Monotone Comparative Statics

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Problem with convention methods of comparative statics

Convention methods of comparative statics could fail because of the assumption on continuity and differentiability. Consider the following example of appeasement problem from Ashworth and Bueno de Mesquita (2006) and Powell (1999) where the actor is maximizing:

\[ u(x) = (1 - x)p(x) + q[1 - p(x)] \]

where \( x \) is the concession that the actor made to the opponent and \( p(x) \) is the probability that this concession will be accepted. If the concession is rejected, conflict breaks out and the probability of the actor winning is \( q \). We are trying to derive comparative statics for the optima of \( x \), \( x^* \) with regard to \( q \) which captures the relative power of the proposer. Without specifying the functional form of \( p(x) \), the first order condition of the the maximization yields:

\[ F(x, q) = -p(x^*) + p'(x^*)(1 - x^*) - qp'(x^*) = 0 \]

where \( x^* \) is the concession that maximizes the payoff. With Implicit Function Theorem, the comparative statics can be derived with:

\[
\frac{\partial x^*}{\partial q} = -\frac{F_q(x, q)}{F_x(x, q)}
\]

(1)

\[
= -\frac{-p'(x)}{-p'(x) + p''(x)(1 - x) - p'(x) - q}
\]

(2)

To use the implicit function theorem to determine the sign of \( \frac{\partial x^*}{\partial q} \), second order differentiability of \( p(x) \) must be guaranteed, which is a somewhat hard assumption to make under many circumstances. To avoid such problem, we can start with a simply specific function of \( f(x) = x \) as Ashworth and Bueno de Mesquita (2006) demonstrates but it is again imposing hard-to-justify assumption on the model that makes extensions very difficult.

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In situations like this where continuity and differentiability assumptions of functions of parameters must be made to derive comparative statics, we can resort to the tool known as Monotone Comparative Statics. We introduce two important concepts in the following.

Single-Crossing Condition:
The function \( f(a, \theta) \) satisfies the single-crossing condition if for all \( a > a' \) and \( \theta > \theta' \), \( f(a, \theta) - f(a', \theta') \geq 0 \) implies \( f(a, \theta) - f(a', \theta) \geq 0 \).

Increasing Differences as Sufficient Condition for Single-Crossing:
For \( f(a, \theta) \), given \( a' < a \), if \( f(a, \theta) - f(a', \theta) \) is increasing in \( \theta \), \( f \) is single-crossing. That is, \( f \) is single-crossing if
\[
\frac{\partial^2 f(a, \theta)}{\partial a \partial \theta} \geq 0.
\]

Monotonicity under Single-Crossing:
If \( f(a, \theta) \) satisfies the single-crossing property and let \( a^* = \arg \max_{a \in A} f(a, \theta) \).

Based on these theorems, to show comparative statics of \( a^* \) with regard to \( \theta \), we only need to find the second crossing derivative of \( f(a, \theta) \) and determine its sign. If the second derivative is negative, we can use the method of reordering to show \( a^* \) is decreasing in \( \theta \).

MCS through Implicit Function Theorem:
To see the rationale of the theorem, consider an optimization problem with a continuously differentiable objective function \( f(x, \theta) \):
\[
\max_{x \in X} f(x, \theta)
\]
To derive comparative statics with regard to \( \theta \), we can apply the Implicit Function Theorem. Given \( x(\theta) \in \arg \max_{x \in X} f(x, \theta) \),
\[
x'(\theta) = -\frac{f_{x \theta}(x, \theta)}{f_{xx}(x, \theta)}.
\]
Provided that the second order condition of \( \max_{x \in X} f(x, \theta) \) derives \( f_{xx}(x, \theta) < 0 \), it is easy to see \( x'(\theta) \) and \( f_{x \theta}(x, \theta) \) always have the same sign. Thus, given verified SOC, we only to check the sign of the cross-derivative between the choice variable and the parameter to derive comparative statics when the objective function is differentiable.

Comparative Statics with Multi-Dimensional Decision

If the decision taken by the actor are multi-dimensional, new tools are needed to derive Monotone Comparative Statics. In the previous appeasement example,
we can allow $q$ to a function of variables such as military build-up or resolve to fight. We introduce the concept of Supermodularity as follows.

**Supermodularity** Given the function $f(a, \theta)$, where $a \in A$ and $\theta \in \Theta$, the function $f : A \times \Theta \to \mathbb{R}$ is supermodular if $A \times \Theta$ is a product set and $f$ has increasing differences for all pairs of arguments of the function.

For example, in a function $f(a_1, a_2, \Theta)$, $f$ is supermodular if

$$\frac{\partial^2 f}{\partial a_1 \partial a_2} > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial a_1 \partial \theta} > 0 \quad \text{and} \quad \frac{\partial^2 f}{\partial a_2 \partial \theta} > 0. \quad (3)$$

**Supermularity as Sufficient Condition for Monotone Comparative Statics:**
Consider the problem $\max_{a \in A} f(a, \theta)$ where $\theta \in \Theta$ is a parameter. If $f$ is supermodular, then the optimal solution $a^*$ has monotone comparative statics.