Optimization with Constraints

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The primary purpose of imposing a constraint is to give due cognizance to certain limiting factors present in the optimization problem. For example, in studying consumers’ behavior, the consumer would purchase infinite number of goods to maximize his or her utility if the budget constraint is not considered. Once the budget constraint that the consumer is facing is incorporated into the question, the problem of optimization may not be captured merely by an objective function. The problem becomes a maximization of the objective function, subject to the budget constraint, which mathematically narrow the domain and hence the range of the objective function. It should noted here that even without any new technique of solution, the constrained maximum can be easily found at times when the constraint can be expressed as a simple linear function. By substituting the expression into the objective function.

But when the constraint is itself a complicated function, or when there are several constraints to consider, however, the technique of substitution and elimination of variables could become a burdensome task and elimination method would in fact be of no avail. In such case we may resort to a method known as the method of Lagrange multiplier, which has distinct analytical advantage.

0.1 Lagrange-Multiplier Method

The essence of the Lagrange-multiplier method is to convert a constrained-extremum problem into a form such that the first-order condition of the free-extremum problem can still be applied. Given an objective function

\[ z = f(x, y) \]

subject to the constraint

\[ g(x, y) = c \]

where \( c \) is a constant, we write the Lagrangian function as

\[ Z = f(x, y) + \lambda [c - g(x, y)] \]

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For stationary value of \( Z \), regarded as a function of the three variables \( \lambda, x \) and \( y \), the necessary condition is

\[
\begin{align*}
Z_\lambda &= c - g(x, y) = 0 \quad (1) \\
Z_x &= f_x - \lambda g_x = 0 \quad (2) \\
Z_y &= f_y - \lambda g_y = 0 \quad (3)
\end{align*}
\]

\( Z_\lambda \) is the first derivative of \( Z \) with regard to \( \lambda \). The notation is similar with \( f_x, g_y \) etc. Since the first equation is simply a restatement of the constraint \( g(x, y) = c \), the stationary values of the Lagrangian function \( Z \) will automatically satisfy the constraint of the original function \( z \). And since the expression \( \lambda[c - g(x, y)] \) is now assuredly zero, the stationary values of \( Z \) is the Lagrangian function must be identical with those of the original objective function \( z \), subject to the constraint.

0.1.1 An Interpretation of the Lagrange Multiplier

The constrained optimization problem can also be solved with the total-differential approach. In the discussion of the free extremum of \( z = f(x, y) \), it was learned that the first-order necessary condition may be stated in terms of the total differential \( dz \) as follows:

\[
dz = f_x dx + f_y dy = 0
\]

This statement remains valid after a constraint \( g(x, y) = c \) is added. However, with the constraint in the picture, we can no longer take both \( dx \) and \( dy \) as “arbitrary” changes as before. For if \( g(x, y) = c \), then \( dg \) must be equal to \( dc \), which is zero since \( c \) is a constant. Hence,

\[
(dg =) g_x dx + g_y dy = 0
\]

and this relation makes \( dx \) and \( dy \) dependent on each other. The first-order necessary condition therefore becomes \( dz = 0 \), subject to \( g = c \), and hence also subject to \( dg = 0 \). Inspecting the two preceding equations, it should be clear that in order to satisfy this necessary condition, we must have

\[
\frac{f_x}{g_x} = \frac{f_y}{g_y}
\]

This condition, together with the constraint \( g(x, y) = c \), will provide two equations from which to find the critical values of \( x \) and \( y \).

This total-differential approach yields the same first-order condition as the Lagrange-multiplier method. Equation 2 and 3 in the result of Lagrange-multiplier can be rewritten as

\[
\frac{f_x}{g_x} = \lambda \text{ and } \frac{f_y}{g_y} = \lambda
\]
Note the Lagrange-multiplier method also gives the value of $\lambda$ as a direct by-product. As it turns out, $\lambda$ provides a measure of the sensitivity of $Z^*$ (and $z^*$) to a shift of the constraint, as we shall presently demonstrate. Therefore the Lagrange-multiplier method offers the advantage of containing certain built-in comparative static information in the solution.

To show that $\lambda$ indeed measures the sensitivity of $Z$ to changes in the constraint, let us perform a comparative-static analysis on the first-order condition. Since $\lambda$, $x$, and $y$ are endogenous, the only available exogenous variable is the constraint parameter $c$.

A change in $c$ would cause a shift of the constraint curve in the $xy$ plane and thereby alter the optimal solution. To do the comparative static analysis, we resort to the implicit-function theorem. Taking $\lambda$, $x$, and $y$ as endogenous variable in the comparative static analysis. Since the optimal value of $Z$ depends on $\lambda^*$, $x^*$, and $y^*$, that is,

$$Z^* = f(x^*, y^*) + \lambda^*[c - g(x^*, y^*)]$$

We can express $\lambda^*$, $x^*$, and $y^*$ all as implicit functions of the parameter $c$

$$\lambda^* = \lambda^*(c) \quad x^* = x^*(c) \quad \text{and} \quad y^*(c)$$

Considering $Z^*$ to be a function of $c$ alone. Differentiating $Z^*$ totally with respect to $c$, we find a simple result

$$\frac{Z^*}{dc} = \lambda^*$$

which validates our claim that the solution value of the Lagrange multiplier constitutes a measure of the effect of a change in the constraint via the parameter $c$ on the optimal value of the objective function.

**0.1.2 n-Variable and Multiconstraint Cases**

The generalization of the Lagrange-multiplier method to $n$ variables can be easily carried out if we write the choice variables in the subscript notation. The objective function will then be in the form

$$z = f(x_1, x_2, \ldots, x_n)$$

subject to the constraint

$$g(x_1, x_2, \ldots, x_n) = c$$

It follows that the Lagrangian function will be

$$Z = f(x_1, x_2, \ldots, x_n) + \lambda[c - g(x_1, x_2, \ldots, x_n)]$$
for which the first-order condition will consist of the following \((n + 1)\) simultaneous equations:

\[
\begin{align*}
Z_y &= c - g(x_1, x_2, \ldots, x_n) = 0 \\
Z_1 &= f_1 - \lambda g_1 = 0 \\
Z_2 &= f_2 - \lambda g_2 = 0 \\
\vdots \\
Z_n &= f_n - \lambda g_n = 0
\end{align*}
\]

When there is more than one constraint, the Lagrange-multiplier method is equally applicable, provided we introduce as many such multipliers as there are constraints in the Lagrangian function. Let an \(n\)-variable function be subject simultaneously to the two constraints

\[
g(x_1, x_2, \ldots, x_n) = c \text{ and } h(x_1, x_2, \ldots, x_n) = d
\]

Then, adopting \(\lambda\) and \(\mu\) as the undetermined multiplier, we may construct a Lagrangian function as follows:

\[
Z = f(x_1, x_2, \ldots, x_n) + \lambda [c - g(x_1, x_2, \ldots, x_n)] - \mu [d - h(x_1, x_2, \ldots, x_n)]
\]

Considering \(\lambda\) and \(\mu\) as choice variables, we now count \((n + 2)\) variables thus the first order condition will in this case consist of the following \((n + 2)\) simultaneous equations

\[
\begin{align*}
Z_y &= c - g(x_1, x_2, \ldots, x_n) = 0 \\
Z_\mu &= d - h(x_1, x_2, \ldots, x_n) = 0 \\
Z_i &= f_i - \lambda g_i - \mu h_i = 0 \quad (i = 1, 2, \ldots, n)
\end{align*}
\]

These should normally enable us to solve for all the \(x_i\) as well as \(\lambda\) and \(\mu\). As before, the first two equation of the necessary condition represent essentially a mere reinstatement of the two constraints.