SET 4 – due 7 October

1) [15 points] In statistical mechanics, the analog of a quantum field variable is called an "order parameter" or a "spin variable" and the idea is that at every point in space there is a variable $\phi(x)$ with some local symmetry properties. For example, for a uniaxial ferromagnet, $\phi(x)$ could be a field which can range over the real numbers. The analog of "mass" is called the "inverse correlation length." The analog of the action is the "Ginsberg-Landau Hamiltonian," a simple example of which is

$$H/T = \int d^D x [\frac{1}{2}(\partial_\mu \phi)^2 + V(\phi)] \tag{1}$$

where the parameters of V are assumed to be temperature-dependent. $V(\phi)$ is usually (but not always) a polynomial in ϕ. Otherwise the system is just path integral quantum field theory in Euclidean space.

(a) [3 points] If the spins couple to an external magnetic field $B(x,t)$ through a coupling $g\phi(x)B(x,t)$, the cross section for elastic scattering of light (recall Jackson) is proportional to

$$\Gamma_{fi} = \langle | \int d^3 x \phi(x) \exp(i(p_i - p_f)x|^2 \rangle . \tag{2}$$

Show how this measures the two-point function. Don't worry about the numerical factors.

(b) [3 points] The "Gaussian approximation" assumes that the functional integral is dominated by the (constant) field configuration(s) $\phi = \phi_m$ which minimize the Hamiltonian (said differently, minimize $V(\phi_m)$), and that one can expand the action density about the minimizing configuration–i.e., $\phi = \phi_m + \delta\phi(x)$. In general, how does the mass asociated with the fluctuation field $\delta\phi(x)$ depend on the properties of $V(\phi)$ for ϕ near to ϕ_m?

(c) [9 points] Show that in the Gaussian approximation $\langle \phi(x) \rangle = \phi_m$. For the G-L potential $V(\phi) = \frac{\mu^2}{2}\phi^2 + \frac{\lambda}{4!}\phi^4$, find $\langle \phi(x) \rangle$ as a function of μ^2, for μ^2 varying from a negative to a positive value. (The quantity μ^2 is just a parameter in the action!) Find the mass of the fluctuation field as a function of μ^2 also.

2) [10 points] This is about the physics of the "snail diagram" (shown below – it was one of the divergent diagrams that brought us to a screeching halt). Assuming that we could figure out how to make sense of it, what physics does it correspond to? The answer comes when you realize that the way to actually see a mass is to observe structure in a scattering amplitude, that is, the real mass is the value of squared momentum running through a propagator, at which the propagator has a pole. Imagine that we could compute the snail, and call the answer $-i\Sigma$, a constant. Then imagine a series of "one particle irreducible" diagrams, also shown in the figure; the first term is the propagator

$$-i\Delta_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} \tag{3}$$

p.2) the "snail" 

the series: 

the second term is

$$-i\Delta_F(p)(-i\Sigma) - i\Delta_F(p) \tag{4}$$

and so on. Sum the geometric series to make the "summed-up propagator." Then, where is its pole? (b) For a slight twist, the snail is just a constant but more complicated graphs will convert $-i\Sigma$ into $-i(p^2S_1 + S_2)$ with S_1 and S_2 both constants. Where is the pole now? Notice that the numerator of the summed up propagator is not unity any more; the residue at the pole has rescaled. Something interesting might be going on here, which we will have to sort out downstream.

3) [10 points] Most Lagrangians in quantum field theories have potentials which are polynomials in the field variables, or in powers of derivatives, or both. But not all. Here is one such system:

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 + \ldots \tag{5}$$

where

$$\mathcal{L}_1 = \frac{F^2}{4}(\partial^\mu U)^\dagger(\partial_\mu U) \tag{6}$$

and

$$\mathcal{L}_2 = \frac{F^2}{4}\chi(U + U^\dagger) \tag{7}$$

The field $\phi(x,t)$ is encoded in the variable

$$U = \exp(i\frac{\sqrt{2}}{F}\phi(x,t)). \tag{8}$$

The quantity F has dimensions of energy to match those of the field $\phi(x,t)$. The other constant, χ, has units of energy^2. The Lagrangian is written this way to expose the "shift symmetry" of $\mathcal{L}_1$, $\phi/F \to \phi/F + \alpha$. What is the physics of this model? Explicitly, I mean the mass-squared, M_0^2 of the phi particle, and a potential interaction term, both from $\mathcal{L}_2$? To answer this, you have to realize that everything we do with scalar fields assumes that the kinetic term is

$$K = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi \tag{9}$$

with the coefficient $1/2$ fixed to its "canonical value." One can also imagine adding two higher order terms,

$$\mathcal{L}_{ho} = L_4(\partial^\mu U)^\dagger(\partial_\mu U)(\chi(U + U^\dagger) + L_6(\chi(U + U^\dagger))^2 \tag{10}$$

How do these terms affect the mass of the ϕ particle? (Usually the literature answers this question assuming that L_4 and L_6 are small.)