SET 5 – due 21 October

People don't solve the Dirac equation all that often, but the first two problems show the most interesting properties of a nonrelativistic Dirac particle, its fine structure terms and its magnetic moment.

1) [10 points] Write the Dirac field for a particle in an external potential which is the fourth component of a four vector as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{1}$$

The Dirac equation in this case is

$$E\psi = (\alpha \cdot p + \beta m + V)\psi. \tag{2}$$

Assume $E = m + E'$ and E'/m is small. Solve for ψ_2 in terms of ψ_1. This will be an expansion in $1/m$. Obtain a Schrödinger equation for ψ_1, keeping all the terms through order $1/m^2$ which give the leading nontrivial corrections to the usual (spinless, nonrelativistic) Schrödinger equation. Identify the various terms. Let V be spherically symmetric, so

$$\vec{\nabla} V(r) = \frac{1}{r}\frac{\partial V}{\partial r}\hat{r}. \tag{3}$$

If V is a world scalar (i.e. transforms under Lorentz transformations as a scalar and appears in H as βV), what happens to the $L \cdot S$ term? (You do not need more than a sentence or two to answer this question.)

2) [10 points] Redo (1) for a free particle in an external constant magnetic field. Make the minimal substitution $\vec{p} \to \vec{p} - e\vec{A}$ where $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$ and show that the B-dependent part of the effective Hamiltonian is $H = -\frac{e}{2m}\sigma \cdot B$ – i.e., that the electron's g-factor is 2.

3) [10 points] A Majorana fermion written as a Dirac particle is

$$\psi^M = \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L^* \end{pmatrix}. \tag{4}$$

Writing the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{5}$$

as

$$\begin{pmatrix} -m & i\sigma^\mu \partial_\mu \\ i\bar{\sigma}^\mu \partial_\mu & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L^* \end{pmatrix} = 0, \tag{6}$$

where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\sigma)$, show that the dispersion relation for a free Majorana particle is $E^2 = p^2 + m^2$.