

Chapter 8

Laws of large numbers

We first discuss the notion of “tail equivalence” of a sequence of random variables. Here, the Borel-Cantelli Lemma is very useful. Recall that it says that if $\{E_n\}_{n \in \mathbb{N}}$ is a sequence of events with $\sum_{n=1}^{\infty} P(E_n) < \infty$, then $P\left(\limsup_{n \rightarrow \infty} E_n\right) = 0$.

Definition 8.1. *Two sequences of random variables $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ are tail equivalent if*

$$\sum_{n=1}^{\infty} P(\{\omega : X_n(\omega) \neq Y_n(\omega)\}) = \sum_{n=1}^{\infty} P(\{\omega : X_n(\omega) - Y_n(\omega) \neq 0\}) = \sum_{n=1}^{\infty} P(A_n) < \infty,$$

where $A_n = \{\omega : X_n(\omega) - Y_n(\omega) \neq 0\}$.

Theorem 8.1. *Suppose $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ are tail equivalent. Then,*

1. $\sum_{n=1}^{\infty} (X_n - Y_n)$ converges almost surely,
2. $\sum_{n=1}^{\infty} X_n$ converges as $\iff \sum_{n=1}^{\infty} Y_n$ converges as,
3. If there exists $a_n \rightarrow \infty$ and a random variable X such that $\frac{1}{a_n} \sum_{j=1}^n X_j \xrightarrow{as} X$, then $\frac{1}{a_n} \sum_{j=1}^n Y_j \xrightarrow{as} X$.

Proof. 1. By tail equivalence and the Borel-Cantelli Lemma $P\left(\limsup_{n \rightarrow \infty} A_n\right) = 0$. Now, recall that $\limsup_{n \rightarrow \infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{m=n}^{\infty} A_m = \bigcap_{n=1}^{\infty} C_n$, where $C_n := \bigcup_{m=n}^{\infty} A_m$. Consequently,

$$\left(\limsup_{n \rightarrow \infty} A_n\right)^c = \left(\bigcap_{n=1}^{\infty} C_n\right)^c = \bigcup_{n=1}^{\infty} C_n^c = \bigcup_{n=1}^{\infty} \left(\bigcup_{m=n}^{\infty} A_m\right)^c = \bigcup_{n=1}^{\infty} \bigcap_{m=n}^{\infty} A_m^c = \liminf_{n \rightarrow \infty} A_n^c.$$

Thus,

$$P\left(\liminf_{n \rightarrow \infty} \{\omega : X_n(\omega) = Y_n(\omega)\}\right) = P\left(\liminf_{n \rightarrow \infty} A_n^c\right) = 1 - P\left(\limsup_{n \rightarrow \infty} A_n\right) = 1.$$

Since $\liminf_{n \rightarrow \infty} A_n^c = \{\omega : \sum_{n=1}^{\infty} I_{A_n}(\omega) < \infty\}$, $P(\{\omega : \sum_{n=1}^{\infty} I_{A_n}(\omega) < \infty\}) = 1$. Hence, there exists a set of ω 's which occurs with probability 1, and in this set $X_n(\omega) = Y_n(\omega)$ for all but finitely many n . That is, for $\omega \in \{\omega : \sum_{n=1}^{\infty} I_{A_n}(\omega) < \infty\}$ there are only finitely many n for which $I_{\{X_n(\omega) \neq Y_n(\omega)\}}(\omega) = 1$. That is, there exists $N(\omega)$ such that for all $n > N(\omega)$, $I_{\{X_n(\omega) \neq Y_n(\omega)\}}(\omega) = 0$. Hence, in this same set,

$$\sum_{n=1}^{\infty} X_n(\omega) - \sum_{n=1}^{\infty} Y_n(\omega) = \sum_{n=1}^{N(\omega)} (X_n(\omega) - Y_n(\omega)) < \infty \text{ almost surely.}$$

2. Note that

$$\begin{aligned} \sum_{n=1}^{\infty} Y_n(\omega) &= \sum_{n=1}^{\infty} X_n(\omega) + \sum_{n=1}^{\infty} Y_n(\omega) - \sum_{n=1}^{\infty} X_n(\omega) \\ &= \sum_{n=1}^{\infty} X_n(\omega) - \sum_{n=1}^{\infty} (X_n(\omega) - Y_n(\omega)). \end{aligned}$$

If $\sum_{n=1}^{\infty} X_n(\omega)$ converges *as* and X_n and Y_n are tail equivalent, then both terms on the right side of the equality converge *as*, hence $\sum_{n=1}^{\infty} Y_n(\omega) < \infty$ *as*. Similarly, writing

$$\begin{aligned} \sum_{n=1}^{\infty} X_n(\omega) &= \sum_{n=1}^{\infty} X_n(\omega) + \sum_{n=1}^{\infty} Y_n(\omega) - \sum_{n=1}^{\infty} Y_n(\omega) \\ &= \sum_{n=1}^{\infty} Y_n(\omega) - \sum_{n=1}^{\infty} (Y_n(\omega) - X_n(\omega)). \end{aligned}$$

we conclude $\sum_{n=1}^{\infty} X_n(\omega) < \infty$ *as*.

3. Write

$$\begin{aligned} \frac{1}{a_n} \sum_{j=1}^n Y_j(\omega) &= \frac{1}{a_n} \sum_{j=1}^n (Y_j(\omega) - X_j(\omega) + X_j(\omega)) \\ &= \frac{1}{a_n} \sum_{j=1}^n (Y_j(\omega) - X_j(\omega)) + \frac{1}{a_n} \sum_{j=1}^n X_j(\omega) \\ &= \frac{1}{a_n} \sum_{j=1}^{N-1} (Y_j(\omega) - X_j(\omega)) + \frac{1}{a_n} \sum_{j=N}^n (Y_j(\omega) - X_j(\omega)) + \frac{1}{a_n} \sum_{j=1}^n X_j(\omega). \end{aligned}$$

As $n \rightarrow \infty$ the last term converges as to $X(\omega)$ by assumption. The second term converges to zero since $Y_j(\omega)$ and $X_j(\omega)$ are tail equivalent (and by 1), and the first term goes to 0 as $a_n \rightarrow \infty$. Hence, $\frac{1}{a_n} \sum_{j=1}^n Y_j(\omega) \xrightarrow{as} X(\omega)$.

■

The following definition and associated notation will be useful.

Definition 8.2. Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of random variables defined on (Ω, \mathcal{F}, P) and $\{s_n\}_{n \in \mathbb{N}}$ be a sequence in $(0, \infty)$. We write,

1. $X_n = O_p(s_n)$ if for all $\epsilon > 0$ and $n \in \mathbb{N}$, there exists $B_\epsilon > 0$ such that

$$P\left(\left\{\omega : \frac{|X_n(\omega)|}{s_n} > B_\epsilon\right\}\right) < \epsilon$$

2. $X_n = o_p(s_n)$ if $\frac{X_n}{s_n} \xrightarrow{p} 0$.

Theorem 8.2. (General Law of Large Numbers) Suppose $\{X_n\}_{n \in \mathbb{N}}$ is a sequence of independent random variables defined on (Ω, \mathcal{F}, P) and $S_n = \sum_{j=1}^n X_j$. If

1. $\sum_{j=1}^n P(\{\omega : |X_j(\omega)| > n\}) \rightarrow 0$ as $n \rightarrow \infty$ and

2. $\frac{1}{n^2} \sum_{j=1}^n E(X_j^2 I_{\{\omega : |X_j| \leq n\}}) \rightarrow 0$ as $n \rightarrow \infty$,

then $\frac{S_n}{n} - \frac{1}{n} \sum_{j=1}^n E(X_j I_{\{\omega : |X_j| \leq n\}}) \xrightarrow{p} 0$.

Proof. Let $T_{n,j}(\omega) = X_j(\omega) I_{\{\omega : |X_j| \leq n\}}$ and $S'_n(\omega) = \sum_{j=1}^n T_{n,j}(\omega)$. Note that $\{\omega : X_j(\omega) \neq T_{n,j}(\omega)\} = \{\omega : |X_j(\omega)| > n\}$ and by assumption $\sum_{j=1}^n P(\{\omega : T_{n,j}(\omega) \neq X_j(\omega)\}) \rightarrow 0$ as $n \rightarrow \infty$. Note also that

$$|S_n(\omega) - S'_n(\omega)| = \left| \sum_{j=1}^n X_j(\omega) - \sum_{j=1}^n T_{n,j}(\omega) \right| \leq \sum_{j=1}^n |X_j(\omega) - T_{n,j}(\omega)|.$$

Thus, for all $\epsilon > 0$,

$$\begin{aligned} \{\omega : |S_n(\omega) - S'_n(\omega)| > \epsilon\} &\subset \left\{ \omega : \sum_{j=1}^n |X_j(\omega) - T_{n,j}(\omega)| > \epsilon \right\} \\ &\subset \bigcup_{j=1}^n \{\omega : |X_j(\omega) - T_{n,j}(\omega)| > \epsilon/n\}. \end{aligned}$$

Consequently,

$$\begin{aligned} P(\{\omega : |S_n(\omega) - S'_n(\omega)| > \epsilon\}) &\leq \sum_{j=1}^n P(\{\omega : |X_j(\omega) - T_{n,j}(\omega)| > \epsilon/n\}) \\ &\leq \sum_{j=1}^n P(\{\omega : |X_j| > n\}). \end{aligned}$$

Taking limits on both sides as $n \rightarrow \infty$, we have that $S_n - S'_n \xrightarrow{p} 0$ since by assumption 1 $\sum_{j=1}^n P(\{\omega : |X_j| > n\}) \rightarrow 0$.

Now, since $\{X_n\}_{n \in \mathbb{N}}$ is an independent sequence $E((T_{n,k} - E(T_{n,k}))(T_{n,l} - E(T_{n,l}))) = 0$ and consequently $V(S'_n) = \sum_{j=1}^n V(T_{n,j}) \leq \sum_{j=1}^n E(T_{n,j}^2)$. Note also that for given n

$$E(T_{n,j}^2) = \int_{\Omega} X_j^2 I_{\{\omega : |X_j| \leq n\}} dP \leq n^2 \int_{\Omega} dP = n^2.$$

Consequently, since $V(S'_n)$ exists for every n , by Chebyshev's Inequality (Remark [5.1](#)),

$$P\left(\left\{\omega : \left|\frac{S'_n - E(S'_n)}{n}\right| > \epsilon\right\}\right) \leq \frac{V(S'_n)}{n^2 \epsilon^2} \leq \frac{1}{n^2 \epsilon^2} \sum_{j=1}^n E(X_j^2 I_{\{\omega : |X_j| \leq n\}}).$$

Taking limits on both sides as $n \rightarrow \infty$ and by the assumption that $\frac{1}{n^2} \sum_{j=1}^n E(X_j^2 I_{\{\omega : |X_j| \leq n\}}) \rightarrow 0$, we have $\frac{S'_n}{n} - \frac{E(S'_n)}{n} \xrightarrow{p} 0$. Now, since

$$\frac{S_n}{n} - E\left(\frac{S'_n}{n}\right) = \frac{S_n}{n} - \frac{S'_n}{n} + \frac{S'_n}{n} - E\left(\frac{S'_n}{n}\right)$$

we can immediately conclude that $\frac{S_n}{n} - E\left(\frac{S'_n}{n}\right) = o_p(1)$. Finally, from the definition of S'_n we have that $\frac{S_n}{n} - \frac{1}{n} \sum_{j=1}^n E(X_j I_{\{\omega : |X_j| \leq n\}}) = o_p(1)$. ■

We note $E(X_j) < \infty$ or $E(X_j^2) < \infty$ are not required for Theorem [8.2](#). The following are examples of how Theorem [8.2](#) can be used.

Example 8.1. Let $\{X_n\}_{n \in \mathbb{N}}$ be an independent and identically distributed sequence of random variables with $E(X_n) = \mu$, $E(X_n^2) \leq C < \infty$. Then, we verify condition 1 by noting that the identical distribution assumption and Markov's Inequality give

$$\sum_{j=1}^n P(|X_j| > n) = nP(|X_1| > n) \leq n \frac{E(X_1^2)}{n^2} = \frac{1}{n} E(X_1^2) \leq \frac{C}{n}.$$

Taking limits on both sides as $n \rightarrow \infty$ gives $\lim_{n \rightarrow \infty} \sum_{j=1}^n P(|X_j| > n) = 0$. For condition 2, note that by the identical distribution assumption

$$\frac{1}{n^2} \sum_{j=1}^n E(X_j^2 I_{\{|X_j| \leq n\}}) = \frac{1}{n} E(X_1^2 I_{\{|X_1| \leq n\}}) \leq \frac{1}{n} E(X_1^2) \leq \frac{C}{n}.$$

Again, taking limits on both sides as $n \rightarrow \infty$ gives $\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{j=1}^n E(X_j^2 I_{\{|X_j| \leq n\}}) = 0$. Finally, observe that

$$\frac{\sum_{j=1}^n E(X_j I_{\{|X_j| \leq n\}})}{n} = E(X_1 I_{\{|X_1| \leq n\}}) \rightarrow E(X_1) = \mu$$

as $n \rightarrow \infty$ by Lebesgue's dominated convergence theorem. Thus, $\frac{1}{n} S_n \xrightarrow{p} \mu$.

Example 8.2. Let $\{X_n\}_{n \in \mathbb{N}}$ be an independent and identically distributed sequence with $E(|X_1|) \leq C < \infty$ and let $E(X_1) = \mu$. For condition 1, note that

$$\sum_{j=1}^n P(|X_j| > n) = nP(|X_1| > n) = E(n I_{\{\omega: |X_1| > n\}}).$$

But since $n I_{\{\omega: |X_1| > n\}} \leq |X_1| I_{\{\omega: |X_1| > n\}}$, we have that

$$\sum_{j=1}^n P(|X_j| > n) \leq E(|X_1| I_{\{\omega: |X_1| > n\}})$$

Consequently, $\lim_{n \rightarrow \infty} \sum_{j=1}^n P(|X_j| > n) \leq \lim_{n \rightarrow \infty} E(|X_1| I_{\{\omega: |X_1| > n\}}) = 0$. And since $E(|X_1|) < C$, $\lim_{n \rightarrow \infty} E(|X_1| I_{\{\omega: |X_1| > n\}}) = 0$.

For condition 2, note that by the identical distribution assumption

$$\begin{aligned} \frac{1}{n^2} \sum_{j=1}^n E(X_j^2 I_{\{\omega: |X_j| \leq n\}}) &= \frac{1}{n} E(X_1^2 I_{\{\omega: |X_1| \leq n\}}) \\ &= \frac{1}{n} (E(X_1^2 I_{\{\omega: |X_1| \leq \epsilon \sqrt{n}\}}) + E(X_1^2 I_{\{\omega: \epsilon \sqrt{n} \leq |X_1| \leq n\}})) \text{ for any } \epsilon \in (0, 1) \end{aligned}$$

Since $E(X_1^2 I_{\{\omega: |X_1| \leq \epsilon \sqrt{n}\}}) = \int_{\Omega} X_1^2 I_{\{\omega: |X_1| \leq \epsilon \sqrt{n}\}} dP \leq n\epsilon^2 \int_{\Omega} dP = n\epsilon^2$, we have

$$\begin{aligned} \frac{1}{n^2} \sum_{j=1}^n E(X_j^2 I_{\{\omega: |X_j| \leq n\}}) &\leq \epsilon^2 + \frac{1}{n} E(|X_1| |X_1| I_{\{\omega: \epsilon \sqrt{n} \leq |X_1| \leq n\}}) \leq \epsilon^2 + \frac{1}{n} E(n |X_1| I_{\{\omega: \epsilon \sqrt{n} \leq |X_1| \leq n\}}) \\ &\leq \epsilon^2 + E(|X_1| I_{\{\omega: \epsilon \sqrt{n} \leq |X_1| \leq n\}}) \end{aligned}$$

Taking limits on both sides as $n \rightarrow \infty$, and noting that $E(|X_1|) < C$, we have that

$$\lim_{n \rightarrow \infty} E(|X_j| I_{\{\omega: \epsilon \sqrt{n} \leq |X_j| \leq n\}}) = 0.$$

And, since ϵ can be made arbitrarily small, $\lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{j=1}^n E(X_j^2 I_{\{\omega: |X_j| \leq n\}}) = 0$. Consequently, $\frac{S_n}{n} - E(X_1 I_{\{\omega: |X_1| \leq n\}}) \xrightarrow{p} 0$. Lastly, note that

$$\lim_{n \rightarrow \infty} \left(\int_{\Omega} X_1 dP - \int_{\Omega} X_1 I_{\{|X_1| \leq n\}} dP \right) = \int_{\Omega} X_1 dP - \lim_{n \rightarrow \infty} \int_{\Omega} X_1 I_{\{|X_1| \leq n\}} dP = E(X_1) - E(X_1) = 0$$

by the previous example. Hence,

$$\frac{S_n}{n} - E(X_1) = \frac{S_n}{n} + E(X_1 I_{\{|X_1| \leq n\}}) - E(X_1 I_{\{|X_1| \leq n\}}) - E(X_1) = o_p(1) + o(1) = o_p(1).$$

Example 8.3. Suppose $\{X_n\}_{n \in \mathbb{N}}$ is an independent and identically distributed sequence with

$\lim_{x \rightarrow \infty} xP(|X_1| > x) = 0$. For condition 1, given the identically distributed assumption, we have

$\sum_{j=1}^n P(|X_j| > n) = nP(|X_1| > n) \rightarrow 0$ by assumption. For condition 2, note that

$$\begin{aligned}
\frac{1}{n^2} \sum_{j=1}^n E(X_j^2 I_{\{\omega: |X_j| \leq n\}}) &= \frac{1}{n} E(X_1^2 I_{\{\omega: |X_1| \leq n\}}) = \frac{1}{n} \int_{|x| \leq n} x^2 dF_{X_1}(x) \\
&= \frac{2}{n} \int_{|x| \leq n} \left(\int_0^{|x|} s ds \right) dF_{X_1}(x) = \frac{2}{n} \int_0^n s \left(\int_{s < |x| \leq n} dF_{X_1}(x) \right) ds \\
&= \frac{2}{n} \int_0^n s (P(|X_1| \leq n) - P(|X_1| < s)) ds \\
&= \frac{2}{n} \int_0^n s (1 - P(|X_1| > n) - 1 + P(|X_1| \geq s)) ds \\
&= \frac{2}{n} \int_0^n s (P(|X_1| \geq s) - P(|X_1| > n)) ds \\
&= \frac{2}{n} \int_0^n \tau(s) ds - 2P(|X_1| > n) \frac{1}{n} \int_0^n s ds, \text{ where } \tau(s) = sP(|X_1| > s) \\
&= \frac{2}{n} \int_0^n \tau(s) ds - 2P(|X_1| > n) \frac{1}{n} \frac{n^2}{2} \\
&= \frac{2}{n} \int_0^n \tau(s) ds - nP(|X_1| > n) = \frac{2}{n} \int_0^n \tau(s) ds - \tau(n).
\end{aligned}$$

Since, $\tau(n) \rightarrow 0$ as $n \rightarrow \infty$, we have that for all $\epsilon > 0$ there exists N_ϵ such that if $n > N_\epsilon$, $\tau(n) \leq \epsilon$. Consequently,

$$\frac{1}{n} \int_0^n \tau(s) ds = \frac{1}{n} \int_0^{N_\epsilon} \tau(s) ds + \frac{1}{n} \int_{N_\epsilon}^n \tau(s) ds \leq \frac{1}{n} \int_0^{N_\epsilon} \tau(s) ds + \epsilon.$$

Taking limits on both sides as $n \rightarrow \infty$ gives $\frac{1}{n} \int_0^n \tau(s) ds \rightarrow 0$. Then, $\frac{S_n}{n} - E(X_1 I_{|X_1| \leq n}) \xrightarrow{p} 0$.

If $\{X_j\}_{j \in \mathbb{N}}$ with $E(X_j) < \infty$, $E(X_j^2) < M < \infty$ for all j , we have that

$$\frac{1}{n} S_n - \frac{1}{n} E(S_n) = \frac{1}{n} \sum_{j=1}^n (X_j - E(X_j)) := \frac{1}{n} \sum_{j=1}^n Z_j$$

where $E(Z_j) = 0$. If $E(Z_i Z_j) = 0$ for all $i \neq j$, then

$$E \left(\left(\frac{1}{n} \sum_{j=1}^n Z_j \right)^2 \right) = \frac{1}{n^2} \sum_{j=1}^n E(Z_j^2) < \frac{M}{n} \rightarrow 0 \text{ as } n \rightarrow \infty.$$

Hence, $\frac{1}{n} \sum_{j=1}^n Z_j \xrightarrow{\mathcal{L}^2} 0$, and by Theorem 7.8 $\frac{1}{n} \sum_{j=1}^n Z_j \xrightarrow{p} 0$. In fact, $\frac{1}{n} \sum_{j=1}^n Z_j \xrightarrow{as} 0$ as shown in the next theorem.

Theorem 8.3. Let $\{X_j\}_{j \in \mathbb{N}}$ with $E(X_j) < \infty$, $E(X_j^2) < M < \infty$ for all j , and assume that $E((X_j - E(X_j))(X_i - E(X_i))) = 0$ for all $i \neq j$. Then, letting $Z_j = X_j - E(X_j)$ and $S_n = \sum_{j=1}^n Z_j$, we have

$$\frac{1}{n} S_n \xrightarrow{as} 0.$$

Proof. For all $\epsilon > 0$ and by Chebyshev's Inequality

$$P(|S_n| > n\epsilon) \leq \frac{M}{n\epsilon^2}.$$

Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges we can't use the Borel-Cantelli Lemma directly. However, if we consider the subsequence S_{n^2} , we have

$$\sum_{n=1}^{\infty} P(\{\omega : |S_{n^2}| > n^2\epsilon\}) \leq \sum_{n=1}^{\infty} \frac{M}{n^2\epsilon^2} < \infty,$$

since $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. Hence, $P\left(\limsup_{n \rightarrow \infty} \{\omega : |S_{n^2}(\omega)| > n^2\epsilon\}\right) = 0$ and we have $\frac{S_{n^2}}{n^2} \xrightarrow{as} 0$.

Now, let

$$D_n := \max_{n^2 \leq k < (n+1)^2} |S_k - S_{n^2}|$$

and note that

$$\frac{|S_k|}{k} \leq \frac{|S_k|}{n^2} = \frac{|S_k - S_{n^2} + S_{n^2}|}{n^2} \leq \frac{|S_k - S_{n^2}|}{n^2} + \frac{|S_{n^2}|}{n^2} \leq \frac{D_n}{n^2} + \frac{|S_{n^2}|}{n^2}.$$

Now, since $P(\max_{1 \leq k \leq m} |W_k| \geq \epsilon) \leq \sum_{k=1}^m P(|W_k| \geq \epsilon)$ and using Markov's Inequality

$$\begin{aligned} P(D_n \geq n^2\epsilon) &\leq \sum_{l=1}^{2n} P\left(\left|\sum_{j=1}^l Z_{n^2+j}\right| \geq n^2\epsilon\right) \leq \sum_{l=1}^{2n} \frac{1}{n^4\epsilon^2} E\left(\left(\sum_{j=1}^l Z_{n^2+j}\right)^2\right) \\ &= \sum_{l=1}^{2n} \frac{1}{n^4\epsilon^2} \sum_{j=1}^l E(Z_{n^2+j}^2) \leq \frac{4n^2 M}{n^4\epsilon^2} = \frac{4M}{n^2\epsilon^2}. \end{aligned}$$

Then, we have $\sum_{n=1}^{\infty} P(D_n \geq n^2\epsilon) \leq \frac{4M}{\epsilon^2} \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$, and by the Borel-Cantelli Lemma $\frac{D_n}{n^2} \xrightarrow{as} 0$. Since, as $n \rightarrow \infty$ we have that $k \rightarrow \infty$, $\frac{|S_k|}{k} \xrightarrow{as} 0$. ■

We now state Markov's Law of Large Numbers.

Theorem 8.4. (Markov's LLN) Let $\{X_n\}_{n \in \mathbb{N}}$ be a sequence of independent random variables with $E(X_n) = \mu_n$. If for some $\delta > 0$ we have $\sum_{n=1}^{\infty} \frac{E|X_n - \mu_n|^{1+\delta}}{n^{1+\delta}} < \infty$, then

$$\frac{1}{n}S_n - \frac{1}{n} \sum_{i=1}^n \mu_i \xrightarrow{as} 0.$$

Proof. Chung (1974, A Course in Probability Theory, pp. 125-126). ■

8.1 Exercises

1. Let U and V be two points in an n -dimensional unit cube, i.e., $[0, 1]^n$ and X_n be the Euclidean distance between these two points which are chosen independently and uniformly. Show that $\frac{X_n}{\sqrt{n}} \xrightarrow{P} \frac{1}{\sqrt{6}}$.

2. Show that if $\{X_j\}_{j \in \mathbb{N}}$ be a sequence of random variables with $E(X_j) = 0$ and $\sum_{j=1}^{\infty} \frac{1}{a_j^p} E(|X_j|^p) < \infty$ for some $p \geq 1$ and a sequence of positive constants $\{a_j\}_{j \in \mathbb{N}}$. Then,

$$\sum_{j=1}^{\infty} P(|X_j| > a_j) < \infty \text{ and } \sum_{j=1}^{\infty} \frac{1}{a_j} |E(X_j I_{\{\omega: |X_j| \leq a_j\}})| < \infty.$$

Furthermore, for any $r \geq p$,

$$\sum_{j=1}^{\infty} \frac{1}{a_j^r} E(|X_j|^r I_{\{\omega: |X_j| \leq a_j\}}) < \infty.$$

Use this result to prove Theorem 8.4 in your class notes with convergence in probability.

3. Let $\{X_i\}_{i=2,3,\dots}$ be a sequence of independent random variables such that

$$P(X_i = i) = P(X_i = -i) = \frac{1}{2i \log i}, \quad P(X_i = 0) = 1 - \frac{1}{i \log i}$$

Show that $\frac{1}{n} \sum_{i=2}^n X_i \xrightarrow{P} 0$.