1. (25 points) Consider a particle moving along a path $\mathbf{r}(t)$ such that its speed is always $\sqrt{37}$, the curvature is always $1/37$, the acceleration is $\mathbf{a}(t) = -\cos(t)\,\mathbf{i} + 0\,\mathbf{j} - \sin(t)\,\mathbf{k}$, and the unit tangent to the path is $\mathbf{T}(t) = (1/\sqrt{37})(-\sin(t)\,\mathbf{i} + 6\,\mathbf{j} + \cos(t)\,\mathbf{k})$.

   (a) Calculate the velocity, $\mathbf{V}(t)$.

   (b) Calculate the unit normal to the path, $\mathbf{N}(t)$.

   (c) Calculate the unit binormal to the path, $\mathbf{B}(t)$.

   (d) At time $t^\star = 2\pi$ the particle is located at $(1, 12\pi, 0)$. Determine $\mathbf{r}(t)$.

   (e) Determine the constants a, b, and c, if one were to write the position vector at time $t^\star$ in the form $\mathbf{r}(t^\star) = a\,\mathbf{T}(t^\star) + b\,\mathbf{N}(t^\star) + c\,\mathbf{B}(t^\star)$.

2. (25 points) Suppose a probe is launched into deep space at time $t = 0$ from its mother ship to search for other lifeforms that understand vector calculus. The probe travels along with velocity $\mathbf{v}_p(t) = -8\sin(3t)\,\mathbf{i} + 6\,\mathbf{j} + 8\cos(3t)\,\mathbf{k}$. The mother ship travels along the path $\mathbf{r}_m(t) = (3\,t^2/\pi)\,\mathbf{i} + 6t\,\mathbf{j} + \tan(3t)\,\mathbf{k}$.

   (a) At time $t = \pi/2$ the probe is located at $(0, 3\pi, -(8/3))$. Find the position vector of the probe, $\mathbf{r}_p(t)$, for all time t.

   (b) The antenna of the space probe is permanently aimed at a very distant star, Sol. As a result, the antenna is always oriented in the direction $\mathbf{d}(t) = -\cos(3t)\,\mathbf{i} + 0\,\mathbf{j} + -\sin(3t)\,\mathbf{k}$. The probe can only communicate with the mother ship when its antenna is pointed directly at the mother ship. Assuming $t > 0$, what is the *first* time that the probe will be able to communicate with the mother ship?

   (c) At what other times (for $t > 0$) will the probe be able to communicate with the mother ship?

3. (25 points) Remember that room ten units long in the x, y, and z directions, where the walls of the room were the four planes, $x = 0$, $x = 10$, $y = 0$, and $y = 10$, and the floor and ceiling are $z = 0$ and $z = 10$, respectively? Well, you are back in that room. The original, relatively large, flat triangular mirror that was mounted in one of the corners of the ceiling has been replaced by a very, very small flat mirror that fits right into the corner of room at location $(10, 10, 10)$. Again, you are sitting at location $(5, 0, 0)$ playing with your green laser pointer. When you aim your laser pointer directly at the mirror, the laser beam reflects off the small mirror and hits the corner located at $(0, 0, 0)$.

   (a) Determine a direction vector from the laser pointer to the mirror, $\mathbf{D}_1$.

   (b) Determine a direction vector, $\mathbf{D}_2$, from the the mirror, to the location location $(0, 0, 0)$.

   (c) Determine a normal direction vector, $\mathbf{D}_3$, to the plane formed by the surface of the mirror. (Hint: an incoming ray of light, and the surface normal where the ray hits the surface, form a plane. The reflected ray is in the same plane. Also, the angle between the incoming ray and the normal is the same as the angle between the normal and the reflected ray.)

   (d) Determine the standard equation of the plane formed by the surface of the mirror.

   (e) Determine the angle between the incoming and outgoing laser beams. You may leave your answer in terms of a trig function.

4. (25 points) For each of the surfaces described below (a through e), list the possible equations (1 through 8) that would fit the description. Note that there might be more than one equation that fits the description. Also, there might not be any equations that fit the description, in which case you should state NONE.

(a) A set of cones parallel to the y-axis.

(b) Hyperboloid of two sheets parallel to the x-axis.

(c) Paraboloid parallel to the x-axis.

(d) Hyperboloid of one sheet parallel to the z-axis.

(e) A hyperbolic paraboloid.

(1) $\frac{z^2}{4} + \frac{y^2}{16} = \frac{x^2}{25} - 1$

(2) $\frac{y^2}{16} = \frac{x}{25} + \frac{z^2}{4}$

(3) $\frac{x^2}{16} + \frac{z^2}{4} = \frac{y^2}{9}$

(4) $\frac{y^2}{4} + \frac{z^2}{16} = \frac{x}{9}$

(5) $\frac{x^2}{16} + \frac{y^2}{9} - \frac{z^2}{4} = 1$

(6) $\frac{x^2}{16} + \frac{y^2}{9} = \frac{z^2}{4} - 1$

(7) $\frac{y^2}{9} + \frac{z^2}{4} + 9 = \frac{x}{9}$

(8) $\frac{z^2}{4} = \frac{x}{5} + \frac{y^2}{9}$

Projections, and distances from a point to a line and a plane

$$\text{proj}_{\mathbf{A}}\mathbf{B} = \left(\frac{\mathbf{A}\cdot\mathbf{B}}{\mathbf{A}\cdot\mathbf{A}}\right)\mathbf{A} \qquad d = \frac{|\overrightarrow{PS}\times\mathbf{v}|}{|\mathbf{v}|} \qquad d = \left|\overrightarrow{PS}\cdot\frac{\mathbf{n}}{|\mathbf{n}|}\right|$$

Arc length, Frenet formulas, and tangential and normal acceleration components

$$ds = |\mathbf{v}|\,dt \qquad \mathbf{T} = \frac{d\mathbf{r}}{ds} = \frac{\mathbf{v}}{|\mathbf{v}|} \qquad \mathbf{N} = \frac{d\mathbf{T}/ds}{|d\mathbf{T}/ds|} = \frac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|} \qquad \mathbf{B} = \mathbf{T}\times\mathbf{N}$$

$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N} \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N} \qquad \kappa = \left|\frac{d\mathbf{T}}{ds}\right| = \frac{|\mathbf{v}\times\mathbf{a}|}{|\mathbf{v}|^3} = \frac{|f''(x)|}{[\,1+(f'(x))^2\,]^{3/2}} = \frac{|\dot{x}\ddot{y}-\dot{y}\ddot{x}|}{(\dot{x}^2+\dot{y}^2)^{3/2}} \qquad \tau = -\frac{d\mathbf{B}}{ds}\cdot\mathbf{N}$$

$$\mathbf{a} = a_N\mathbf{N} + a_T\mathbf{T} \qquad a_T = \frac{d|\mathbf{v}|}{dt} \qquad a_N = \kappa|\mathbf{v}|^2 = \sqrt{|\mathbf{a}|^2 - a_T^2}$$