If you're going to use units...at least use them correctly.

First, we need to discuss a few basics about the meaning of, and differences between, the terms **dimensions** and **units**. The dimension of a variable, say x, tells you what type of physical quantity, or perhaps property, the variable represents. For example, if l is the length of a piece of string, the dimension of l would be *length* and it tells you something about a physical property associated with the size of the string. We typically use L to represent the concept of the dimension of a length measurement. Or if y is the average live span of a wolf living in the wild, then the dimension of y would be *time*, and we might use T to represent that. We can measure things like length and time in different ways. For example, I can tell you that a piece of string is $l = 1$ foot long, or $l = 12$ inches long. The use of feet or inches tells you the units that I am using to quantify the dimension associated with the variable l.

We have many different units that we commonly use to quantify length, such as inches, feet, meters, smoots, light-years, parsecs, angstroms, and so on. In general, we have many units with which to quantify a particular dimension. Some common human-relatable time units are seconds, minutes, hours, days, weeks, months and years.

Some common symbols used to denote dimensions are: L for length, T for time, M for mass, F for force, and Θ for temperature. There are, indeed, more than this short list, but it is a good starting place. Sometimes we need a convenient way to identify the dimensions of variables. Frequently people use a set of square brackets, such as $[l] = L$, to indicate that l is a variable describing a length. Similarly, one might write $[y] = T$ to indicate that the variable y has time as its dimension.

Recall the definition of an angle θ in radians (see Figure 1 below). Since θ is defined as an arc length along a section of a circle divided by the radius of the cicrcle, it has no dimensions associated with it. (It is a length divided by a length.) So in things like $\sin(x)$, $\sin(3x)$, $\sin(\pi x)$, the arguments x, $3x$, and πx have to be dimensionless. In $\cos(at^2)$ the term at^2 must also be dimensionless. In addition from Figure 2 you can see that all of the circular trigonometric functions have a return value that is dimensionless because they are defined as a ratio of lengths on a right triangle. So, as a consequence, functions like $\sin(x)$ and $\tan(t)$, not only have arguments that are dimensionless, but the values returned are also dimensionless. (Actually, the same statement is also true for other functions like $\exp(x)$, $\ln(x)$ and so on.)

However, if in the expression $\cos(at^2)$ I told you that t has the units of time, then a would need to have the units of $1/\text{time}^2$. Let's denote the units of something with a set of square brackets [ ]. Then the units of time would be refered to as $[t] = \text{time}$ in which case the units of a would be denoted $[a] = 1/\text{time}^2$. Another example might be $\tan(\pi t^2)$ where t is time with units $[t] = \text{time}$. Then in this case $[\pi] = 1/\text{time}^2$. This is not exactly what you might have expected. To simplify notation even further let's use M, L, and T to denote some appropriate units of mass, length, and time, respectively. We could also denote the units of force as F. However for temperature we might have to think a bit harder since we have already used T for units of time. Different disciplines have their own conventions symbols denoting units. Poke around in the literature to see the details.

Unexpected problems can arise if you don't pay attention to the units. Consider the following example. If I write the expression $\sin(\pi t) + \cos(\pi t^2)$ where $[t] = \text{T}$, then we have a serious problem because the π in the $\sin(\pi t)$ term must have the units $1/T$ but the π in the $\cos(\pi t^2)$ term must have the units $1/T^2$. (And here you thought all along that π was always just a number with no units.)

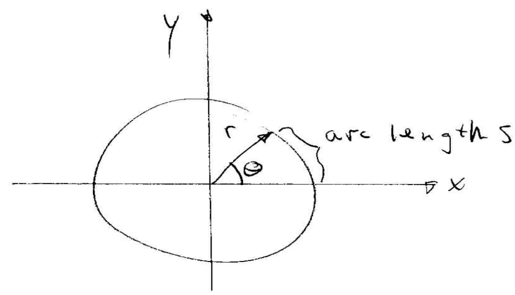
So then, how can one fix this problem? Well, one way is instead of writing $\sin(\pi t) + \cos(\pi t^2)$ we could write $\sin(at) + \cos(bt^2)$ where $a = \pi/T$ and $b = \pi/T^2$, with T being some unit of time, say seconds, or fortnights, or whatever, that matches up the units of t. Now we have $[a] = 1/T$, $[b] = 1/T^2$, and we again have the traditional dimensionless number π.

Another approach might be to define a *dimensionless variable*, for example $\tau = at$, with the understanding that $[a] = 1/T$ and $[t] = 1/T$. Then something like $\sin(at)$ could be written as $\sin(\tau)$. (Be careful though because $\cos(bt^2)$ would need to be something different, specifically $\cos\left(\frac{b}{a^2}\tau^2\right)$ where $[b] = 1/T^2$ and b/a^2 is a dimensionless quantity.) Note that if I wanted to calculate $\frac{d\sin(at)}{dt}$, then I could invoke the chain rule as

$$\frac{d\sin(at)}{dt} = \frac{d\sin(\tau)}{dt} = \frac{d\sin(\tau)}{d\tau} \cdot \frac{d\tau}{dt} = \cos(\tau) \cdot a = \cos(at) \cdot a,$$

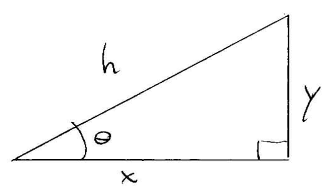
which is, of course, the same as $\frac{d\sin(at)}{dt} = \cos(at) \cdot a$. Clearly the units on both sides of the equation above are $1/T$.

Figure 1



In radians $\theta \equiv s/R$

Figure 2



$\sin\theta \equiv Y/h$

$\cos\theta \equiv x/h$

$\tan\theta \equiv Y/x$