CHAPTER X

THE SPECTRAL THEOREM OF GELFAND

DEFINITION A Banach algebra is a complex Banach space $A$ on which there is defined an associative multiplication $\times$ for which:

1. $x \times (y + z) = x \times y + x \times z$ and $(y + z) \times x = y \times x + z \times x$ for all $x, y, z \in A$.
2. $x \times (\lambda y) = \lambda x \times y = (\lambda x) \times y$ for all $x, y \in A$ and $\lambda \in \mathbb{C}$.
3. $\|x \times y\| \leq \|x\| \|y\|$ for all $x, y \in A$.

We call the Banach algebra commutative if the multiplication in $A$ is commutative.

An involution on a Banach algebra $A$ is a map $x \rightarrow x^*$ of $A$ into itself that satisfies the following conditions for all $x, y \in A$ and $\lambda \in \mathbb{C}$.

1. $(x + y)^* = x^* + y^*$.
2. $(\lambda x)^* = \overline{\lambda} x^*$.
3. $(x^*)^* = x$.
4. $(x \times y)^* = y^* \times x^*$.
5. $\|x^*\| = \|x\|$.

We call $x^*$ the adjoint of $x$. A subset $S \subseteq A$ is called selfadjoint if $x \in S$ implies that $x^* \in S$.

A Banach algebra $A$ on which there is defined an involution is called a Banach *-algebra.

An element of a Banach *-algebra is called selfadjoint if $x^* = x$. If a Banach *-algebra $A$ has an identity $I$, then an element $x \in A$, for which $x \times x^* = x^* \times x = I$, is called a unitary element of $A$. A selfadjoint
element \( x \), for which \( x^2 = x \), is called a projection in \( A \). An element \( x \) that commutes with its adjoint \( x^* \) is called a normal element of \( A \).

A Banach algebra \( A \) is a \( C^* \)-algebra if it is a Banach *-algebra, and if the equation
\[
\| x \times x^* \| = \| x \| ^2
\]
holds for all \( x \in A \). A sub \( C^* \)-algebra of a \( C^* \)-algebra \( A \) is a subalgebra \( B \) of \( A \) that is a closed subset of the Banach space \( A \) and is also closed under the adjoint operation.

REMARK. We ordinarily write \( xy \) instead of \( x \times y \) for the multiplication in a Banach algebra. It should be clear that the axioms for a Banach algebra are inspired by the properties of the space \( B(H) \) of bounded linear operators on a Hilbert space \( H \).

EXERCISE 10.1. (a) Let \( A \) be the set of all \( n \times n \) complex matrices, and for \( M = [a_{ij}] \in A \) define
\[
\| M \| = \sqrt{\sum_{i=1}^{n} \sum_{j=1}^{n} |a_{ij}|^2}.
\]
Prove that \( A \) is a Banach algebra with identity \( I \). Verify that \( A \) is a Banach *-algebra if \( M^* \) is defined to be the complex conjugate of the transpose of \( M \). Give an example to show that \( A \) is not a \( C^* \)-algebra.

(b) Suppose \( H \) is a Hilbert space. Verify that \( B(H) \) is a \( C^* \)-algebra. Using as \( H \) the Hilbert space \( \mathbb{C}^2 \), give an example of an element \( x \in B(H) \) for which \( \| x^2 \| \neq \| x \| ^2 \). Observe that this example is not the same as that in part a. (The norms are different.)

(c) Verify that \( L^1(\mathbb{R}) \) is a Banach algebra, where multiplication is defined to be convolution. Show further that, if \( f^* \) is defined to be \( \overline{f(-x)} \), then \( L^1(\mathbb{R}) \) is a Banach *-algebra. Give an example to show that \( L^1(\mathbb{R}) \) is not a \( C^* \)-algebra.

(d) Verify that \( C_0(\Delta) \) is a Banach algebra, where \( \Delta \) is a locally compact Hausdorff space, the algebraic operations are pointwise, and the norm on \( C_0(\Delta) \) is the supremum norm. Show further that \( C_0(\Delta) \) is a \( C^* \)-algebra, if we define \( f^* \) to be \( \overline{f} \). Show that \( C_0(\Delta) \) has an identity if and only if \( \Delta \) is compact.

(e) Let \( A \) be an arbitrary Banach algebra. Prove that the map \( (x, y) \to xy \) is continuous from \( A \times A \) into \( A \).

(f) Let \( A \) be a Banach algebra. Suppose \( x \in A \) satisfies \( \| x \| < 1 \). Prove that \( 0 = \lim_n x^n \).
(g) Let $M$ be a closed subspace of a Banach algebra $A$, and assume that $M$ is a two-sided ideal in (the ring) $A$; i.e., $xy \in M$ and $yx \in M$ if $x \in A$ and $y \in M$. Prove that the Banach space $A/M$ is a Banach algebra and that the natural map $\pi : A \to A/M$ is a continuous homomorphism of the Banach algebra $A$ onto the Banach algebra $A/M$.

(h) Let $A$ be a Banach algebra with identity $I$ and let $x$ be an element of $A$. Show that the smallest subalgebra $B$ of $A$ that contains $x$ coincides with the set of all polynomials in $x$, i.e., the set of all elements $y$ of the form $y = \sum_{j=0}^{n} a_j x^j$, where each $a_j$ is a complex number and $x^0 = I$. We denote this subalgebra by $[x]$ and call it the subalgebra of $A$ generated by $x$.

(i) Let $A$ be a Banach $*$-algebra. Show that each element $x \in A$ can be written uniquely as $x = x_1 + ix_2$, where $x_1$ and $x_2$ are selfadjoint. Show further that if $A$ contains an identity $I$, then $I^* = I$. If $A$ is a $C^*$-algebra with identity, and if $U$ is a unitary element in $A$, show that $\|U\| = 1$.

(j) Let $x$ be a selfadjoint element of a $C^*$-algebra $A$. Prove that $\|x^n\| = \|x\|^n$ for all nonnegative integers $n$. HINT: Do this first for $n = 2^k$.

EXERCISE 10.2. (Adjoining an Identity) Let $A$ be a Banach algebra, and let $B$ be the complex vector space $A \times \mathbb{C}$. Define a multiplication on $B$ by

$$(x, \lambda) \times (x', \lambda') = (xx' + \lambda x' + x' \lambda, \lambda),$$

and set $||(x, \lambda)|| = \|x\| + |\lambda|$.

(a) Prove that $B$ is a Banach algebra with identity.

(b) Show that the map $x \to (x, 0)$ is an isometric isomorphism of the Banach algebra $A$ onto an ideal $M$ of $B$. Show that $M$ is of codimension 1; i.e., the dimension of $B/M$ is 1. (This map $x \to (x, 0)$ is called the canonical isomorphism of $A$ into $B$.)

(c) Conclude that every Banach algebra is isometrically isomorphic to an ideal of codimension 1 in a Banach algebra with identity.

(d) Suppose $A$ is a Banach algebra with identity, and let $B$ be the Banach algebra $A \times \mathbb{C}$ constructed above. What is the relationship, if any, between the identity in $A$ and the identity in $B$?

(e) If $A$ is a Banach $*$-algebra, can $A$ be imbedded isometrically and isomorphically as an ideal of codimension 1 in a Banach $*$-algebra?

THEOREM 10.1. Let $x$ be an element of a Banach algebra $A$ with identity $I$, and suppose that $\|x\| = \alpha < 1$. Then the element $I - x$ is
invertible in $A$ and

$$(I - x)^{-1} = \sum_{n=0}^{\infty} x^n.$$  

PROOF. The sequence of partial sums of the infinite series $\sum_{n=0}^{\infty} x^n$ forms a Cauchy sequence in $A$, for

$$\| \sum_{n=0}^{j} x^n - \sum_{n=0}^{k} x^n \| = \| \sum_{n=k+1}^{j} x^n \|$$

$$\leq \sum_{n=k+1}^{j} \| x^n \|$$

$$\leq \sum_{n=k+1}^{j} \| x \|^n$$

$$= \sum_{n=k+1}^{j} \alpha^n.$$  

We write

$$y = \sum_{n=0}^{\infty} x^n = \lim_{j} \sum_{n=0}^{j} x^n = \lim_{j} S_j.$$  

Then

$$(I - x)y = \lim_{j} (I - x)S_j$$

$$= \lim_{j} (I - x) \sum_{n=0}^{j} x^n$$

$$= \lim_{j} (I - x^{j+1})$$

$$= I,$$  

by part f of Exercise 10.1, showing that $y$ is a right inverse for $I - x$. That $y$ also is a left inverse follows similarly, whence $y = (I - x)^{-1}$, as desired.

EXERCISE 10.3. Let $A$ be a Banach algebra with identity $I$.

(a) If $x \in A$ satisfies $\|x\| < 1$, show that $I + x$ is invertible in $A$.

(b) Suppose $y \in A$ is invertible, and set $\delta = 1/\|y^{-1}\|$. Prove that $x$ is invertible in $A$ if $\|x - y\| < \delta$. HINT: Write $x = y(I + y^{-1}(x - y))$.  

(c) Conclude that the set of invertible elements in $A$ is a nonempty, proper, open subset of $A$.

(d) Prove that the map $x \mapsto x^{-1}$ is continuous on its domain. HINT: $y^{-1} - x^{-1} = y^{-1}(x - y)x^{-1}$.

(e) Let $x$ be an element of $A$. Show that the infinite series

$$\sum_{n=0}^{\infty} \frac{x^n}{n!}$$

converges to an element of $A$. Define

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$ 

Show that

$$e^{x+y} = e^x e^y$$

if $xy = yx$. Compare with part c of Exercise 8.13.

(f) Suppose in addition that $A$ is a Banach $^*$-algebra and that $x$ is a selfadjoint element of $A$. Prove that $e^{ix}$ is a unitary element of $A$. Compare with part d of Exercise 8.13.

**THEOREM 10.2.** (Mazur’s Theorem) Let $A$ be a Banach algebra with identity $I$, and assume further that $A$ is a division ring, i.e., that every nonzero element of $A$ has a multiplicative inverse. Then $A$ consists of the complex multiples $\lambda I$ of the identity $I$, and the map $\lambda \mapsto \lambda I$ is a topological isomorphism of $\mathbb{C}$ onto $A$.

**PROOF.** Assume false, and let $x$ be an element of $A$ that is not a complex multiple of $I$. This means that each element $x\lambda = x - \lambda I$ has an inverse.

Let $f$ be an arbitrary element of the conjugate space $A^*$ of $A$, and define a function $F$ of a complex variable $\lambda$ by

$$F(\lambda) = f(x\lambda^{-1}) = f((x - \lambda I)^{-1}).$$

We claim first that $F$ is an entire function of $\lambda$. Thus, let $\lambda$ be fixed. We use the factorization formula

$$y^{-1} - z^{-1} = y^{-1}(z - y)z^{-1}.$$
We have
\[
F(\lambda + h) - F(\lambda) = f(x_{\lambda+h}^{-1}) - f(x_\lambda^{-1}) \\
= f(x_{\lambda+h}^{-1}(x_\lambda - x_{\lambda+h}x_\lambda^{-1})) \\
= hf(x_{\lambda+h}^{-1}x_\lambda^{-1}).
\]
So,
\[
\lim_{h \to 0} \frac{F(\lambda + h) - F(\lambda)}{h} = f(x_\lambda^{-2}).
\]
and \(F\) is differentiable everywhere. See part d of Exercise 10.3.

Next, observe that
\[
\lim_{\lambda \to \infty} F(\lambda) = \lim_{\lambda \to \infty} f((x - \lambda I)^{-1}) \\
= \lim_{\lambda \to \infty} (1/\lambda) f((x/\lambda - I)^{-1}) \\
= 0.
\]
Therefore, \(F\) is a bounded entire function, and so by Liouville’s Theorem, \(F(\lambda) = 0\) identically. Consequently, \(f(x_0^{-1}) = f(x^{-1}) = 0\) for all \(f \in A^*\). But this would imply that \(x^{-1} = 0\), which is a contradiction.

We introduce next a dual object for Banach algebras that is analogous to the conjugate space of a Banach space.

**DEFINITION.** Let \(A\) be a Banach algebra. By the structure space \(\Delta\) of \(A\) we mean the set \(\Delta\) of all nonzero continuous algebra homomorphisms (linear and multiplicative) \(\phi : A \to \mathbb{C}\). The structure space is a (possibly empty) subset of the conjugate space \(A^*\), and we think of \(\Delta\) as being equipped with the inherited weak* topology.

**THEOREM 10.3.** Let \(A\) be a Banach algebra, and let \(\Delta\) denote its structure space. Then \(\Delta\) is locally compact and Hausdorff. Further, if \(A\) is a separable Banach algebra, then \(\Delta\) is second countable and metrizable. If \(A\) contains an identity \(I\), then \(\Delta\) is compact.

**PROOF.** \(\Delta\) is clearly a Hausdorff space since the weak* topology on \(A^*\) is Hausdorff.

Observe next that if \(\phi \in \Delta\), then \(\|\phi\| \leq 1\). Indeed, for any \(x \in A\), we have
\[
|\phi(x)| = |\phi(x^n)|^{1/n} \leq \|\phi\|^{1/n} \|x\| \to \|x\|,
\]
implying that \(\|\phi\| \leq 1\), as claimed. It follows then that \(\Delta\) is contained in the closed unit ball \(B_1^{\ast}\) of \(A^*\). Since the ball \(B_1^{\ast}\) in \(A^*\) is by Alaoglu’s
Theorem compact in the weak* topology, we could show that $\Delta$ is compact by verifying that it is closed in $\overline{B_1}$. This we can do if $A$ contains an identity $I$. Thus, let $\{\phi_\alpha\}$ be a net of elements of $\Delta$ that converges in the weak* topology to an element $\phi \in \overline{B_1}$. Since this convergence is pointwise convergence on $A$, it follows that $\phi(xy) = \phi(x)\phi(y)$, for all $x, y \in A$, whence $\phi$ is a homomorphism of the algebra $A$ into $C$. Also, since every nonzero homomorphism of $A$ must map $I$ to 1, it follows that $\phi(I) = 1$, whence $\phi$ is not the 0 homomorphism. Hence, $\phi \in \Delta$, as desired.

We leave the proof that $\Delta$ is always locally compact to the exercises.

Of course, if $A$ is separable, then the weak* topology on $B_1$ is compact and metrizable, so that $\Delta$ is second countable and metrizable in this case, as desired.

EXERCISE 10.4. Let $A$ be a Banach algebra.
(a) Suppose that the elements of the structure space $\Delta$ of $A$ separate the points of $A$. Prove that $A$ is commutative.
(b) Suppose $A$ is the algebra of all $n \times n$ complex matrices as defined in part a of Exercise 10.1. Prove that the structure space $\Delta$ of $A$ is the empty set if $n > 1$.
(c) If $A$ has no identity, show that $\Delta$ is locally compact. HINT: Show that the closure of $\Delta$ in $\overline{B_1}$ is contained in the union of $\Delta$ and $\{0\}$, whence $\Delta$ is an open subset of a compact Hausdorff space.
(d) Let $B$ be the Banach algebra with identity constructed from $A$ as in Exercise 10.2, and identify $A$ with its canonical isomorphic image in $B$. Prove that every element $\phi$ in the structure space $\Delta_A$ of $A$ has a unique extension to an element $\phi'$ in the structure space $\Delta_B$ of $B$. Show that there exists a unique element $\phi_0 \in \Delta_B$ whose restriction to $A$ is identically 0. Show further that the above map $\phi \mapsto \phi'$ is a homeomorphism of $\Delta_A$ onto $\Delta_B - \{\phi_0\}$.

DEFINITION. Let $A$ be a Banach algebra and let $\Delta$ be its structure space. For each $x \in A$, define a function $\hat{x}$ on $\Delta$ by

$$\hat{x}(\phi) = \phi(x).$$

The map $x \mapsto \hat{x}$ is called the Gelfand transform of $A$, and the function $\hat{x}$ is called the Gelfand transform of $x$.

EXERCISE 10.5. Let $A$ be the Banach algebra $L^1(\mathbb{R})$ of part c of Exercise 10.1, and let $\Delta$ be its structure space.
(a) If \( \lambda \) is any real number, define \( \phi_\lambda : A \to \mathbb{C} \) by

\[
\phi_\lambda(f) = \int f(x)e^{-2\pi i \lambda x} \, dx.
\]

Show that \( \phi_\lambda \) is an element of \( \Delta \).

(b) Let \( \phi \) be an element of \( \Delta \), and let \( h \) be the \( L^\infty \) function satisfying

\[
\phi(f) = \int f(x)\overline{h(x)} \, dx.
\]

Prove that \( h(x + y) = h(x)h(y) \) for almost all pairs \((x, y) \in \mathbb{R}^2\). HINT: Show that

\[
\int \int f(x)g(y)\overline{h(x + y)} \, dydx = \int \int f(x)g(y)\overline{h(x)}\overline{h(y)} \, dydx
\]

for all \( f, g \in L^1(\mathbb{R}) \).

(c) Let \( \phi \) and \( h \) be as in part b, and let \( f \) be an element of \( L^1(\mathbb{R}) \) for which \( \phi(f) \neq 0 \). Write \( f_x \) for the function defined by \( f_x(y) = f(x + y) \).

Show that the map \( x \to \phi(f_x) \) is continuous, and that

\[
h(x) = \frac{\phi(f_x)}{\phi(f)}
\]

for almost all \( x \). Conclude that \( h \) may be chosen to be a continuous function in \( L^\infty(\mathbb{R}) \), in which case \( h(x + y) = h(x)h(y) \) for all \( x, y \in \mathbb{R} \).

(d) Suppose \( h \) is a bounded continuous map of \( \mathbb{R} \) into \( \mathbb{C} \), which is not identically 0 and which satisfies \( h(x + y) = h(x)h(y) \) for all \( x \) and \( y \). Show that there exists a real number \( \lambda \) such that \( h(x) = e^{2\pi i \lambda x} \) for all \( x \).

HINT: If \( h \) is not identically 1, show that there exists a smallest positive number \( \delta \) for which \( h(\delta) = 1 \). Show then that \( h(\delta/2) = -1 \) and \( h(\delta/4) = \pm i \). Conclude that \( \lambda = \pm(1/\delta) \) depending on whether \( h(\delta/4) = i \) or \( -i \).

(e) Conclude that the map \( \lambda \to \phi_\lambda \) of part a is a homeomorphism between \( \mathbb{R} \) and the structure space \( \Delta \) of \( L^1(\mathbb{R}) \). HINT: To prove that the inverse map is continuous, suppose that \( \{\lambda_n\} \) does not converge to \( \lambda \). Show that there exists an \( f \in L^1(\mathbb{R}) \) such that \( \int f(x)e^{-2\pi i \lambda_n x} \, dx \) does not approach \( \int f(x)e^{-2\pi i \lambda x} \, dx \).

(f) Show that, using the identification of \( \Delta \) with \( \mathbb{R} \) in part e, that the Gelfand transform on \( L^1(\mathbb{R}) \) and the Fourier transform on \( L^1(\mathbb{R}) \) are identical. Conclude that the Gelfand transform is 1-1 on \( L^1(\mathbb{R}) \).
THEOREM 10.4. Let $A$ be a Banach algebra. Then the Gelfand transform of $A$ is a norm-decreasing homomorphism of $A$ into the Banach algebra $C(\Delta)$ of all continuous complex-valued functions on $\Delta$.

EXERCISE 10.6. (a) Prove Theorem 10.4.
(b) If $A$ is a Banach algebra without an identity, show that each function $\hat{x}$ in the range of the Gelfand transform is an element of $C^0(\Delta)$. HINT: The closure of $\Delta$ in $B_1$ is contained in the union of $\Delta$ and $\{0\}$.

DEFINITION. Let $A$ be a Banach algebra with identity $I$, and let $x$ be an element of $A$. By the resolvent of $x$ we mean the set $\text{res}_A(x)$ of all complex numbers $\lambda$ for which $\lambda I - x$ has an inverse in $A$. By the spectrum $\text{sp}_A(x)$ of $x$ we mean the complement of the resolvent of $x$; i.e., $\text{sp}_A(x)$ is the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - x$ does not have an inverse in $A$. We write simply $\text{res}(x)$ and $\text{sp}(x)$ when it is unambiguous what the algebra $A$ is.

By the spectral radius (relative to $A$) of $x$ we mean the extended real number $\|x\|_{\text{sp}}$ defined by

$$\|x\|_{\text{sp}} = \sup_{\lambda \in \text{sp}_A(x)} |\lambda|.$$ 

EXERCISE 10.7. Let $A$ be a Banach algebra with identity $I$, and let $x$ be an element of $A$.
(a) Show that the resolvent $\text{res}_A(x)$ of $x$ is open in $\mathbb{C}$, whence the spectrum $\text{sp}_A(x)$ of $x$ is closed.
(b) Show that the spectrum of $x$ is nonempty, whence the spectral radius of $x$ is nonnegative. HINT: Make an argument similar to the proof of Mazur’s theorem.
(c) Show that $\|x\|_{\text{sp}} \leq \|x\|$, whence the spectrum of $x$ is compact. HINT: If $\lambda \neq 0$, then $\lambda I - x = \lambda(I - (x/\lambda))$.
(d) Show that there exists a $\lambda \in \text{sp}_A(x)$ such that $\|x\|_{\text{sp}} = |\lambda|$; i.e., the spectral radius is attained.
(e) (Spectral Mapping Theorem) If $p(z)$ is any complex polynomial, show that $\text{sp}_A(p(x)) = p(\text{sp}_A(x))$; i.e., $\mu \in \text{sp}_A(p(x))$ if and only if there exists a $\lambda \in \text{sp}_A(x)$ such that $\mu = p(\lambda)$. HINT: Factor the polynomial $p(z) - \mu$ as

$$p(z) - \mu = c \prod_{i=1}^{n} (z - \lambda_i),$$
whence
\[ p(x) - \mu I = e \prod_{i=1}^{n} (x - \lambda_i I). \]

Now, the left hand side fails to have an inverse if and only if some one of the factors on the right hand side fails to have an inverse.

**THEOREM 10.5.** Let \( A \) be a commutative Banach algebra with identity \( I \), and let \( x \) be an element of \( A \). Then the spectrum \( sp_A(x) \) of \( x \) coincides with the range of the Gelfand transform \( \hat{x} \) of \( x \). Consequently, we have
\[ \|x\|_{sp} = \|\hat{x}\|_{\infty}. \]

**PROOF.** If there exists a \( \phi \) in the structure space \( \Delta \) of \( A \) for which \( \hat{x}(\phi) = \lambda \), then
\[ \phi(\lambda I - x) = \lambda - \phi(x) = \lambda - \hat{x}(\phi) = 0, \]
from which it follows that \( \lambda I - x \) cannot have an inverse. Hence, the range of \( \hat{x} \) is contained in \( sp(x) \).

Conversely, let \( \lambda \) be in the spectrum of \( x \). Let \( J \) be the set of all multiples \( (\lambda I - x)y \) of \( \lambda I - x \) by elements of \( A \). Then \( J \) is an ideal in \( A \), and it is a proper ideal since \( \lambda I - x \) has no inverse (\( I \) is not in \( J \)). By Zorn's Lemma, there exists a maximal proper ideal \( M \) containing \( J \). Now the closure of \( M \) is an ideal. If this closure of \( M \) is all of \( A \), then there must exist a sequence \( \{m_n\} \) of elements of \( M \) that converges to \( I \). But, since the set of invertible elements in \( A \) is an open set, it must be that some \( m_n \) is invertible. But then \( M \) would not be a proper ideal. Therefore, \( M \) is proper, and since \( M \) is maximal it follows that \( M \) is itself closed.

Now \( A/M \) is a Banach algebra by part g of Exercise 10.1. Also, since \( M \) is maximal, we have that \( A/M \) is a field. By Mazur's Theorem (Theorem 10.2), we have that \( A/M \) is topologically isomorphic to the set of complex numbers. The natural map \( \pi : A \to A/M \) is then a continuous nonzero homomorphism of \( A \) onto \( \mathbb{C} \), i.e., \( \pi \) is an element of \( \Delta \). Further, \( \pi(\lambda I - x) = 0 \) since \( \lambda I - x \in J \subseteq M \). Hence, \( \hat{x}(\pi) = \lambda \), showing that \( \lambda \) belongs to the range of \( \hat{x} \).

**EXERCISE 10.8.** Suppose \( A \) is a commutative Banach algebra with identity \( I \), and let \( \Delta \) be its structure space. Assume that \( x \) is an element of \( A \) for which the subalgebra \( [x] \) generated by \( x \) is dense in \( A \). (See part h of Exercise 10.1.) Prove that \( \hat{x} \) is a homeomorphism of \( \Delta \) onto the spectrum \( sp_A(x) \) of \( x \).
THEOREM 10.6. Let $A$ be a commutative $C^*$-algebra with identity $I$. Then, for each $x \in A$, we have $\hat{x}^* = \overline{\hat{x}}$.

PROOF. The theorem will follow if we show that $\hat{x}$ is real-valued if $x$ is selfadjoint. (Why?) Thus, if $x$ is selfadjoint, and if $U = e^{ix} = \sum_{n=0}^{\infty} (ix)^n / n!$, then we have seen in part f of Exercise 10.2 and part i of Exercise 10.1 that $U$ is unitary and that $\|U\| = \|U^{-1}\| = 1$. Therefore, if $\phi$ is an element of the structure space $\Delta$ of $A$, then $\|\phi(U)\| = \|\phi(U^{-1})\| \leq 1$, and this implies that $|\phi(U)| = 1$. On the other hand,

$$
\phi(U) = \sum_{n=0}^{\infty} (i\phi(x))^n / n! = e^{i\phi(x)}.
$$

But $|e^{it}| = 1$ if and only if $t$ is real. Hence, $\hat{x}(\phi) = \phi(x)$ is real for every $\phi \in \Delta$.

The next result is an immediate consequence of the preceding theorem.

THEOREM 10.7. If $x$ is a selfadjoint element of a commutative $C^*$-algebra $A$ with identity, then the spectrum $\text{sp}_A(x)$ of $x$ is contained in the set of real numbers.

EXERCISE 10.9. (A Formula for the Spectral Radius) Let $A$ be a Banach algebra with identity $I$, and let $x$ be an element of $A$. Write $\text{sp}(x)$ for $\text{sp}_A(x)$.

(a) If $n$ is any positive integer, show that $\mu \in \text{sp}(x^n)$ if and only if there exists a $\lambda \in \text{sp}(x)$ such that $\mu = \lambda^n$, whence

$$
\|x\|_{\text{sp}} = \|x^n\|_{\text{sp}}^{1/n}.
$$

Conclude that

$$
\|x\|_{\text{sp}} \leq \liminf \|x^n\|_{\text{sp}}^{1/n}.
$$

(b) If $f$ is an element of $A^*$, show that the function $\lambda \rightarrow f((\lambda I - x)^{-1})$ is analytic on the (open) resolvent $\text{res}(x)$ of $x$. Show that the resolvent contains all $\lambda$ for which $|\lambda| > \|x\|_{\text{sp}}$.

(c) Let $f$ be in $A^*$. Show that the function $F(\mu) = \mu f((I - \mu x)^{-1})$ is analytic on the disk of radius $1/\|x\|_{\text{sp}}$ around 0 in $\mathbb{C}$. Show further that

$$
F(\mu) = \sum_{n=0}^{\infty} f(x^n) \mu^{n+1}
$$
on the disk of radius $1/\|x\|$ and hence also on the (possibly) larger disk of radius $1/\|x\|_{sp}$.

(d) Using the Uniform Boundedness Principle, show that if $|\mu| < 1/\|x\|_{sp}$, then the sequence $\{\mu^{n+1}x^n\}$ is bounded in norm, whence

$$\lim \sup \|x^n\|^{1/n} \leq 1/|\mu|$$

for all such $\mu$. Show that this implies that

$$\lim \sup \|x^n\|^{1/n} \leq \|x\|_{sp}.$$  

(e) Derive the spectral radius formula:

$$\|x\|_{sp} = \lim \|x^n\|^{1/n}.$$  

(f) Suppose that $A$ is a $C^\ast$-algebra and that $x$ is a selfadjoint element of $A$. Prove that

$$\|x\| = \sup_{\lambda \in \text{sp}(x)} |\lambda| = \|x\|_{sp}.$$  

**THEOREM 10.8.** (Gelfand’s Theorem) Let $A$ be a commutative $C^\ast$-algebra with identity $I$. Then the Gelfand transform is an isometric isomorphism of the Banach algebra $A$ onto $C(\Delta)$, where $\Delta$ is the structure space of $A$.

**PROOF.** We have already seen that $x \rightarrow \hat{x}$ is a norm-decreasing homomorphism of $A$ into $C(\Delta)$. We must show that the transform is an isometry and is onto.

Now it follows from part f of Exercise 10.9 and Theorem 10.4 that $\|x\| = \|\hat{x}\|_{\infty}$ whenever $x$ is selfadjoint. For an arbitrary $x$, write $y = x^*x$. Then

$$\|x\| = \sqrt{\|y\|}$$

$$= \sqrt{\|y\|_{\infty}}$$

$$= \sqrt{\|x^*x\|_{\infty}}$$

$$= \sqrt{\|x^*\hat{x}\|_{\infty}}$$

$$= \sqrt{\|\hat{x}\|_{\infty}^2}$$

$$= \|\hat{x}\|_{\infty},$$
showing that the Gelfand transform is an isometry.

By Theorem 10.6, we see that the range \( \hat{A} \) of the Gelfand transform is a subalgebra of \( C(\Delta) \) that separates the points of \( \Delta \) and is closed under complex conjugation. Then, by the Stone-Weierstrass Theorem, \( \hat{A} \) must be dense in \( C(\Delta) \). But, since \( A \) is itself complete, and the Gelfand transform is an isometry, it follows that \( \hat{A} \) is closed in \( C(\Delta) \), whence is all of \( C(\Delta) \).

**EXERCISE 10.10.** Let \( A \) be a commutative \( C^* \)-algebra with identity \( I \), and let \( \Delta \) denote its structure space. Verify the following properties of the Gelfand transform on \( A \).

(a) \( x \) is invertible if and only if \( \hat{x} \) is never 0.
(b) \( x = yy^* \) if and only if \( \hat{x} \geq 0 \).
(c) \( x \) is a unitary element of \( A \) if and only if \( |\hat{x}| \equiv 1 \).
(d) \( A \) contains a nontrivial projection if and only if \( \Delta \) is not connected.

**EXERCISE 10.11.** Let \( A \) and \( B \) be commutative \( C^* \)-algebras, each having an identity, and let \( \Delta_A \) and \( \Delta_B \) denote their respective structure spaces. Suppose \( T \) is a (not a priori continuous) homomorphism of the algebra \( A \) into the algebra \( B \). If \( \phi \) is any linear functional on \( B \), define \( T'(\phi) \) on \( A \) by

\[
T'(\phi) = \phi \circ T.
\]

(a) Suppose \( \phi \) is a positive linear functional on \( B \); i.e., \( \phi(xx^*) \geq 0 \) for all \( x \in B \). Show that \( \phi \) is necessarily continuous.
(b) Prove that \( T' \) is a continuous map of \( \Delta_B \) into \( \Delta_A \).
(c) Show that \( \hat{x}(T'(\phi)) = \hat{T}(x)(\phi) \) for each \( x \in A \).
(d) Show that \( ||T'(x)|| \leq ||x|| \) and conclude that \( T \) is necessarily continuous.
(e) Prove that \( T' \) is onto if and only if \( T \) is 1-1. HINT: \( T \) is not 1-1 if and only if there exists a nontrivial continuous function on \( \Delta_A \) that is identically 0 on the range of \( T' \).
(f) Prove that \( T' \) is 1-1 if and only if \( T \) is onto.
(g) Prove that \( T' \) is a homeomorphism of \( \Delta_B \) onto \( \Delta_A \) if and only if \( T \) is an isomorphism of \( A \) onto \( B \).

**EXERCISE 10.12.** (Independence of the Spectrum)

(a) Suppose \( B \) is a commutative \( C^* \)-algebra with identity \( I \), and that \( A \) is a sub-\( C^* \)-algebra of \( B \) containing \( I \). Let \( x \) be an element of \( A \). Prove that \( \text{sp}_A(x) = \text{sp}_B(x) \). HINT: Let \( T \) be the injection map of \( A \) into \( B \).
(b) Suppose \( C \) is a (not necessarily commutative) \( C^* \)-algebra with identity \( I \), and let \( x \) be a normal element of \( C \). Suppose \( A \) is the smallest
sub-$C^*$-algebra of $C$ that contains $x$, $x^*$, and $I$. Prove that $\text{sp}_A(x) = \text{sp}_C(x)$. HINT: If $\lambda \in \text{sp}_A(x)$, and $\lambda I - x$ has an inverse in $C$, let $B$ be the smallest sub-$C^*$-algebra of $C$ containing $x, I, and (\lambda I - x)^{-1}$. Then use part a.

(c) Let $H$ be a separable Hilbert space, and let $T$ be a normal element of $B(H)$. Let $A$ be the smallest sub-$C^*$-algebra of $B(H)$ containing $T, T^*$, and $I$. Show that the spectrum $\text{sp}(T)$ of the operator $T$ coincides with the spectrum $\text{sp}_A(T)$ of $T$ thought of as an element of $A$.

THEOREM 10.9. (Spectral Theorem) Let $H$ be a separable Hilbert space, let $A$ be a separable, commutative, sub-$C^*$-algebra of $B(H)$ that contains the identity operator $I$, and let $\Delta$ denote the structure space of $A$. Write $B$ for the $\sigma$-algebra of Borel subsets of $\Delta$. Then there exists a unique $H$-projection-valued measure $p$ on $(\Delta, B)$ such that for every operator $S \in A$ we have

$$S = \int \hat{S} \, dp.$$ 

That is, the inverse of the Gelfand transform is the integral with respect to $p$.

PROOF. Since $A$ contains $I$, we know that $\Delta$ is compact and metrizable. Since the inverse $T$ of the Gelfand transform is an isometric isomorphism of the Banach algebra $C(\Delta)$ onto $A$, we see that $T$ satisfies the three conditions of Theorem 9.7.

1. $T(fg) = T(f)T(g)$ for all $f, g \in C(\Delta)$.
2. $T(f) = [T(f)]^*$ for all $f \in C(\Delta)$.
3. $T(1) = I$.

The present theorem then follows immediately from Theorem 9.7.

THEOREM 10.10. (Spectral Theorem for a Bounded Normal Operator) Let $T$ be a bounded normal operator on a separable Hilbert space $H$. Then there exists a unique $H$-projection-valued measure $p$ on $(C, B)$ such that

$$T = \int f \, dp = \int f(\lambda) \, dp(\lambda),$$

where $f(\lambda) = \lambda$. (We also use the notation $T = \int \lambda \, dp(\lambda)$.) Furthermore, $p_{\text{sp}(T)} = I$; i.e., $p$ is supported on the spectrum of $T$.

PROOF. Let $A_0$ be the set of all elements $S \in B(H)$ of the form

$$S = \sum_{i=0}^{n} \sum_{j=0}^{m} a_{ij} T^i T^{*j},$$
where each $a_{ij} \in \mathbb{C}$, and let $A$ be the closure in $B(H)$ of $A_0$. We have that $A$ is the smallest sub-$C^*$-algebra of $B(H)$ that contains $T$, $T^*$, and $I$. It follows that $A$ is a separable commutative sub-$C^*$-algebra of $B(H)$ that contains $I$. If $\Delta$ denotes the structure space of $A$, then, by Theorem 10.9, there exists a unique projection-valued measure $q$ on $(\Delta, \mathcal{B})$ such that

$$S = \int \hat{S} \, dq = \int \hat{S}(\phi) \, dq(\phi)$$

for every $S \in A$.

Note next that the function $\hat{T}$ is 1-1 on $\Delta$. For, if $\hat{T}(\phi_1) = \hat{T}(\phi_2)$, then $\hat{T}^*(\phi_1) = \hat{T}^*(\phi_2)$, and hence $\hat{S}(\phi_1) = \hat{S}(\phi_2)$ for every $S \in A_0$. Therefore, $\hat{S}(\phi_1) = \hat{S}(\phi_2)$ for every $S \in A$, showing that $\phi_1 = \phi_2$. Hence, $\hat{T}$ is a homeomorphism of $\Delta$ onto the subset $sp_A(T)$ of $\mathbb{C}$. By part c of Exercise 10.12, $sp_A(T) = sp(T)$.

Define a projection-valued measure $p = \hat{T}_* q$ on $sp(T)$ by

$$p_E = \hat{T}_* q_E = q_{\hat{T}^{-1}(E)}.$$ 

See part c of Exercise 9.3. Then $p$ is a projection-valued measure on $(\mathbb{C}, \mathcal{B})$, and $p$ is supported on $sp(T)$.

Now, let $f$ be the identity function on $\mathbb{C}$, i.e., $f(\lambda) = \lambda$. Then, by Exercise 9.13, we have that

$$\int \lambda \, dp(\lambda) = \int f \, dp = \int (f \circ \hat{T}) \, dq = \int \hat{T} \, dq = T,$$

as desired.

Finally, let us show that the projection-valued measure $p$ is unique. Suppose $p'$ is another projection-valued measure on $(\mathbb{C}, \mathcal{B})$, supported on $sp(T)$, such that

$$T = \int \lambda \, dp'(\lambda) = \int \lambda \, dp(\lambda).$$

It follows also that

$$T^* = \int \lambda \, dp'(\lambda) = \int \lambda \, dp(\lambda).$$
Then, for every function $P$ of the form

$$P(\lambda) = \sum_{i=1}^{n} \sum_{j=1}^{m} c_{ij} \lambda^i \bar{\lambda}^j,$$

we have

$$\int P(\lambda) \, dp'(\lambda) = \int P(\lambda) \, dp(\lambda).$$

Whence, by the Stone-Weierstrass Theorem,

$$\int f(\lambda) \, dp'(\lambda) = \int f(\lambda) \, dp(\lambda)$$

for every continuous complex-valued function $f$ on $\text{sp}(T)$. If $q' = \hat{T}_*^{-1} p'$ is the projection-valued measure on $\Delta$ defined by

$$q'_E = p'_{T(E)},$$

then, for any continuous function $g$ on $\Delta$, we have

$$\int g \, dq' = \int (g \circ \hat{T}_*^{-1}) \, dp'$$
$$= \int (g \circ \hat{T}_*^{-1}) \, dp$$
$$= \int (g \circ \hat{T}_*^{-1} \circ \hat{T}) \, dq$$
$$= \int g \, dq.$$

So, by the uniqueness assertion in the general spectral theorem, we have that $q' = q$. But then

$$p' = \hat{T}_* q' = \hat{T}_* q = p,$$

and the uniqueness is proved.

**DEFINITION.** The projection-valued measure $p$, associated as in the above theorem to a normal operator $T$, is called the *spectral measure* for $T$.

The next result is an immediate consequence of the preceding theorem.
THEOREM 10.11. (Spectral Theorem for a Bounded Selfadjoint Operator) Let $H$ be a separable Hilbert space, and let $T$ be a selfadjoint element in $B(H)$. Then there exists a unique projection-valued measure $p$ on $(\mathbb{R}, B)$ for which $T = \int \lambda dp(\lambda)$. Further, $p$ is supported on the spectrum of $T$.

REMARK. A slightly different notation is frequently used to indicate the spectral measure for a selfadjoint operator. Instead of writing $T = \int \lambda dp(\lambda)$, one often writes $T = \int \lambda dE\lambda$. Also, such a projection-valued measure is sometimes referred to as a resolution of the identity.

EXERCISE 10.13. Let $T$ be a normal operator in $B(H)$ and let $p$ be its spectral measure.

(a) If $U$ is a nonempty (relatively) open subset of $\text{sp}(T)$, show that $p_U \neq 0$. If $U$ is an infinite set, show that the range of $p_U$ is infinite-dimensional.

(b) Show that if $E$ is a closed subset of $\mathbb{C}$ for which $p_E = I$, then $E$ contains $\text{sp}(T)$. Conclude that the smallest closed subset of $\mathbb{C}$ that supports $p$ is the spectrum of $T$.

(c) If $T$ is invertible, show that the function $1/\lambda$ is bounded on $\text{sp}(T)$ and that $T^{-1} = \int (1/\lambda) dp(\lambda)$.

(d) If $\text{sp}(T)$ contains at least two distinct points, show that $T = T_1 + T_2$, where $T_1$ and $T_2$ are both nonzero normal operators and $T_1 \circ T_2 = 0$.

(e) Suppose $S$ is a bounded operator on $H$ that commutes with every projection $p_E$ for $E$ a Borel subset of $\text{sp}(T)$. HINT: Do this first for open subsets of $\text{sp}(T)$, and then consider the collection of all sets $E$ for which $p_E S = S p_E$. (It is a monotone class.)

(f) Suppose $S$ is a bounded operator that commutes with $T$. Let $E = \text{sp}(T) \cap B_\epsilon(\lambda_0)$, where $\epsilon > 0$ and $\lambda_0$ is a complex number. Show that, if $x$ belongs to the range of $p_E$, then $S(x)$ also belongs to the range of $p_E$, implying that $S$ commutes with $p_E$. (Use part b of Exercise 9.11.) Deduce the Fuglede-Putnam-Rosenbloom Theorem: If a bounded operator $S$ commutes with a bounded normal operator $T$, then $S$ also commutes with $T^*$.

EXERCISE 10.14. Let $T$ be a normal operator on a separable Hilbert space $H$, let $A$ be a sub-$C^*$-algebra of $B(H)$ that contains $T$ and $I$, let $f$ be a continuous complex-valued function on the spectrum $\text{sp}(T)$ of $T$, and suppose $S$ is an element of $A$ for which $S = f \circ T$.

(a) Show that the spectrum $\text{sp}(S)$ of $S$ equals $f(\text{sp}(T))$. Compare this result with the spectral mapping theorem (part e of Exercise 10.7).
(b) Let \( p^T \) denote the spectral measure for \( T \) and \( p^S \) denote the spectral measure for \( S \). In the notation of Exercises 9.3 and 9.13, show that

\[
p^S = f_*(p^T).
\]

**HINT:** Show that \( S = \int \lambda d f_*(p^T)(\lambda) \), and then use the uniqueness assertion in the Spectral Theorem for a normal operator.

(c) Apply parts a and b to describe the spectral measures for \( S = q(T) \) for \( q \) a polynomial and \( S = e^T \).

**EXERCISE 10.15.** Let \( p \) be an \( H \)-projection-valued measure on the Borel space \((S, B)\). If \( f \) is an element of \( L^\infty(p) \), define the **essential range** of \( f \) to be the set of all \( \lambda \in \mathbb{C} \) for which

\[
p_{f^{-1}(B_\epsilon(\lambda))} \neq 0
\]

for every \( \epsilon > 0 \).

(a) Let \( f \) be an element of \( L^\infty(p) \). If \( T \) is the bounded normal operator \( \int f \, dp \), show that the spectrum of \( T \) coincides with the essential range of \( f \). See part e of Exercise 9.10.

(b) Let \( f \) be an element of \( L^\infty(p) \), and let \( T = \int f \, dp \). Prove that the spectral measure \( q \) for \( T \) is the projection-valued measure \( f_*p \). See Exercises 9.3 and 9.13.

**EXERCISE 10.16.** Let \((S, \mu)\) be a \( \sigma \)-finite measure space. For each \( f \in L^\infty(\mu) \), let \( m_f \) denote the multiplication operator on \( L^2(\mu) \) given by \( m_f g = fg \). Let \( p \) denote the canonical projection-valued measure on \( L^2(\mu) \).

(a) Prove that the operator \( m_f \) is a normal operator and that

\[
m_f = \int f \, dp.
\]

Find the spectrum \( \text{sp}(m_f) \) of \( m_f \).

(b) Using \( S = [0, 1] \) and \( \mu \) as Lebesgue measure, find the spectrum and spectral measures for the following \( m_f \)'s:

(1) \( f = \chi_{[0,1/2]} \),
(2) \( f(x) = x \),
(3) \( f(x) = x^2 \),
(4) \( f(x) = \sin(2\pi x) \), and
(5) \( f \) is a step function \( f = \sum_{i=1}^n a_i \chi_{I_i} \), where the \( a_i \)'s are complex numbers and the \( I_i \)'s are disjoint intervals.
(c) Let $S$ and $\mu$ be as in part b. Compute the spectrum and spectral measure for $m_f$ if $f$ is the Cantor function.

**DEFINITION.** We say that an operator $T \in B(H)$ is **diagonalizable** if it can be represented as the integral of a function with respect to a projection-valued measure. That is, if there exists a Borel space $(S, B)$ and an $H$-projection-valued measure $p$ on $(S, B)$ such that $T = \int f \, dp$ for some bounded $B$-measurable function $f$. A collection $B$ of operators is called **simultaneously diagonalizable** if there exists a projection-valued measure $p$ on a Borel space $(S, B)$ such that each element of $B$ can be represented as the integral of a function with respect to $p$.

**REMARK.** Theorem 10.11 and Theorem 10.10 show that selfadjoint and normal operators are diagonalizable. It is also clear that simultaneously diagonalizable operators commute.

**EXERCISE 10.17.** (a) Let $H$ be a separable Hilbert space. Suppose $B$ is a commuting, separable, selfadjoint subset of $B(H)$. Prove that the elements of $B$ are simultaneously diagonalizable.

(b) Let $H$ be a separable Hilbert space. Show that a separable, selfadjoint collection $S$ of operators in $B(H)$ is simultaneously diagonalizable if and only if $S$ is contained in a commutative sub-$C^∗$-algebra of $B(H)$.

(c) Let $A$ be an $n \times n$ complex matrix for which $a_{ij} = a_{ji}$. Use the Spectral Theorem to show that there exists a unitary matrix $U$ such that $UAU^{-1}$ is diagonal. That is, use the Spectral Theorem to prove that every Hermitian matrix can be diagonalized.

One of the important consequences of the spectral theorem is the following:

**THEOREM 10.12.** (Stone’s Theorem) Let $t \rightarrow U_t$ be a map of $\mathbb{R}$ into the set of unitary operators on a separable Hilbert space $H$, and suppose that this map satisfies:

1. $U_{t+s} = U_t \circ U_s$ for all $t, s \in \mathbb{R}$.
2. The map $t \rightarrow (U_t(x), y)$ is continuous for every pair $x, y \in H$.

Then there exists a unique projection-valued measure $p$ on $(\mathbb{R}, B)$ such that

$$U_t = \int e^{-2\pi i \lambda t} \, dp(\lambda)$$

for each $t \in \mathbb{R}$.
PROOF. For each \( f \in L^1(\mathbb{R}) \), define a map \( L_f \) from \( H \times H \) into \( \mathbb{C} \) by
\[
L_f(x, y) = \int_{\mathbb{R}} f(s)(U_s(x), y) \, ds.
\]
It follows from Theorem 8.5 (see the exercise below) that for each \( f \in L^1(\mathbb{R}) \) there exists a unique element \( T_f \in B(H) \) such that
\[
L_f(x, y) = (T_f(x), y)
\]
for all \( x, y \in H \). Let \( B \) denote the set of all operators on \( H \) of the form \( T_f \) for \( f \in L^1(\mathbb{R}) \). Again by the exercise below, it follows that \( B \) is a separable commutative selfadjoint subalgebra of \( B(H) \).

We claim first that the subspace \( H_0 \) spanned by the vectors of the form \( y = T_f(x) \), for \( f \in L^1(\mathbb{R}) \) and \( x \in H \), is dense in \( H \). Indeed, if \( z \in H \) is orthogonal to every element of \( H_0 \), then
\[
0 = (T_f(z), z) = \int_{\mathbb{R}} f(s)(U_s(z), z) \, ds
\]
for all \( f \in L^1(\mathbb{R}) \), whence
\[
(U_s(z), z) = 0
\]
for almost all \( s \in \mathbb{R} \). But, since this is a continuous function of \( s \), it follows that
\[
(U_s(z), z) = 0
\]
for all \( s \). In particular,
\[
(z, z) = (U_0(z), z) = 0,
\]
proving that \( H_0 \) is dense in \( H \) as claimed.

We let \( A \) denote the smallest sub-\( C^* \)-algebra of \( B(H) \) that contains \( B \) and the identity operator \( I \), and we denote by \( \Delta \) the structure space of \( A \). We see that \( A \) is the closure in \( B(H) \) of the set of all elements of the form \( \lambda I + T_f \), for \( \lambda \in \mathbb{C} \) and \( f \in L^1(\mathbb{R}) \). So \( A \) is a separable commutative \( C^* \)-algebra. Again, by Exercise 10.18 below, we have that the map \( T \) that sends \( f \in L^1(\mathbb{R}) \) to the operator \( T_f \) is a norm-decreasing homomorphism of the Banach \( * \)-algebra \( L^1(\mathbb{R}) \) into the \( C^* \)-algebra \( A \). Recall from Exercise 10.5 that the structure space of the Banach algebra
$L^1(\mathbb{R})$ is identified, specifically as in that exercise, with the real line $\mathbb{R}$. With this identification, we define $T' : \Delta \to \mathbb{R}$ by

$$T'(\phi) = \phi \circ T.$$ 

Because the topologies on the structures spaces of $A$ and $L^1(\mathbb{R})$ are the weak$^*$ topologies, it follows directly that $T'$ is continuous. For each $f \in L^1(\mathbb{R})$ we have the formula

$$\hat{f}(T'(\phi)) = [T'(\phi)](f) = \phi(T_f) = \hat{T}_f(\phi).$$

By the general Spectral Theorem, we let $q$ be the unique projection-valued measure on $\Delta$ for which

$$S = \int S(\phi) \, dq(\phi)$$

for all $S \in A$, and we set $p = T'_* q$. Then $p$ is a projection-valued measure on $(\mathbb{R}, \mathcal{B})$, and we have

$$\int \hat{f} \, dp = \int (\hat{f} \circ T') \, dq$$

$$= \int \hat{f}(T'(\phi)) \, dq(\phi)$$

$$= \int \hat{T}_f(\phi) \, dq(\phi)$$

$$= T_f$$

for all $f \in L^1(\mathbb{R})$.

Now, for each $f \in L^1(\mathbb{R})$ and each real $t$ we have

$$(U_t(T_f(x)), y) = \int_\mathbb{R} f(s)(U_t(U_s(x)), y) \, ds$$

$$= \int_\mathbb{R} f(s)(U_{t+s}(x), y) \, ds$$

$$= \int_\mathbb{R} f_{-t}(s)(U_s(x), y) \, ds$$

$$= (T_{f_{-t}}(x), y)$$

$$= (|\int e^{-2\pi i \lambda t} \hat{f}(\lambda) \, dp(\lambda)|(x), y)$$

$$= (|\int e^{-2\pi i \lambda t} \hat{f}(\lambda) \, dp(\lambda)|(T_f(x)), y),$$
where $f_{-t}$ is defined by $f_{-t}(x) = f(x - t)$. So, because the set $H_0$ of all vectors of the form $T_f(x)$ span a dense subspace of $H$,

$$U_t = \int e^{-2\pi i \lambda t} \, dp(\lambda),$$

as desired.

We have left to prove the uniqueness of $p$. Suppose $\tilde{p}$ is a projection-valued measure on $(\mathbb{R}, B)$ for which $U_t = \int e^{-2\pi i \lambda t} \, d\tilde{p}(\lambda)$ for all $t$. Now for each vector $x \in H$, define the two measures $\mu_x$ and $\tilde{\mu}_x$ by

$$\mu_x(E) = (p_E(x), x)$$

and

$$\tilde{\mu}_x(E) = (\tilde{p}_E(x), x).$$

Our assumption on $\tilde{p}$ implies then that

$$\int e^{-2\pi i \lambda t} \, d\mu_x(\lambda) = \int e^{-2\pi i \lambda t} \, d\tilde{\mu}_x(\lambda)$$

for all real $t$. Using Fubini’s theorem we then have for every $f \in L^1(\mathbb{R})$ that

$$\int \hat{f}(\lambda) \, d\mu_x(\lambda) = \int \int f(t) e^{-2\pi i \lambda t} \, dt \, d\mu_x(\lambda)$$

$$= \int f(t) \int e^{-2\pi i \lambda t} \, d\mu_x(\lambda) \, dt$$

$$= \int f(t) \int e^{-2\pi i \lambda t} \, d\tilde{\mu}_x(\lambda) \, dt$$

$$= \int \hat{f}(\lambda) \, d\tilde{\mu}_x(\lambda).$$

Since the set of Fourier transforms of $L^1$ functions is dense in $C_0(\mathbb{R})$, it then follows that

$$\int g \, d\mu_x = \int g \, d\tilde{\mu}_x$$

for every $g \in C_0(\mathbb{R})$. Therefore, by the Riesz representation theorem, $\mu_x = \tilde{\mu}_x$. Consequently, $p = \tilde{p}$ (see part d of Exercise 9.2), and the proof is complete.

**Exercise 10.18.** Let the map $t \rightarrow U_t$ be as in the theorem above.

(a) Prove that $U_0$ is the identity operator on $H$ and that $U_t^* = U_{-t}$ for all $t$. 


(b) If \( f \in L^1(\mathbb{R}) \), show that there exists a unique element \( T_f \in B(H) \) such that
\[
\int_{\mathbb{R}} f(s)(U_s(x), y) \, ds = (T_f(x), y)
\]
for all \( x, y \in H \). HINT: Use Theorem 8.5.

(c) Prove that the assignment \( f \mapsto T_f \) defined in part b satisfies
\[
\|T_f\| \leq \|f\|_1
\]
for all \( f \in L^1(\mathbb{R}) \),
\[
T_{f*} g = T_f \circ T_g
\]
for all \( f, g \in L^1(\mathbb{R}) \) and
\[
T_{f*} = T_f^*
\]
for all \( f \in L^1(\mathbb{R}) \), where
\[
f^*(s) = \overline{f(-s)}.
\]

(d) Conclude that the set of all \( T_f \)'s, for \( f \in L^1(\mathbb{R}) \), is a separable commutative selfadjoint algebra of operators.

EXERCISE 10.19. Let \( H \) be a separable Hilbert space, let \( A \) be a separable, commutative, sub-C*-algebra of \( B(H) \), assume that \( A \) contains the identity operator \( I \), and let \( \Delta \) denote the structure space of \( A \). Let \( x \) be a vector in \( H \) and let \( M \) be the closure of the set of all vectors \( T(x) \), for \( T \in A \). That is, \( M \) is a cyclic subspace for \( A \). Prove that there exists a finite Borel measure \( \mu \) on \( \Delta \) and a unitary operator \( U \) of \( L^2(\mu) \) onto \( M \) such that
\[
U^{-1} \circ T \circ U = m_{\mu}
\]
for every \( T \in A \). HINT: Let \( G \) denote the inverse of the Gelfand transform of \( A \). Define a positive linear functional \( L \) on \( C(\Delta) \) by \( L(f) = (\overline{G(f)}(x), x) \), use the Riesz Representation Theorem to get a measure \( \mu \), and then define \( U(f) = [G(f)](x) \) on the dense subspace \( C(\Delta) \) of \( L^2(\mu) \).