Homework 8 - Answers

1. Using the hint

$$P\left(\left| \frac{1}{n} \sum_{i=1}^{n} w_i - \mu \right| > \epsilon \right) < \frac{1}{\epsilon^2} E\left\{ \left(\frac{1}{n} \sum_{i=1}^{n} w_i - \mu \right)^2 \right\}.$$

$$= \frac{1}{\epsilon^2} V\left(\frac{1}{n} \sum_{i=1}^{n} w_i \right)$$

$$= \frac{1}{\epsilon^2} \frac{1}{n^2} n V(w_i)$$

$$= \frac{1}{\epsilon^2} \frac{1}{n} \sigma^2$$

Taking limits as $n \to \infty$ on both sides gives.

$$\lim_{n \to \infty} P\left(\left| \frac{1}{n} \sum_{i=1}^{n} w_i - \mu \right| > \epsilon \right) = 0$$

Hence, by definition of convergence in probability

$$\frac{1}{n} \sum_{i=1}^{n} w_i \xrightarrow{P} \mu.$$

2 (a) Let $\{x_i^T\}_{i=1,2,\dots,n}$ be an IID sequence of vectors such that $E(X_{ik}^2) < \infty$. Then, $E(X_{ik} X_{ik'}) < \infty$

Since, $\frac{X^TX}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i^T X_i$, where $x_i^T x_i = [X_{ik} X_{ik'}]_{k,k'=1}^{K+1}$

By Kolmogorov's LLN $\frac{X^TX}{n} \xrightarrow{P} E(X_i^T X_i) = Q$.

(b) Similarly, if $\{U_i\}_{i=1,2,\dots,n}$ is IID with $E(U_i|X_i) = 0$

then, $\frac{X^Tu}{n} = \frac{1}{n} \sum_{i=1}^{n} X_i^T u_i$ and $E(x_i^T u_i) = 0$.

Hence, by Kolmogorov's LLN $\frac{X^Tu}{n} \xrightarrow{P} 0$. (a vector).

(c) $\hat{\beta} = \beta + \left(\frac{X^TX}{n}\right)^{-1} \frac{X^Tu}{n}$

Since $\frac{X^TX}{n} \xrightarrow{P} Q$ and $\frac{X^Tu}{n} \xrightarrow{P} 0$, then

$$\hat{\beta} \xrightarrow{P} \beta + Q^{-1} \cdot 0 = \beta$$

(d) $\sqrt{n}(\hat{\beta} - \beta) = \left(\frac{X^TX}{n}\right)^{-1} \frac{X^Tu}{\sqrt{n}}$

By Lèvy's CLT, $\frac{X^Tu}{\sqrt{n}} \xrightarrow{d} N(0, Q)$. Since, for any $W_n \xrightarrow{P} W$ and $V_n \xrightarrow{d} V$ we have $W_n V_n \xrightarrow{d} W V$.

Then, $\sqrt{n}(\hat{\beta} - \beta) \xrightarrow{d} Q^{-1} Z$, $Z \sim N(0, Q)$.

Hence, $Q^{-1} Z \sim N(0, Q^{-1} Q Q^{-1}) := N(0, \sigma^2 Q^{-1})$.

3. From Lévy's CLT

$$\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sqrt{\frac{\sigma^2}{n}}} \xrightarrow{d} Z \sim N(0,1).$$

Since $s^2 \xrightarrow{P} \sigma^2$

$$\frac{\frac{1}{n}\sum_{i=1}^{n} X_i - \mu}{\sqrt{\frac{s^2}{n}}} \xrightarrow{d} Z$$

Thus, under $H_0: \mu = c$

$$t_n = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i - c}{\sqrt{s^2/n}} \xrightarrow{d} Z.$$

Then, if $|t_n| \leq t^*_{1-\frac{\alpha}{2}}$ accept H_0, where $t^*_{1-\alpha/2}$ is the $1-\frac{\alpha}{2}$ quantile associated with a standard normal, and $\alpha \in (0,1)$. Otherwise, reject H_0.