SET 2 – due 16 September

Free spinless fermions in one dimension on a lattice of L lattice sites and lattice spacing a have a "hopping" or "tight-binding" Hamiltonian

$$H = -t_0 \sum_{m=0}^{L-1} c^\dagger_{m+1} c_m + c^\dagger_{m-1} c_m \tag{1}$$

where c_m annihilates an electron at site m and $c^\dagger_{m+1}$ creates an electron at site $m+1$. Assume that the sites form a ring of physical length aL where "a" is the lattice spacing.
(a) [4 points] Defining

$$c_k = \frac{1}{\sqrt{L}} \sum_m e^{ikam} c_m \tag{2}$$

where $k = 2\pi n_k/aL$, $n_k = -L/2, -L/2+1 \ldots L/2-1$, find an expression for H in momentum space. Sketch $E(k)$ vs. k. Note that for the case of periodic boundary conditions (particles on a ring) there is a useful identity

$$\frac{1}{L} \sum_k e^{ika(m-n)} = \delta_{m,n}. \tag{3}$$

(b)[5 points] Now define $|k_F|$ from $E(|k_F|) = 0$ and define particle and hole operators $c_{\pm(k_F+\kappa)} = b_{\pm\kappa}$ (annihilates a particle at κ) and $c_{\pm(k_F-\kappa)} = d^\dagger_{\pm\kappa}$ (creates a hole at κ). Measuring energies in terms of κ show

$$H = 2t_0 \sum_{\kappa=-k_F}^{\kappa=k_F} (b^\dagger_\kappa b_\kappa + d^\dagger_\kappa d_\kappa) \sin(|\kappa a|) + C \tag{4}$$

where C is a constant.

(c) [5 points] Show $[N, H] = 0$ where $N = \sum_m c^\dagger_m c_m$ (this is easiest in momentum space).

(d) [5 points] If we have a ground state consisting of a half-filled band, $c_k|0\rangle = 0$ for $|k| > k_F$ and $c^\dagger_k|0\rangle = 0$ ifor $|k| < k_F$, describe the first few excited states of the system with the same N as the ground state in terms of electrons and then in terms of particles and holes.

e) [6 points] If phonons are also present, an electron-phonon interaction might be $H_I = \sum_n c_n^\dagger c_n \phi_n$ with ϕ the phonon field. Writing

$$\phi_n = \sum_p a_p e^{-ipan} + a_p^\dagger e^{ipan}$$

write out H_I in momentum space and describe the different kinds of interaction terms you get in terms of electrons and in terms of particle/hole creation and annihilation.

Comments:

- Many variations on this Hamiltonian are used in condensed matter physics
- Remember this problem set when we come to the Dirac equation