Topological games

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I’ll mostly talk about games of length $\omega$ of the following type:

There are two players, $I$ and $II$. In the $n$th round, $I$ chooses a set $I_n$ (in some restricted collection of sets defined by the “rules” of the game). $II$ responds by choosing a set $J_n$.

$I$: $I_0, I_1, \ldots, I_n, \ldots$

$II$: $J_0, J_1, \ldots, J_n, \ldots$

There is a round for every $n \in \omega$, and then the game is over.

$I$ wins the game if the sequence $I_0, J_0, I_1, J_1, \ldots$ of plays of the game satisfies a certain condition (e.g., $\bigcap_{n \in \omega} J_n = \emptyset$); otherwise $II$ wins.

In a topological game, the sets $I_n$ and $J_n$ of course are topological objects, e.g., points in a space $X$, closed subsets of a space, an open cover of a space, etc.
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A *strategy for Player I* is a function $\sigma$ whose domain is the set of finite initial segments $I_0, J_0, \ldots, I_n, J_n$ of plays of the game ending with a play by Player II.

The strategy $\sigma$ is a *winning strategy for Player I* if Player I wins every sequence $I_0, J_0, I_1, J_1, \ldots$ of legal plays of the game in which $I_n = \sigma(I_0, J_0, \ldots, I_{n-1}, J_{n-1})$ for every $n \in \omega$. 

Wlog, a strategy for Player I may be considered to be a function whose domain is the set of finite sequences $J_0, J_1, \ldots$ of plays by Player II, since given a strategy $\sigma$ as above, and $J_0, \ldots, J_n$, there is a unique way to fill in the plays $I_0 = \sigma(\emptyset)$, $I_1 = \sigma(I_0, J_0)$, etc. of Player I.

A (winning) strategy for Player II is defined mutatis mutandis.

Obviously, I and II cannot both have a winning strategy, and it is possible that neither has. A game is *determined* if one of the players has a winning strategy, otherwise it is *undetermined*.
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Banach-Mazur game

Banach-Mazur game BM(X) on X:

Players E and NE alternately choose nonempty open sets in X:

E: \(U_0 \supset V_0 \supset U_1 \supset V_1 \ldots\)

NE: \(V_0 \supset U_0 \supset V_1 \supset U_1 \ldots\)

E wins if \(\bigcap_{n \in \omega} U_n = \emptyset\).

Theorem (Oxtoby)

A space \(X\) is a Baire space iff E has no winning strategy in BM(X).

If \(G(X)\) is a game on \(X\), we’ll write “\(I^{\uparrow} G(X)\)” to mean “Player I has a winning strategy in \(G(X)\)”.

So:

\(X\) is Baire iff \(E \not^{\uparrow} BM(X)\).
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$\text{NE} \uparrow \text{BM}(X_\alpha)$ for each $\alpha \Rightarrow \text{NE} \uparrow \text{BM}(\prod_{\alpha \in \kappa} X_\alpha)$
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Conjecture (Galvin)

Converse is true: $\text{NE} \uparrow \text{BM}(X) \iff X^\kappa$ with box topology Baire $\forall \kappa$
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Conjecture is consistent if there is a proper class of measurables;
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There are metric Baire spaces $X$, $Y$ with $X \times Y$ not Baire.

$\text{NE} \uparrow \text{BM}(X_\alpha)$ for each $\alpha$ $\Rightarrow$ $\text{NE} \uparrow \text{BM}(\prod_{\alpha \in \kappa} X_\alpha)$

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Converse is true: $\text{NE} \uparrow BM(X) \iff X^\kappa$ with box topology Baire $\forall \kappa$.

Conjecture is consistent if there is a proper class of measurables; true in ZFC?
A strategy of a player is *stationary* if that player's next move depends only on the previous move of his opponent.
Stationary strategies

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Galvin and Telgarsky obtained a result which says that for games in a certain class, call it *STAT*, if Player I has a winning strategy, then she has a stationary winning strategy.
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BM(X) in this class. If E has a winning strategy in the BM(X), then E has a stationary winning strategy.
Idea of proof:

Let \( \sigma \) be a winning strategy for E. Fix a well-order \( \prec \) of the collection of all open sets. Suppose NE plays \( V_n = V \) in round \( n \). Look at the set \( \mathcal{P}(V) \) of all partial legal plays \( (U_0, V_0, \ldots, U_k, V_k) \) of the game with E using \( \sigma \) and with \( V_k = V \). One may check:

(i) The lexicographic order on \( \mathcal{P}(V) \) is a well-order.

(ii) Given \( V_n = V \) is NE's move in round \( n \), let \( \tau(V) = \sigma(U_0, V_0, \ldots, V_k) \), where \( (U_0, V_0, \ldots, V_k) \) is the least element of \( \mathcal{P}(V) \) ending in \( V \). Then \( \tau \) defines a stationary winning strategy for E.
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$\tau(V) = \sigma(V_0, V_1, \ldots, V_k = V)$, where $(V_0, V_1, \ldots, V_k = V)$ is the least element of $P(V)$ ending in $V$. Then $\tau$ defines a stationary winning strategy for E.
Example (Debs)

∃X with NE↑BM(X) but NE has no stationary winning strategy.
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$\exists X$ with $\text{NE} \uparrow \text{BM}(X)$ but $\text{NE}$ has no stationary winning strategy.

$\text{NE}$ has strategy depending on last two moves of $E$. 

Open: must they all?

Question (Telgarsky)

$\exists X$ with $\text{NE} \uparrow \text{BM}(X)$ but $\text{NE}$ has winning strategy based on last 3 moves of $E$ but not last 2?

Remark: Galvin and Telgarsky, Debs: $\text{NE} \uparrow \text{BM}(X) \implies \text{NE}$ has winning strategy based on last move of opponent and his own last move.
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Remark: Galvin and Telgarsky, Debs: \( \text{NE} \uparrow \text{BM}(X) \Rightarrow \text{NE has winning strategy based on last move of opponent and his own last move.} \)
Let $X$ be a space, and let $K$ be a closed hereditary class of spaces.
Telgársky’s game

Let $X$ be a space, and let $\mathbb{K}$ be a closed hereditary class of spaces. We define the game $G(\mathbb{K}, X)$. There are two players, I and II.
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Let $X$ be a space, and let $\mathbb{K}$ be a closed hereditary class of spaces. We define the game $G(\mathbb{K}, X)$. There are two players, I and II. Player I begins by choosing a nonempty closed subset $A_0$ of $X$ such that $A_0 \in \mathbb{K}$.
Let $X$ be a space, and let $\mathcal{K}$ be a closed hereditary class of spaces. We define the game $G(\mathcal{K}, X)$. There are two players, I and II.

**Player I** begins by choosing a nonempty closed subset $A_0$ of $X$ such that $A_0 \in \mathcal{K}$.

II responds by choosing a closed set $B_0 \subset X \setminus A_0$.

Player I then chooses a nonempty closed $A_1 \subset B_0$ with $A_1 \in \mathcal{K}$.

II chooses a closed $B_1 \subset B_0 \setminus A_1$ etc.
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I: \( A_0 \in K \quad A_1 \subset B_0, A_1 \in K \quad \ldots \quad A_n \subset B_{n-1}, A_n \in K \)

II: \( B_0 \subset X \setminus A_0 \quad B_1 \subset B_0 \setminus A_1 \quad \ldots \quad B_n \subset B_{n-1} \setminus A_n \)
I: $A_0 \in \mathbb{K}, A_1 \subset B_0, A_1 \in \mathbb{K}, \ldots, A_n \subset B_{n-1}, A_n \in \mathbb{K}$

II: $B_0 \subset X \setminus A_0, B_1 \subset B_0 \setminus A_1, \ldots, B_n \subset B_{n-1} \setminus A_n$

We say I wins the game if $\bigcap_{n \in \omega} B_n = \emptyset$; otherwise II wins.
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The space $X$ is said to be $\mathbb{K}$-like if Player I has a winning strategy in $G(\mathbb{K}, X)$ (i.e., if $I \uparrow G(\mathbb{K}, X)$).
I: $A_0 \in \mathcal{K}$, $A_1 \subset B_0$, $A_1 \in \mathcal{K}$, \ldots, $A_n \subset B_{n-1}$, $A_n \in \mathcal{K}$

II: $B_0 \subset X \setminus A_0$, $B_1 \subset B_0 \setminus A_1$, \ldots, $B_n \subset B_{n-1} \setminus A_n$

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The space $X$ is said to be \textbf{K-like} if Player I has a winning strategy in $G(\mathcal{K}, X)$ (i.e., if I ↑ $G(\mathcal{K}, X)$).

Trivially, $\mathcal{K}$ is contained in the class of K-like spaces.
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**Theorem (Telgarsky)**

If $X$ is paracompact and $\mathcal{DC}$-like, then $X \times Y$ is paracompact for all paracompact spaces $Y$. 
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**Theorem (Telgarsky)**

If $X$ is paracompact and $\mathcal{DC}$-like, then $X \times Y$ is paracompact for all paracompact spaces $Y$.

(Sub)paracompact scattered spaces, more generally $\mathcal{C}$-scattered (every closed subspace has a point of local compactness), and spaces with a $\sigma$-closure-preserving cover by compact sets, are $\mathcal{DC}$-like.
Telgarsky’s Conjecture

$X \times Y$ is paracompact $\forall$ paracompact $Y$ iff I has a winning strategy in $G(\mathcal{DC}, X)$ (i.e., $X$ is $\mathcal{DC}$-like)
Telgarsky’s Conjecture

$X \times Y$ is paracompact $\forall$ paracompact $Y$ iff I has a winning strategy in $G(\mathbb{D}C, X)$ (i.e., $X$ is $\mathbb{D}C$-like)

Theorem (Alster, 2006)

Telgärsky’s Conjecture holds if $X$ has a base of cardinality $\leq \aleph_1$
Telgärsky’s game is in STAT. Player I has a winning strategy iff she has a stationary winning strategy.
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Thus, $X$ is $\mathbb{K}$-like iff there is a function $\sigma : \mathcal{C}(X) \to \mathbb{K}$, where $\mathcal{C}(X)$ is the collection of nonempty closed subsets of $X$, such that
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Thus, $X$ is $\mathbb{K}$-like iff there is a function $\sigma : C(X) \to \mathbb{K}$, where $C(X)$ is the collection of nonempty closed subsets of $X$, such that

1. $\sigma(C) \subset C$;

2. $\sigma(C) \in \mathbb{K}$;

3. If $X = B_1 \supset B_0 \supset B_1 \supset \ldots$ is a decreasing sequence of closed sets such that for each $n \in \omega$, $B_n \cap \sigma(B_{n-1}) = \emptyset$, then $\bigcap_{n \in \omega} B_n = \emptyset$. 
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$$X = B_{-1} \supset B_0 \supset B_1 \cdots \supset B_n \supset \cdots$$

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A space $X$ is a **$D$-space** if, given an open nbhd $N(x)$ for each $x \in X$, there is a closed discrete $D \subset X$ such that $N(D) = \{N(x) : x \in D\}$ covers $X$. Compact or $\sigma$-compact implies $D$. Open question: Do any of the other standard covering properties (e.g., (Lindel"of, paracompact, metacompact, submetacompact,...) imply $D$?
A space $X$ is a **D-space** if, given an open nbhd $N(x)$ for each $x \in X$, there is a closed discrete $D \subset X$ such that $N(D) = \{N(x) : x \in D\}$ covers $X$.

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Do any of the other standard covering properties (e.g., (Lindelöf, paracompact, metacompact, submetacompact,...) imply $D$?
$X$ is said to be **Menger** if, given open covers $U_0, U_1, \ldots$, there are finite $F_n \subset U_n$ such that $\bigcup_{n \in \omega} F_n$ covers $X$. 
$X$ is said to be **Menger** if, given open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$, there are finite $\mathcal{F}_n \subset \mathcal{U}_n$ such that $\bigcup_{n \in \omega} \mathcal{F}_n$ covers $X$.

$\sigma$-compact $\Rightarrow$ Menger $\Rightarrow$ Lindelöf
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Irrationals are not Menger.
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$MA(\kappa) \Rightarrow$ any $X \subset \mathbb{R}$ with $|X| \leq \kappa$ is Menger.
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(Fremlin-Miller) $\text{ZFC} \Rightarrow \exists$ non-$\sigma$-compact Menger $X \subset \mathbb{R}$
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\textbf{Theorem (Aurich)}

Menger spaces are $D$-spaces.
$X$ is said to be **Menger** if, given open covers $\mathcal{U}_0, \mathcal{U}_1, \ldots$, there are finite $\mathcal{F}_n \subset \mathcal{U}_n$ such that $\bigcup_{n \in \omega} \mathcal{F}_n$ covers $X$.

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**Theorem (Aurichi)**

Menger spaces are $D$-spaces.

Not clear how to do a direct proof. A game characterization of Menger, due to Hurewicz, provides an easy proof.
Menger game $M(X)$:

In round $n$, Player I chooses open cover $U_n$ of $X$. II responds with finite $F_n \subset U_n$. II wins if $\bigcup_{n \in \omega} F_n$ covers $X$. Equivalently, I's cover is closed under finite unions, and II chooses $U_n \in U_n$. II wins if $\bigcup_{n \in \omega} U_n = X$. Easy: $I \not\uparrow M(X) \Rightarrow X$ is Menger (Hurewicz) $X$ is Menger iff $I \not\uparrow M(X)$.
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Easy: $I \uparrow M(X) \Rightarrow X$ Menger
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Equivalently, I's cover is closed under finite unions, and II chooses $U_n \in U_n$. II wins if $\bigcup_{n \in \omega} U_n = X$.
Easy: $I \nmid M(X) \Rightarrow X$ Menger

**Theorem (Hurewicz)**

$X$ is Menger iff $I \nmid M(X)$
Proof of Aurichi’s theorem

Assume \( X \) is a Menger space. Let \( \mathcal{N} \) be a neighborhood assignment on \( X \). Let Player I's first play be \( \{ \mathcal{N}(x) : x \in X \} \). Player II responds with \( \{ \mathcal{N}(x) : x \in F_0 \} \), where \( F_0 \in [X]^{< \omega} \). Let \( V_0 = \bigcup \{ \mathcal{N}(x) : x \in F_0 \} \), and let I then play \( \{ V_0 \cup \mathcal{N}(x) : x \in X \setminus V_0 \} \). Then similarly, if II's reply is \( \{ V_0 \cup \mathcal{N}(x) : x \in F_1 \} \), where \( F_1 \in [X \setminus V_0]^{< \omega} \), let \( V_1 = V_0 \cup \bigcup \{ \mathcal{N}(x) : x \in F_1 \} \) and let I play \( \{ V_1 \cup \mathcal{N}(x) : x \in X \setminus V_1 \} \), and so on.

This defines a strategy for Player I.
**Proof of Aurichi’s theorem**

Assume $X$ Menger.
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**Proof of Aurichi’s theorem**

Assume $X$ Menger.

Let $N$ be a neighborhood assignment on $X$.

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Player II responds with $\{N(x) : x \in F_0\}$, $F_0 \in [X]<\omega$.

Let $V_0 = \bigcup\{N(x) : x \in X\}$, and let I then play $\{V_0 \cup N(x) : x \in X \setminus V_0\}$.
Then similarly, if II’s reply is $\{V_0 \cup N(x) : x \in F_1\}$, where $F_1 \in [X \setminus V_0]<\omega$,
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Then similarly, if II’s reply is $\{ V_0 \cup N(x) : x \in F_1 \}$, where $F_1 \in [X \setminus V_0]^{<\omega}$, let $V_1 = V_0 \cup \bigcup \{ N(x) : x \in F_1 \}$ and let I play

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$X$ is Menger, so this can’t be a winning strategy.
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Therefore there is some play of the game with $I$ using this strategy such that, if $F_0, F_1, \ldots$ code the plays of $II$, then

$$X = \bigcup_{n \in \omega} V_n = \bigcup \{N(x) : x \in \bigcup_{n \in \omega} F_n\}.$$
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$X = \bigcup_{n \in \omega} V_n = \bigcup \{ N(x) : x \in \bigcup_{n \in \omega} F_n \}$.

Let $D = \bigcup_{n \in \omega} F_n$. Then $N(D)$ covers $X$. 
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Let $D = \bigcup_{n \in \omega} F_n$. Then $N(D)$ covers $X$.

Since for each $n$, we have $F_n \subset V_n$ and $F_{n+1} \cap V_n = \emptyset$, it is easy to check that $D$ is a closed discrete subset of $X$. Hence $X$ is a $D$-space. □
Proof of Hurewicz’s theorem

To show: Menger $\Rightarrow I \uparrow M(X)$
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To show: Menger $\Rightarrow I \uparrow M(X)$
Suppose $X$ Menger, and consider any fixed strategy by I. We show it can be defeated.

$X$ is Lindelöf, so wlog, I always chooses a countable open cover. Since $\neg I$ is choosing a finite subcollection, it doesn’t hurt I to be restricted to increasing open covers.

In this case, it doesn’t harm $\neg I$ to be restricted to choosing a single element of the cover.

Finally, if $\neg I$ chooses $U$, I may as well make the first member of the increasing open cover that is his response contain $U$.

To summarize: I chooses countable increasing open cover, each member of which contains $\neg I$’s previous move. $\neg I$ chooses a member of I’s cover. Want to show that $\neg I$ can defeat I’s strategy.
To show: $\text{Menger} \implies I \nuparrow M(X)$

Suppose $X$ Menger, and consider any fixed strategy by $I$. We show it can be defeated.

$X$ is Lindelöf, so wlog, $I$ always chooses a countable open cover.
Proof of Hurewicz’s theorem

To show: Menger $\Rightarrow I \nuparrow M(X)$
Suppose $X$ Menger, and consider any fixed strategy by $I$. We show it can be defeated.

$X$ is Lindelöf, so wlog, $I$ always chooses a countable open cover.
Since $II$ is choosing a finite subcollection, it doesn’t hurt $I$ to be restricted to increasing open covers.
To show: Menger $\Rightarrow I \not\uparrow M(X)$
Suppose $X$ Menger, and consider any fixed strategy by $I$. We show it can be defeated.

$X$ is Lindelöf, so wlog, $I$ always chooses a countable open cover. Since $II$ is choosing a finite subcollection, it doesn't hurt $I$ to be restricted to increasing open covers.

In this case, it doesn't harm $II$ to be restricted to choosing a single element of the cover.
Proof of Hurewicz’s theorem

To show: Menger ⇒ I ↑ M(X)
Suppose X Menger, and consider any fixed strategy by I. We show it can be defeated.
X is Lindelöf, so wlog, I always chooses a countable open cover.
Since II is choosing a finite subcollection, it doesn’t hurt I to be restricted to increasing open covers.
In this case, it doesn’t harm II to be restricted to choosing a single element of the cover.
Finally, if II chooses \( U \), I may as well make the first member of the increasing open cover that is his response contain \( U \).
Proof of Hurewicz’s theorem

To show: Menger ⇒ I ∉ M(X)
Suppose X Menger, and consider any fixed strategy by I. We show it can be defeated.
X is Lindelöf, so wlog, I always chooses a countable open cover.
Since II is choosing a finite subcollection, it doesn’t hurt I to be restricted to increasing open covers.
In this case, it doesn’t harm II to be restricted to choosing a single element of the cover.
Finally, if II chooses $U$, I may as well make the first member of the increasing open cover that is his response contain $U$.
To summarize: I chooses countable increasing open cover, each member of which contains II’s previous move. II chooses a member of I’s cover.
Proof of Hurewicz’s theorem

To show: Menger $\Rightarrow I \uparrow M(X)$

Suppose $X$ Menger, and consider any fixed strategy by I. We show it can be defeated.

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To summarize: I chooses countable increasing open cover, each member of which contains II’s previous move. II chooses a member of I’s cover. Want to show that II can defeat I’s strategy.
Let $\{U_n\}_n$ be I's first move using the strategy. If II responds with $U_n$, let $\{U_{nm}\}_m$ be I's next move. Then if II plays $U_{nm}$, let $\{U_{nmk}\}_k$ be I's reply. In this way we define a "game tree" $\{U_\sigma\}_\sigma \in \omega < \omega$. (Let $U_\emptyset = \emptyset$.)

We need to show that there is a play of the game, i.e., a branch of the game tree, for which the corresponding open sets cover. That is, we want $f: \omega \to \omega$ such that $X = \bigcup_{n \in \omega} U_{f \upharpoonright n}$.

A naive idea is to apply the Menger property to the countably many covers $\{U_{\sigma n}\}_n$. There is a choice of one member of each that covers. But this doesn't get you a branch that covers!
Let \( \{ U_n \}_n \) be I’s first move using the strategy.

\[
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In this way we define a “game tree” \( \{ U_\sigma \}_{\sigma \in \omega < \omega} \). (Let \( U_\emptyset = \emptyset \).) We need to show that there is a play of the game, i.e., a branch of the game tree, for which the corresponding open sets cover.
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In this way we define a “game tree” \( \{ U_\sigma \}_{\sigma \in \omega < \omega} \). (Let \( U_\emptyset = \emptyset \).) We need to show that there is a play of the game, i.e., a branch of the game tree, for which the corresponding open sets cover. That is, we want \( f : \omega \to \omega \) such that \( X = \bigcup_{n \in \omega} U_f \upharpoonright n \). A naive idea is to apply the Menger property to the countably many covers \( \{ U_\sigma \}_{\sigma \in \omega < \omega} \). There is a choice of one member of each that covers. But this doesn’t get you a branch that covers!
Instead, we use the game tree to define covers \( \{ V_k^n \}_k \) as follows.

Let \( V_0^k = U_k \); then \( \{ V_0^k \}_k \) is increasing open cover.

Next let \( V_1^k = U_k \cap U_0^k \cap U_1^k \cap \ldots \cap U_{k-1}^k \).

Since we have assumed \( U_{ik} \supset U_i \), note that \( V_1^k = \bigcap_{\sigma \in \omega \leq 1} U_\sigma^k \), i.e.,

\( V_1^k \) is the intersection of all \( k \)th terms of all of I's plays from rounds 0 and 1.

Claim. \( \{ V_1^k \}_k \) is an increasing open cover.

Increasing: \( V_{k+1}^k = U_{k+1} \cap \bigcap_{i \leq k} U_i \supset U_k \cap \left( \bigcap_{i < k} U_{ik} \right) \cap U_{k+1}^k = V_1^k \).

Cover: Let \( x \in X \). There is \( k_0 \) with \( x \in U_{k_0}^0 \).

For each \( i < k_0 \), there is \( l_i \) such that \( x \in U_{il_i} \).

Let \( k \) be greater than \( k_0 \) and the \( l_i \)’s.

Then \( V_1^k \) is the intersection of (i) \( U_k^k \); (ii) \( \bigcap_{i < k_0} U_{ik} \), and (iii) \( \bigcap_{k_0 \leq i < k} U_{ik} \).

Now \( x \) is in (i) since \( k \geq k_0 \), \( x \) is in (ii) since \( k \geq l_i \) for all \( i < k_0 \), and \( x \) is in (iii) since \( U_{ik} \supset U_{k_0}^0 \) for all \( i \geq k_0 \).

So \( x \in V_1^k \).
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So $x \in V^1_k$. 

Similarly, for each $n$, $\{V_k^n\}_k$ is an increasing open cover, where

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Since $X$ is Menger, there is $f : \omega \to \omega$ such that $X = \bigcup_{n \in \omega} V^n_{f(n)}$. 

But $V^n_{f(n)} \subset U(f \upharpoonright n) \downharpoonright f(n) = U_{f(n) + 1}$.

So $X = \bigcup_{n \in \omega} U_{f(n) + 1}$, which corresponds to a play of the game in which I’s strategy has been defeated.
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TUTORIAL: Topological games, lecture II
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In the $n$th round, $O$ chooses an open $O_n \supset H$, and $P$ chooses a point $p_n \in O_n$. The game continues until $p_n \to H$ in the sense that every open superset of $H$ contains $p_n$ for all but finitely many $n \in \omega$.

If $O \uparrow G(H, X)$, we call $H$ a W-set in $X$. 
Convergence game

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P. Sharma proved $X$ is a $w$-space iff for each $x \in X$:

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This showed the class of $w$-spaces equivalent to a class introduced by Arhangel’skii (Fréchet $\alpha_2$-spaces).
Theorem

1. First-countable $\Rightarrow$ W-space $\Rightarrow$ w-space $\Rightarrow$ Fréchet;
2. W-spaces are hereditary and countably productive (closed under $\Sigma$-products, even).

Consider the game $G(H, X)$ played in a compact Hausdorff space $X$, where $H \subset X$ is closed. O will have a winning strategy if $H$ has a countable "outer base", i.e., there is a countable collection of open supersets of $H$ such that every open superset of $H$ contains one. By compactness, $H$ has a countable outer base iff $H$ is $G_\delta$.

A classical result:

Theorem (Schneider) A compact Hausdorff space $X$ is metrizable iff the diagonal $\Delta$ of $X$ is $G_\delta$ in $X^2$. 

Gary Gruenhage Auburn University ( ) Topological games May 31, 2010 26 / 54
Theorem

1. First-countable $\Rightarrow$ W-space $\Rightarrow$ w-space $\Rightarrow$ Fréchet;
2. W-spaces are hereditary and countably productive (closed under $\Sigma$-products, even).

Consider the game $G(H, X)$ played in a compact Hausdorff space $X$, where $H \subset X$ is closed.
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$\Omega$ will have a winning strategy if $H$ has a countable “outer base”, i.e., there is a countable collection of open supersets of $H$ such that every open superset of $H$ contains one. By compactness, $H$ has a countable outer base iff $H$ is $G_\delta$. 

A classical result:

Theorem (Schneider)

A compact Hausdorff space $X$ is metrizable iff the diagonal $\Delta$ of $X$ is $G_\delta$ in $X^2$. 

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O can have a winning strategy in $G(H, X)$ for compact $X$ without $H$ being $G_\delta$: 

consider one-point compactification of uncountable discrete space.

To say that O has winning strategy in $G(\Delta, X^2)$ is weaker than to say the diagonal is $G_\delta$ in $X^2$.

Definition. Compact $X$ is Corson compact iff $X$ is homeo to subspace of $\Sigma_{\mathbb{R}^\kappa} = \{ x \in \mathbb{R}^\kappa : |\{ \alpha < \kappa : x(\alpha) \neq 0 \} | \leq \omega \}$ or equivalently $X$ has a point-countable $T_0$-separating cover by open $F_\sigma$'s.

Theorem (G.G., 1984) A compact space $X$ is Corson compact iff $O \uparrow G(H, X)$. 

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**Theorem (G.G., 1984)**

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Eberlein compact: homeo to weakly compact subset of Banach space

\[ \text{Eberlein} \Rightarrow \text{Corson} \]

Theorem (G.G., 1986)

A compact \( X \) is Eberlein compact iff \( O \) has a winning strategy in \( G(\Delta, X^2) \) which depends only on \( P \)'s last move and the number of the move (i.e., a Markov strategy).

Strong Eberlein = scattered Eberlein

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DEF (Tkachuk): A space $X$ is **monotonically monolithic** if one can assign to each $F \in [X]<\omega$ a countable collection $\mathcal{N}(F)$ of subsets of $X$ such that

1. $F \subseteq F' \Rightarrow \mathcal{N}(F) \subseteq \mathcal{N}(F')$;
2. If $U$ open and $x \in A \cap U$ there is $F \in [A]<\omega$ and $N \in \mathcal{N}(F)$ with $x \in N \subseteq U$.

(That is, $\bigcup F \in [A]<\omega \mathcal{N}(F)$ includes a network at every point of $A$.)

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D-space: $x \in N(x)^o \ \forall x \in X \Rightarrow \exists$ closed discrete $D$ with $\{N(x) : x \in D\}$ covering $X$
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Question (Tkachuk)

- Monotonically monolithic compact $\Rightarrow$ Corson compact?
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Yes:

Theorem (G.G., 2010)

If $X$ is compact and monotonically monolithic, then $X$ is Corson compact.
Lemma

Suppose $X$ is compact and monotonically monolithic. Then $O$ has a winning strategy in $G(H, X)$ for any closed $H \subset X$. 

To prove the Theorem from the Lemma:

- $X$ compact monotonically monolithic
- $\Rightarrow$ ditto for $X^2$
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$O$ looks at $\mathcal{N} \left( \{ p_0 \} \right) = \{ N_{00}, N_{01}, \ldots \}$

$O$ chooses open $O_1 \supset H$ s.t. $O_1 \cap N_{00} = \emptyset$ if such $O_1$ exists; else $O_1 = X$.

$P$ chooses $p_1 \in O_1$.

Let $\mathcal{N} \left( \{ p_0, p_1 \} \right) = \{ N_{10}, N_{11}, \ldots \}$

$O$ chooses $O_2 \subset O_1$ s.t., whenever possible for $i, j < 2$, $O_2 \cap N_{ij} = \emptyset$.

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Let $q \in U$ open, $\bar{U} \cap H = \emptyset$.

$\exists k \in \omega$ with $N \in \mathcal{N}(\{p_i\}_{i \leq k})$ and $q \in N \subset U$

$q \not\in O \Rightarrow q$ not limit of $\{p_n\}_n$. 

Contradiction.
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Use the point-countable $T_0$-separating open cover characterization. Can get such if can show $X^2 \setminus \Delta$ is metalindelöf, i.e., every open cover has a point-countable open refinement.
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Then we have:

**Theorem**

Let $X$ be compact and countably tight, and $H$ closed. Then $O$ has a winning strategy in $G(H, X)$ iff $X \setminus H$ is metalindelöf.
It is useful to view the game as a game in $X \setminus H$, with players K and P.
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In the $n^{th}$ round, K chooses a compact $K_n \subset X \setminus H$ (the complement of a play by O), and P responds with a point $p_n \notin K_n$. 

K wins if $p_n \to \infty$ (i.e., $\{p_n : n \in \omega\}$ is closed discrete in $X \setminus H$). 

Replacing $X \setminus H$ with $X$, let us denote this game by $G_{K,P}(X)$. 

Then the result becomes:

**Theorem**

Let $X$ be locally compact and countably tight. Then K has a winning strategy in $G_{K,P}(X)$ iff $X$ is metalindel"of.

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Proof. If $X$ is metalindelöf, then there is a point-countable cover $\mathcal{U}$ of $X$ by open sets with compact closures.
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$K$ wins by looking at the countably many members of $\mathcal{U}$ containing $P$'s chosen point at each round, and choosing an increasing sequence of compact sets that eventually cover every one of these members of $\mathcal{U}$. It is easy to check that this wins for $K$. 
Outline of proof, elementary submodel style

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Now suppose $K$ has a winning strategy $\sigma$, and let $\mathcal{U}$ be a cover of $X$ by open sets with compact closures. Let $M$ be an elementary submodel (of some sufficiently large $H(\theta)$) with $X, \mathcal{U}, \sigma \in M$. 
**Key Claim.** \( M \cap X \subset \bigcup (M \cap U) \).
Key Claim. $\overline{M \cap X} \subset \bigcup (M \cap U)$.

Proof of Key Claim. Suppose $p \in \overline{M \cap X} \setminus \bigcup (M \cap U)$. Let $p \in U_p \in \mathcal{U}$. Since $M$ also contains a finite subset of $U$ covering $\bigcup U_0$, we have $p \not\in \bigcup U_0$. So there exists $p_{n+1} \in U_p \cap (M \cap X) \setminus \bigcup U_0$.

It follows that if $K$ uses the strategy $\sigma$, $P$ can always choose a point in $U_p \cap (M \cap X)$. But then $K$ loses the game, a contradiction which completes the proof of Key Claim.
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Proof of Key Claim. Suppose \( p \in \overline{M \cap X} \setminus \bigcup (M \cap \mathcal{U}) \). Let \( p \in U_p \in \mathcal{U} \).

Suppose \( F = \{p_0, p_1, \ldots, p_n\} \subset U_p \cap (M \cap X) \). Then \( \sigma(F) \) is compact and in \( M \) so there exists a finite \( \mathcal{U}_0 \subset \mathcal{U} \) in \( M \) covering \( \sigma(F) \).
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Since $M$ also contains a finite subset of $\mathcal{U}$ covering $\overline{U_0}$, we have $p \notin \overline{U_0}$. So there exists $p_{n+1} \in U_p \cap (M \cap X) \setminus \overline{U_0}$.
Key Claim. $\overline{M \cap X} \subset \bigcup (M \cap \mathcal{U})$.

Proof of Key Claim. Suppose $p \in \overline{M \cap X} \setminus \bigcup (M \cap \mathcal{U})$. Let $p \in U_p \in \mathcal{U}$. Suppose $F = \{p_0, p_1, \ldots, p_n\} \subset U_p \cap (M \cap X)$. Then $\sigma(F)$ is compact and in $M$ so there exists a finite $\mathcal{U}_0 \subset \mathcal{U}$ in $M$ covering $\sigma(F)$.

Since $M$ also contains a finite subset of $\mathcal{U}$ covering $\overline{\mathcal{U}_0}$, we have $p \notin \overline{\mathcal{U}_0}$. So there exists $p_{n+1} \in U_p \cap (M \cap X) \setminus \overline{\mathcal{U}_0}$.

It follows that if $K$ uses the strategy $\sigma$, $P$ can always choose a point in $U_p \cap (M \cap X)$. But then $K$ loses the game, a contradiction which completes the proof of Key Claim.
Since there is an $M$ with $\mathcal{U} \subset M$, the next claim completes the proof of the theorem.
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**Claim 2.** There is a point-countable open refinement $\mathcal{V}_M$ of $M \cap \mathcal{U}$ covering $\bigcup (M \cap \mathcal{U})$. 
Since there is an $M$ with $\mathcal{U} \subset M$, the next claim completes the proof of the theorem.

**Claim 2.** There is a point-countable open refinement $\mathcal{V}_M$ of $M \cap \mathcal{U}$ covering $\bigcup (M \cap \mathcal{U})$.

**Proof of Claim 2.** By induction on $|M| = \kappa$. Write $M = \bigcup \{ M_\alpha : \alpha < \kappa \}$ and use Key Claim to put together point-countable refinements of $M_\alpha \cap \mathcal{U}$. 
$G_{K,L}(X)$ is defined just like $G_{K,P}(X)$, except that $P$, who will be renamed $L$, chooses compact sets instead of points, i.e., $L$’s $n^{th}$ play is a compact set $L_n$ missing $K$’s previous move $K_n$. 

$K$ wins iff $\{L_i\}_{i \in \omega}$ is a discrete collection.

$G_{K,L}(X)$ is the same as $G_{K,L}(X)$, except that $K$ wins iff $\{L_i\}_{i \in \omega}$ has a discrete open expansion.

It is easy to see that $K$ has a winning strategy in any locally compact $\sigma$-compact space: $K$ simply chooses at the $n^{th}$ play the $n^{th}$ set in an increasing sequence of compact sets whose interiors cover the space.

It is nearly as easy to see that $K$ wins if $X$ is a topological sum of locally compact $\sigma$-compact spaces, i.e., whenever $X$ is locally compact and paracompact. The next theorem shows we have an equivalence:
$G_{K,L}(X)$ is defined just like $G_{K,P}(X)$, except that $P$, who will be renamed $L$, chooses compact sets instead of points, i.e., $L$’s $n^{th}$ play is a compact set $L_n$ missing $K$’s previous move $K_n$.

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\(G^{o}_{K,L}(X)\) is the same as \(G_{K,L}(X)\), except that \(K\) wins iff \(\{L_i\}_{i \in \omega}\) has a discrete open expansion.

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\( G_{K,L}(X) \) is defined just like \( G_{K,P}(X) \), except that \( P \), who will be renamed \( L \), chooses compact sets instead of points, i.e., \( L \)'s \( n^{th} \) play is a compact set \( L_n \) missing \( K \)'s previous move \( K_n \).

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It is nearly as easy to see that \( K \) wins if \( X \) is a topological sum of locally compact \( \sigma \)-compact spaces, i.e., whenever \( X \) is locally compact and paracompact. The next theorem shows we have an equivalence:
Let $X$ be a locally compact space. Then the following are equivalent:

1. $K \uparrow G_{K,L}(X)$;
2. $K \uparrow G_{K,L}^o(X)$;
3. $X$ is paracompact.
Why $G_{K,L}^o(X)$? Because it is the most natural one for attacking the following open problem:
Why $G^0_{K,L}(X)$? Because it is the most natural one for attacking the following open problem:

**Question**

For what (completely regular) spaces $X$ is $C_k(X)$ a Baire space?

($C_k(X)$ is the space of continuous real-valued functions on $X$ with the compact-open topology.)
Theorem (McCoy, Ntantu)

1. If $\text{NE} \uparrow \text{BM}(C_k(X))$ then $K \uparrow G_{K,L}^o(X)$;
2. If $C_k(X)$ is Baire, then $L \nuparrow G_{K,L}^o(X)$;
3. If $X$ is locally compact, then $\text{NE} \uparrow \text{BM}(C_k(X))$ iff $K \uparrow G_{K,L}^o(X)$. 

Proof of (2)

Suppose $L \uparrow G_{K,L}^o(X)$.

Claim. $E \uparrow \text{BM}(C_k(X))$ (so $C_k(X)$ not Baire, contradiction).

W.l.o.g., in the $n$th round, Non-empty chooses a basic open set of the form $B(K_n, f_n, \epsilon_n) = \{g \in C_k(X) : \forall x \in K_n |g(x) - f_n(x)| < \epsilon_n\}$, where $K_n$ is compact.

Let $L_n$ be $L$'s response to $K_n$ in $G_{K,L}^o(X)$ using a winning strategy.

Then Empty plays $B(K_n, f_n, \epsilon_n) \cap B(L_n, c_n, 1/3)$, where $c_n = \text{constant n}$.

Suppose $\phi \in \bigcap_{n \in \omega} B(L_n, c_n, 1/3)$.

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Then \( \text{Empty} \) plays \( B(K_n, f_n, \epsilon_n) \cap B(L_n, c_n, 1/3) \), where \( c_n = \text{constant n} \).
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Suppose $\phi \in \bigcap_{n \in \omega} B(L_n, c_n, 1/3)$.
Theorem (McCoy, Ntantu)

1. If NE ↑ BM(\(C_k(X)\)) then K ↑ \(G^o_{K,L}(X)\);
2. If \(C_k(X)\) is Baire, then L ↑ \(\nabla G^o_{K,L}(X)\);
3. If \(X\) is locally compact, then NE ↑ BM(\(C_k(X)\)) iff K ↑ \(G^o_{K,L}(X)\).

Proof of (2) Suppose L ↑ \(G^o_{K,L}(X)\)

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Theorem (Ma, GG)

If $X$ is locally compact, then $C_k(X)$ is Baire iff $L \uparrow G_{K,L}(X)$.
**Theorem (Ma, GG)**

If $X$ is locally compact, then $C_k(X)$ is Baire iff $L \nrightarrow G^o_{K,L}(X)$.

**Question**

Is it true that for any completely regular space $X$, $C_k(X)$ is Baire iff $L \nrightarrow G^o_{K,L}(X)$? That $NE \uparrow BM(C_k(X))$ iff $K \uparrow G^o_{K,L}(X)$?
A non-game theory characterization of “$L$ has no winning strategy in $G_{K,K}^L(X)$”
A non-game theory characterization of “$L$ has no winning strategy in $G_{K,L}^o(X)$”

**Definition.** A collection $\mathcal{L}$ of non-empty compact subsets of $X$ is said to move off the compact sets if for every compact subset $K$ of $X$, there is some $L \in \mathcal{L}$ with $K \cap L = \emptyset$. 
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The space $X$ is said to have the **Moving Off Property (MOP)** iff every collection $\mathcal{L}$ which moves off the compact sets contains an infinite subcollection which has a discrete open expansion.
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**Theorem**

TFAE:

1. $X$ has the MOP;
2. $L \upharpoonright G_{K,L}(X)$.

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Gary Gruenhage  Auburn University  ()  Topological games  May 31, 2010  45 / 54
A non-game theory characterization of “$L$ has no winning strategy in $G_{K,L}(X)$”

**Definition.** A collection $\mathcal{L}$ of non-empty compact subsets of $X$ is said to *move off* the compact sets if for every compact subset $K$ of $X$, there is some $L \in \mathcal{L}$ with $K \cap L = \emptyset$.

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**Theorem**

TFAE:

1. $X$ has the MOP;
2. $L \not\uparrow G_{K,L}(X)$.

**Question**

Does $X$ have MOP iff $C_k(X)$ Baire?
There is some subtlety in determining when $X$ (even locally compact $X$) has the property “L has no winning strategy in $G_{K,L}(X)$".
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Consider spaces $T \cup A$, where $T$ is the Cantor tree and $A$ is a subset of the Cantor set.
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**Theorem (Ma)**

The following are equivalent:

1. $C_k(T \cup A)$ is a Baire space;
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$A \subset \mathbb{R}$ is a $\gamma$-set if, given any collection $\mathcal{U}$ of open sets such that any finite subset of $A$ is contained in some member of $\mathcal{U}$, there are $U_0, U_1, \ldots$ in $\mathcal{U}$ such that $A \subset \bigcup_{n \in \omega} \bigcap_{i \geq n} U_i$. 
Todorcevic showed that it is consistent for there to be two $\gamma$-sets $A_0$ and $A_1$ whose topological sum is not a $\gamma$-set.

Since $C_k(X_0 \oplus X_1) \sim C_k(X_0) \times C_k(X_1)$, Ma obtained the following corollary.

**Corollary**

There are, consistently, two function spaces with the compact-open topology which are Baire but whose product is not.

But we don't know about ZFC examples.

**Question**

Are there examples in ZFC of two Baire function spaces whose product is not Baire?
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Some other games and/or applications

Tel’garsky(1975) The countably metacompact game $CM(X)$
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Theorem

$X \times M$ is normal for every metrizable space $M$ iff $X$ is normal and $II \uparrow CM(X)$. 
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II \( \uparrow S(X) \) if \( \bigcup_{n \in \omega} \mathcal{V}_n \) covers \( X \)

**Theorem**

*(Babinkostova)*

1. II \( \uparrow S(X) \) iff \( X \) is countable dimensional;
2. II has winning strategy in game of length \( k + 1 \) iff \( X \) is \( \leq k \) dimensional.
A space $X$ is *selectively separable* ($SS$) if, given dense sets $D_0, D_1, \ldots$, there are finite $F_i \subset D_i$ with $\bigcup_{n \in \omega} F_n$ dense.
A space $X$ is selectively separable (SS) if, given dense sets $D_0, D_1, \ldots$, there are finite $F_i \subset D_i$ with $\bigcup_{n \in \omega} F_n$ dense.

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A space $X$ is **selectively separable (SS)** if, given dense sets $D_0, D_1, \ldots$, there are finite $F_i \subset D_i$ with $\bigcup_{n \in \omega} F_n$ dense.

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$X$ is $\text{SS}^+$ if II $\uparrow \text{SS}(X)$.
A space $X$ is selectively separable (SS) if, given dense sets $D_0, D_1, \ldots$, there are finite $F_i \subset D_i$ with $\bigcup_{n \in \omega} F_n$ dense.

The game $SS(X)$ (Dow, Barman): In round $n$, I chooses dense $D_n$, II chooses finite $F_n \subset D_n$. II wins if $\bigcup_{n \in \omega} F_n$ is dense.

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Countable $\pi$-base $\Rightarrow SS^+ \Rightarrow SS$

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SS $\not\Rightarrow SS^+$.
Theorem

\((Dow)\) \(X\) countable \(SS^+\) \(\Rightarrow\) II has Markov winning strategy in \(SS(X)\).

So, for each dense \(D\), for each \(n \in \omega\), one can assign finite \(F(D, n) \subset D\) such that, if \(D_0, D_1, \ldots\) are dense, then \(\bigcup_{n \in \omega} F(D_n, n)\) is dense.
Idea of proof.
Let $\sigma$ be winning strategy for II.

For each possible first round reply $F$, choose $D(F)$ dense such that $\sigma(D(F)) = F$.

For each possible second round reply $F'$ of II, choose dense $D(F, F')$ such that $\sigma(D(F), D(F, F')) = F'$.

Etc.

This constructs a (countable) tree of finite sequences of dense sets. Let $t_0, t_1, ...$ be the nodes of the tree. The Markov winning strategy for II is:

Given dense $D$ in round $n$, II plays $\sigma(t_n \langle D \rangle)$.

[Same result for any game with II having only countably many responses, and I's legal plays unchanged during the game.]
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Let $X$ be countable $\Rightarrow$ II has only countably many possible replies.
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SURVEYS


OTHER REFERENCES


