^2 \) and whose cross sections perpendicular to the \( x \)-axis are squares with one side in the \( xy \)-plane.

Since the cross sections are perpendicular to the \( x \)-axis we will have slices of thickness \( \triangle x \). The volume of each slice will be given by \( y \triangle x \) where \( y = x - x^2 \). The Riemann sum approximating the volume is then

\[
\sum (x - x^2)^2 \triangle x
\]

This leads us to computing the volume of the solid by the integral

\[
\int_{0}^{1} (x - x^2)^2 \, dx
\]
14. A certain bacteria is growing in a petri dish of volume $300 \text{ cm}^3$. Assume that each bacterium occupies $1 \times 10^{-12} \text{ cm}^3$, and that the growth rate of the bacteria, starting at time $t = 1$ hour, is given by

$$r(t) = \frac{\sin(t) + 2}{t} \text{ bacteria/hour}.$$

Will the bacteria ever outgrow the petri dish? Explain your answer carefully.

Yes, the bacteria will outgrow the petri dish. We will show that total quantity of bacteria tends to $\infty$ as $t$ goes to $\infty$, so the total volume of the bacteria will eventually exceed the volume of the petri dish, regardless of the size of the bacteria.

Note that the total quantity of bacteria after $b$ hours is equal to (the amount of bacteria after 1 hour) $+ \int_1^b r(t) \, dt$. Thus, we are trying to show that

$$\lim_{b \to \infty} \int_1^b \frac{\sin(t) + 2}{t} \, dt = \infty,$$

i.e. we want to show $\int_1^\infty \frac{\sin(t) + 2}{t} \, dt = \infty$. We use a comparison. Notice that $1 \leq \sin(t) + 2$, so

$$\frac{1}{t} \leq \frac{\sin(t) + 2}{t}.$$  

Thus, $\int_1^\infty \frac{1}{t} \, dt \leq \int_1^\infty \frac{\sin(t) + 2}{t} \, dt$. We know that $\int_1^\infty \frac{1}{t} \, dt$ diverges to $\infty$, so $\int_1^\infty \frac{\sin(t) + 2}{t} \, dt$ must diverge to infinity as well.

15. Show that the volume contained in the solid obtained by rotating the curve $y = e^{-x}$, from $x = 1$ to $\infty$, about the $x$-axis is finite. To calculate the volume of the solid, we slice the solid perpendicular to the $x$-axis to get slices that look approximately like cylinders of height (or width, depending on your point of view) $\Delta x$. Now, we make the following calculations.

- The volume of a slice $\approx \pi r^2 h = \pi e^{-2x} \Delta x$
- The volume of the solid $\approx \sum \pi e^{-2x} \Delta x$
- The volume of the solid $= \lim_{\Delta x \to 0} \sum \pi e^{-2x} \Delta x = \int_1^{\infty} \pi e^{-2x} \, dx$

Thus, we need to show that $\int_1^{\infty} \pi e^{-2x} \, dx$ is finite. We use a comparison. Using the facts that $f(y) = e^y$ is an increasing function of $y$ and that $-2x$ is less than $-x$ for $x \geq 1$, we see that $0 \leq \pi e^{-2x} \leq \pi e^{-x}$ for $x \geq 1$. We conclude that $0 \leq \int_1^{\infty} \pi e^{-2x} \, dx \leq \int_1^{\infty} \pi e^{-x} \, dx$. Since $\int_1^{\infty} \pi e^{-x} \, dx$ is finite (this improper integral can be calculated directly), $\int_1^{\infty} \pi e^{-2x} \, dx$ must also be finite.

16. The density of oil in a circular oil slick on the surface of the ocean at a distance $r$ meters from the center of the slick is given by $\delta(r) = 50/(1 + r)$ kg/m².

(a) If the slick extends from $r = 0$ to $r = 10,000$ m, find a Riemann sum approximating the total mass of the oil in the slick.

$$\sum_{i=1}^{n} 2\pi r_i \frac{50}{1 + r_i} \Delta r$$

(b) Find the exact value of the mass of oil in the slick by turning your sum into an integral, and evaluating it.

$$2\pi \int_0^{10000} \frac{50r}{1 + r} \, dr \approx 3.1387 \cdot 10^6.$$
(c) Within what distance $r$ is half of the oil slick contained? about 5004 meters

17. Let $\{f_n\}$ be the sequence defined recursively by $f_1 = 5$ and $f_n = f_{n-1} + 2n + 4$.

(a) Check that the sequence $g_n$ whose $n$-th term is $g_n = n^2 + 5n - 1$ satisfies this recurrence relation, and that $g_1 = 5$. (This tells us $g_n = f_n$ for all $n$.) We check:

$$g_{n-1} + 2n + 4 = ((n - 1)^2 + 5(n - 1) - 1) + 2n + 4 = n^2 + 5n - 1 = g_n$$

and plainly $g_1 = 5$.

(b) Use the result of part (a) to find $f_{20}$ quickly, with the aid of a calculator. $f_{20} = 20^2 + 5 \cdot 20 - 1 = 499$.

18. Decide whether each of the following sequences converges. If a series converges, what does it converge to? If not, why not?

(a) The sequence whose $n$-th term is $a_n = 1 - \frac{1}{n}$. Converges to 1. (The $\frac{1}{n}$ part goes to zero.)

(b) The sequence whose $n$-th term is $b_n = \sqrt{n + 1} - \sqrt{n}$. Converges to 0:

$$\sqrt{n + 1} - \sqrt{n} = \sqrt{n + 1} - \sqrt{n} \left( \frac{\sqrt{n + 1} + \sqrt{n}}{\sqrt{n + 1} + \sqrt{n}} \right) = \frac{n + 1 - n}{\sqrt{n + 1} + \sqrt{n}} = \frac{1}{\sqrt{n + 1} + \sqrt{n}}$$

and the denominator of this expression grows as $n$ gets large, so it approaches zero.

(c) The sequence whose $n$-th term is $c_n = \cos(\pi n)$. Diverges. The sequence is $\{-1, 1, -1, 1, \cdots\}$ which oscillates.

(d) The sequence $\{d_n\}$, where $d_1 = 2$ and $d_n = 2d_{n-1}$ for $n > 1$.

Diverges. The sequence is $\{2, 4, 8, 16, \cdots\}$ and the terms go to infinity.

19. A ball is dropped from a height of 10 feet and bounces. Assume that there is no air resistance. Each bounce is $\frac{3}{4}$ of the height of the bounce before.

(1) Find an expression for the height to which the ball rises after it hits the floor for the $n$th time. $H(n) = 10 \left( \frac{3}{4} \right)^n$

(2) Find an expression for the total vertical distance the ball has traveled when it hits the floor for the $n$th time. $D(n) = 10 + (2 \cdot 10 \cdot (3/4)) \left( \frac{1 - (3/4)^{n-1}}{1 - (3/4)} \right)$.

(3) Using without proof the fact that a ball dropped from a height of $h$ feet reaches the ground in $\sqrt{h/4}$ seconds: Will the ball bounce forever? If not, how long will it take for the ball to come to rest?

The ball will not bounce forever. The total time it bounces is given by $\sqrt{\frac{h}{4}} + a/(1 - r)$, with $a = (1/2) \cdot \sqrt{10 \cdot (3/4)}$ and $r = \sqrt{3/4}$.

20. Check whether the following series converge or diverge. In each case, justify your answer by either computing the sum or by showing which convergence test you are using, why and how it applies (depending on the case).

(a) $\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)}$ diverges – integral or limit comparison test
(b) \(\sum_{n=1}^{\infty} (-1)^n \frac{n}{(n^2 + 1)^2}\) conditionally converges – alternating series test

(c) \(\sum_{n=1}^{\infty} \frac{n}{(n^2 + 1)^2}\) converges – integral, comparison, or limit comparison test

(d) \(\sum_{n=1}^{\infty} \left(\frac{n + 1}{n}\right)^n\) diverges – \(n\)th term test (that is, the \(n\)th term does not go to zero)

(e) \(\sum_{n=1}^{\infty} \frac{\sqrt{n^3 + 1}}{5n^2}\) diverges – limit comparison test or \(n\)th term test

(f) \(\sum_{n=1}^{\infty} \sin \left(\frac{1}{n^2}\right)\) (hint: consider \(\sum_{n=1}^{\infty} \frac{1}{n^2}\)) converges – limit comparison test

(g) \(\sum_{n=1}^{\infty} \frac{2^n}{n!}\) converges – ratio test

(h) \(\sum_{n=1}^{\infty} \frac{(2n)!}{(n+3)!}\) diverges – ratio test

(i) \(\sum_{n=1}^{\infty} \frac{n!}{(n+2)!}\) converges – comparison or limit comparison test

21. Find the values of \(a\) for which the series converges/diverges:

(a) \(\sum_{n=2}^{\infty} \frac{1}{n(\ln(n))^a}\) \(a > 1\)

(b) \(\sum_{n=1}^{\infty} \frac{1}{(n!)^a}\) \(a > 0\)

22. Consider the series \(\sum_{n=1}^{\infty} (-1)^n \frac{\ln n}{n}\). Are the following statements true or false? Fully justify your answer.

(a) The series converges by limit comparison with the series \(\sum_{n=1}^{\infty} \frac{1}{n}\). False

(b) The series converges by the ratio test. False

(c) The series converges by the integral test. False

(d) The series converges by the alternating series test. True