SET 1–due 4 September

We'll start out reviewing the algebra of creation and annihilation operators.

1) A coherent state $|\lambda\rangle$ of a one-dimensional oscillator is defined through

$$a|\lambda\rangle = \lambda|\lambda\rangle \tag{1}$$

where λ is a complex number.

(a) [5 points] Prove that the normalized coherent state is

$$|\lambda\rangle = e^{-|\lambda|^2/2}e^{\lambda a^\dagger}|0\rangle.$$

(Hint: begin with $|\lambda\rangle = \sum_n c(n)|n\rangle$ and find an iterative solution for the $c(n)$'s.)

(b) [5 points] What is $\langle\lambda|H|\lambda\rangle$? In addition, show that $|c(n)|^2$ is Poisson-distributed in n: what is the most probable n?

(c) [5 points] What is $\langle\lambda|x(t)|\lambda\rangle$? An important little lesson, told (like all lessons) cryptically: States of definite photon number have vanishing electric field; if you want to do all of Jackson starting from quantum electrodynamics, you must work with coherent states.

2) [15 points] One can define "propagators" for quantum mechanical systems. (This is the analog of something we'll see later in the course.) (a) [5 points] Evaluate

$$G(t) = \langle 0|x(t)x(0)|0\rangle\theta(t) + \langle 0|x(0)x(t)|0\rangle\theta(-t)$$

where $|0\rangle$ is the ground state for a simple harmonic oscillator of natural frequency Ω and $x(t)$ is the position operator at time t. It may be helpful to recall that operators evolve according to

$$-i\frac{dO}{dt} = [H, O]. \tag{2}$$

(b) [5 points] What is

$$G(\omega) = \int_{-\infty}^{\infty} \exp(i\omega t)G(t)dt? \tag{3}$$

(You may need to introduce convergence factors $\exp(\pm\epsilon t)$ in the integrals, and take the $\epsilon \to 0$ limit.)

(c) [5 points] Because $\Omega > 0$, the answer you found in part (b) is often written as

$$G(\omega) \sim \frac{1}{\omega^2 - \Omega^2 + i\epsilon} \tag{4}$$

where I have not bothered to write the correct prefactors (but you should!). Check that this expression is correct, and then compute $G(t)$ from $G(\omega)$ by inverting the Fourier transform. This is a little exercise in complex variables for you to review.

1) Coherent states are defined formally as

$$a|\lambda\rangle = \lambda|\lambda\rangle$$

Writing $|\lambda\rangle = \sum_n c_n|n\rangle$, this is

$$a|\lambda\rangle = \sum_n c_n\sqrt{n}\,|n-1\rangle = \lambda\sum c_n|n\rangle$$

$$= \sum_n c_{n+1}\sqrt{n+1}\,|n\rangle = \lambda\sum_n c_n|n\rangle$$

shifting the first sum. Thus

$$c_{n+1} = \frac{\lambda}{\sqrt{n+1}}\,c_n$$

$$|\lambda\rangle = c_0\left\{|0\rangle + \frac{\lambda}{\sqrt{1}}|1\rangle + \frac{\lambda^2|2\rangle}{\sqrt{1\cdot 2}} + \frac{\lambda^3|3\rangle}{\sqrt{3!}} + \cdots\right\}$$

$$= c_0\left\{|0\rangle + \frac{\lambda a^+|0\rangle}{1} + \frac{\lambda^2(a^+)^2|0\rangle}{2!} + \frac{\lambda^3(a^+)^3|0\rangle}{3!} + \cdots\right\}$$

$$= c_0\exp(\lambda a^+)|0\rangle.$$

To normalize, $\langle\lambda|\lambda\rangle = |c_0|^2\left\{1 + |\lambda|^2 + \frac{(|\lambda|^2)^2}{2!} + \cdots\right\}$

so $\langle\lambda|\lambda\rangle = |c_0|^2\exp(|\lambda|^2).$

and the normalized state is $|\lambda\rangle = \exp(-\frac{1}{2}|\lambda|^2)\exp(\lambda a^+)|0\rangle$

b) What is $\langle\lambda|H|\lambda\rangle$? $H = \omega[a^+a + \frac{1}{2}]$ so

$$\langle\lambda|H|\lambda\rangle = \omega\left[\frac{1}{2} + \langle\lambda|a^+a|\lambda\rangle\right] = \omega\left[\frac{1}{2} + |\lambda|^2\right]$$

using $(a|\lambda\rangle)^+ = \langle\lambda|a^+ = \langle\lambda|\lambda^*$.

For the second part of (b),

$$|c_n|^2 = \frac{(|\lambda|^2)}{n!} \exp[-|\lambda|^2]$$

which is the definition of a Poisson distribution,

$$P_n = e^{-x}\frac{x^n}{n!} \quad \text{with } x = |\lambda|^2.$$

The mean value of n is

$$\langle n\rangle = \frac{\sum_n n \cdot x^n e^{-x}/n!}{\sum_n x^n \frac{e^{-x}}{n!}} = e^{-x}\sum_n \frac{x^n}{(n-1)!}$$

$$= e^{-x} \cdot x \cdot \sum_j \frac{x^j}{j!} = x \qquad (j = n-1)$$

$\langle n\rangle = |\lambda|^2$ so if $\langle H\rangle = \omega(\frac{1}{2} + \langle n\rangle)$,

again $\langle H\rangle = \omega(\frac{1}{2} + |\lambda|^2)$

If you want to interpret "most probable" as the n which maximizes P_n, I can only do that using Stirling's approximation $\log n! = n \log n$.

Then $P_n = \exp[n\log x - n\log n] \times P_0$

$$\frac{dP_n}{dn}[n\log x - n\log n] = 0 \quad \text{at } n = \bar{n}$$

$\log x - \log \bar{n} - 1 = 0$; if $x \gg 1$, $\bar{n} \approx x = |\lambda|^2$

(again!)

c) $\langle \lambda | x(t) | \lambda \rangle$ uses $i \frac{dx}{dt} = [x, H]$

or $x(t) = e^{iHt} x e^{-iHt}$:

$$\langle \lambda | x(t) | \lambda \rangle = \langle \lambda | e^{iHt} x e^{-iHt} | \lambda \rangle$$

Use $e^{-iHt}|\lambda\rangle = \sum_n e^{-i\omega(n+\frac{1}{2})t} \frac{\lambda^n}{\sqrt{n!}} e^{-\frac{1}{2}|\lambda|^2} |n\rangle$

The $\frac{1}{2}\omega t$ will cancel between exponentials - drop it. And $x = \frac{1}{\sqrt{2m\omega}}(a + a^+)$ so

$$\langle x(t) \rangle = \frac{e^{-|\lambda|^2}}{\sqrt{2m\omega}} \sum_{j,n} \langle n | \frac{(\lambda^*)^n}{n!} e^{i\omega n t} \frac{\lambda^j}{\sqrt{j!}} e^{-i\omega j t}$$

$$\times (a + a^+) |j\rangle$$

$$= \frac{e^{-|\lambda|^2}}{\sqrt{2m\omega}} \sum_n \langle n | \frac{(\lambda^*)^n}{\sqrt{n!}} e^{i\omega(n-j)t} \frac{\lambda^j}{\sqrt{j!}}$$

$$\times (\sqrt{j}\,|j-1\rangle + \sqrt{j+1}\,|j+1\rangle).$$

There are two terms - one has $j-1=n$ with coefficient $\frac{e^{-i\omega t}(\lambda^*)^n \lambda^j}{\sqrt{n!\,(j-1)!}} = \frac{\lambda(|\lambda^2|)^n}{n!} e^{-i\omega t}$.

The other has $j+1=n$ with coefficient

$$e^{i\omega t}(\lambda^*)^n \lambda^j \sqrt{\frac{j+1}{n!\,j!}} = \frac{\lambda^*(|\lambda^2|)^{n-1}}{(n-1)!} e^{i\omega t}$$

Shift its sum, recognize the exponential,

$$\langle x(t)\rangle = \frac{1}{\sqrt{2m\omega}}\left[\lambda e^{-i\omega t} + \lambda^* e^{i\omega t}\right]$$

doing the sums. This is obviously real.

If λ is real, $\langle x(t)\rangle = \dfrac{2\lambda\cos\omega t}{\sqrt{2m\omega}}$.

In contrast, for energy eigenstates

$\langle n|x|n\rangle = 0.$

2) A propagator:

$$G(t) = \langle 0| x(t) x(0) |0\rangle \theta(t) + \langle 0| x(0) x(t) |0\rangle \times \theta(-t)$$

Again $i \frac{dx}{dt} = [x, H]$, $x(t) = e^{iHt} x(0) e^{-iHt}$.

Write $x(0) = x$.

$$G(t) = \langle 0| e^{iHt} x e^{-iHt} x |0\rangle \theta(t) + \langle 0| x e^{iHt} x e^{-iHt} |0\rangle \theta(-t)$$

$$= \sum_n \langle 0| e^{iHt} x |n\rangle \langle n| e^{-iHt} x |0\rangle \theta(t) + \ldots$$

inserting a complete set of energy eigenstates, $H|n\rangle = \Omega(n+\frac{1}{2})|n\rangle$. Because $\langle 0|x|n\rangle = 0$ if $n \neq 1$ and $\langle 0|x|1\rangle = \frac{1}{\sqrt{2m\Omega}}$ the sum collapses

$$G(t) = |\langle 0|x|1\rangle|^2 \left\{ e^{-i(E_1-E_0)t} \theta(t) + e^{i(E_1-E_0)t} \theta(-t) \right\}$$

and $E_1 - E_0 = \Omega$, the oscillator's natural frequency. So

a) $$G(t) = \frac{1}{2m\Omega} \left[e^{-i\Omega t} \theta(t) + e^{i\Omega t} \theta(-t) \right].$$

Now we need

$$G(\omega) = \frac{1}{2m\Omega} \left[\int_0^\infty e^{i\omega t} e^{-i\Omega t} e^{-\epsilon t} dt + \int_{-\infty}^0 e^{i\omega t} e^{i\Omega t} e^{\epsilon t} dt \right]$$

I Inserted $\exp(\pm \epsilon t)$ factors to define the integral (ϵ is infinitesimal -)

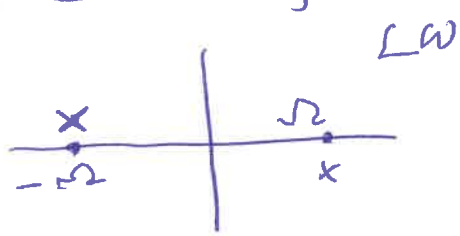
The integrals are easy:

$$G(\omega) = \frac{1}{2m\Omega}\left\{\frac{1}{-i\omega + i\Omega + \epsilon} + \frac{1}{i\omega + i\Omega + \epsilon}\right\}$$

$$= \frac{i}{2m\Omega}\left\{\frac{1}{\omega - \Omega + i\epsilon} + \frac{-1}{\omega + \Omega - i\epsilon}\right\}$$

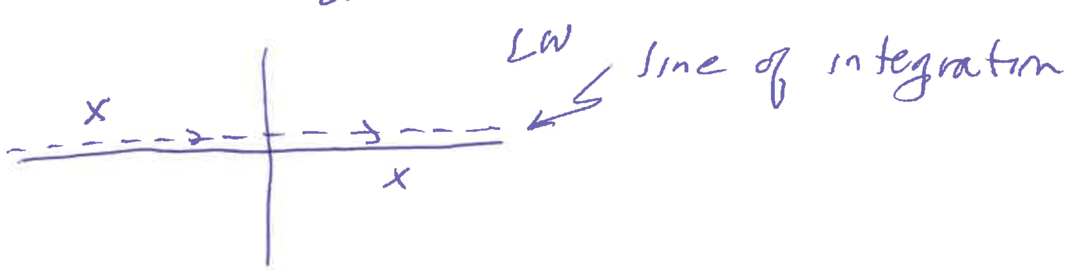
$$= \frac{i}{2m\Omega} \cdot \frac{2\Omega}{\left[\omega^2 - (\Omega - i\epsilon)^2\right]} = \frac{1}{m}\,\frac{i}{\omega^2 - (\Omega - i\epsilon)^2}$$

Note the singularity structure in the complex-ω plane

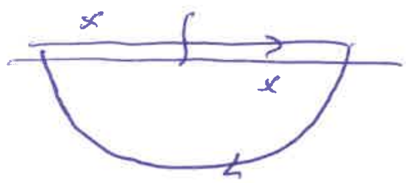


The locations of the poles are offset from the real-ω axis so that we can recover the correct answer when we do the inverse Fourier transform

$$G(t) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, G(\omega)\, e^{-i\omega t}$$



When $t>0$, $e^{-i\omega t}$ at $\omega = -iW$ goes to zero at large W. So we close the contour in the lower half plane, pick up the pole at $\Omega - i\epsilon$,



$$G(t) = \theta(t) \cdot \frac{i}{2m\Omega} \cdot \frac{(-2\pi i)}{2\pi}$$

$$= \frac{1}{2m\Omega}\,\theta(t).$$

and when $t<0$, close the contour in the upper half plane,

$$G(t) = \theta(-t)\left[\frac{-i}{2m\Omega}\right]\left[\frac{2\pi i}{2\pi}\right] = \frac{\theta(-t)}{2m\Omega}.$$

We have recovered the "direct result."

And the sloppy answer? Ω is positive by definition, and $(\Omega - i\epsilon)^2 \simeq \Omega^2 - 2i\epsilon\Omega$

Call $\epsilon' = 2\epsilon\Omega$, $(\Omega - i\epsilon)^2 = \Omega^2 - i\epsilon'$

$$G(\omega) = \frac{1}{m}\,\frac{1}{\omega^2 - \Omega^2 + i\epsilon'}$$

and just drop the prime!