Homework 1. (Answers). Question 1, problems 7, 11, 13 from book.

7. Let $s :=$ salary, $e :=$ exper

$$\log(s) = 10.6 + 0.27e$$

(i) If $e = 0$, $\log(s) = 10.6$ and $s = \exp(10.6)$
$= 40{,}134.84$

(ii) $\log(s') = 10.6 + 0.27 \times (e+5)$
$\log(s) = 10.6 + 0.27 \times e$

Then,

$$\log(s') - \log(s) = 0.27(e+5-e)$$

$$\frac{\Delta s}{s} = \frac{s'-s}{s} \approx \log\frac{s'}{s} = 0.27 \times \underbrace{5}_{\Delta e}.$$ As in A.28

$$100\frac{\Delta s}{s} = 100 \underbrace{0.27}_{\beta_1} \underbrace{\Delta e}_{5} \Rightarrow 100\frac{\Delta s}{s} = 135\%$$

(iii) $\log(s') = 10.6 + 0.27 \times 5 \Rightarrow s' = \exp(11.95) = 154{,}817.15$

$\% \Delta s = 285\%$

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$\%\,\Delta s = 285\%$.

11. (i) $\left.\begin{array}{l} y_1 = \beta_0 + \beta_1 x_1 \\ y_2 = \beta_0 + \beta_1 x_2 \end{array}\right\} \Rightarrow y_1 + y_2 = 2\beta_0 + \beta_1 (x_1 + x_2) \Rightarrow \bar{y} = \beta_0 + \beta_1 \bar{x}.$

(ii) $y_i = \beta_0 + \beta_1 x_i$ for $i = 1, 2, \dots, n$. Then,

$$\bar{y} = \frac{1}{n} \sum_{i=1}^{n} y_i = \frac{n}{n}\beta_0 + \beta_1 \frac{\sum_{i=1}^{n} x_i}{n} = \beta_0 + \beta_1 \bar{x}$$

13. (i) We give a counter exaple $x_i = i$, $n=2$. Then,

$$\sum_{i=1}^{n} \frac{1}{x_i} = 1 + \frac{1}{2} = 1.5$$

$$\sum_{i=1}^{2} x_i = 3 \quad \text{and} \quad \frac{1}{\sum_{i=1}^{2} x_i} = \frac{1}{3}, \quad \text{so} \quad \sum_{i=1}^{n} \frac{1}{x_i} \neq \frac{1}{\sum x_i}.$$

(ii) If $x_i = c > 0$, then $\sum_{i=1}^{n} \frac{1}{x_i} = \sum_{i=1}^{n} \frac{1}{c} = \frac{n}{c}$. Also, $\sum_{i=1}^{n} x_i = nc$

Hence, $\frac{1}{\sum_{i=1}^{n} x_i} = \frac{1}{nc} \neq \frac{n}{c}$. However, if $n = 1$, then $\frac{1}{c} = \frac{1}{c}$.

Homework 1. (Question 2)

1. $$E(X) = \int_0^1 x f(x)\, dx$$

$$= \int_0^1 x\, 2\, (1-x)\, dx = 2 \int_0^1 x(1-x)\, dx = 2 \int_0^1 (x - x^2)\, dx$$

$$= 2\left\{\int_0^1 x\, dx - \int_0^1 x^2\, dx\right\} = 2\left\{ \Big|_0^1 \frac{x^2}{2} - \Big|_0^1 \frac{x^3}{3} \right\}$$

$$= 2\left\{ \frac{1}{2} - \frac{1}{3} \right\} = 2\, \frac{1}{6} = \frac{1}{3}$$

$$V(X) = E(X^2) - [E(X)]^2$$

$$E(X^2) = \int_0^1 x^2\, 2(1-x)\, dx = 2 \int_0^1 (x^2 - x^3)\, dx$$

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$$= 2\left\{ \frac{1}{3} - \frac{1}{4} \right\} = 2 \cdot \frac{1}{12} = \frac{1}{6}$$

So, $$V(X) = \frac{1}{6} - \left[\frac{1}{3}\right]^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

Question 3

(1.) Since different questions do not measure a student's ability perfectly, the same student can get different SAT scores on different dates of the exam.

(2.) If $X \sim \mathcal{N}(5,4)$ then

- $X \leq 6 \Leftrightarrow X-5 \leq 1 \Leftrightarrow \frac{X-5}{2} \leq \frac{1}{2}$.

But $Z = \frac{X-5}{2}$ is a random variable with standard normal density. Hence,

$$P(X \leq 6) = P(Z \leq \tfrac{1}{2}) = 0.6915$$

- $X > 4 \Leftrightarrow X-5 > -1 \Leftrightarrow \frac{X-5}{2} > -\frac{1}{2}$

$$P(X>4) = P(Z > -\tfrac{1}{2}) = 1 - P(Z \leq -\tfrac{1}{2})$$
$$= 0.6915$$

- $|X-5| > 1 \Leftrightarrow X-5 > 1$ or $X-5 < -1$

$\Leftrightarrow X > 6$ or $X < 4$

$$P(|X-5|>1) = P(X>6) + P(X<4).$$
$$= 1 - P(X \leq 6) + 1 - P(X \geq 4)$$
$$= 2 - 0.6915 - 0.6915$$
$$= 0.617$$

(4) $X \in [0,1]$

$$P(X \leqslant x) = F(x) = 3x^2 - 2x^3 \quad \text{for} \quad 0 \leqslant x \leqslant 1$$

$$P(X \geqslant 0.6) = 1 - P(X \leqslant 0.6)$$
$$= 1 - \left[3(0.6)^2 - 2(0.6)^3\right]$$

$$= 0.352$$

(6) $$E(X) = \int_0^3 x \frac{1}{9} x^2 \, dx = \frac{1}{9} \Big|_0^3 \frac{x^4}{4} = \frac{9}{4}$$

11 (i) $E(X) = \frac{1}{2}1 + \frac{1}{2}(-1) = 0$

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$E\left(\frac{1}{X}\right) = \frac{1}{2}1 + \frac{1}{2}\frac{1}{2} = \frac{3}{4}$

(iii) In (i), $g(x) = x^2$ and $E(g(X)) = 1 \neq g(E(X)) = 0$

In (ii), $g(x) = \frac{1}{x}$ and $E(g(X)) = \frac{3}{4} \neq \frac{1}{E(X)} = \frac{1}{\frac{3}{2}} = \frac{2}{3}$

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