SET 3 – due 30 September

1.) [25 points] Consider a system of two kinds of particles 1 and 2 (of masses m_1 and m_2) with an interaction Hamiltonian density

$$\mathcal{H}_I = g\phi_1^\dagger(x)\phi_1(x)\phi_2(x)$$

where ϕ_2 is self-adjoint ($a_2^c = a_2$). Find the scattering amplitude for $\phi_1(k_1) + \phi_1(k_2) \to \phi_1(k_3) + \phi_1(k_4)$ to lowest nontrivial order. Take your time with this, see how all the creation and annihilation operators collapse, and explain everything carefully. Sketch the relevant Feynman graph(s).

2.) [15 points] A useful set of relativistic invariants for the reaction $1+2 \to 3+4$ are $s = (p_1 + p_2)^2$, $t = (p_1 - p_3)^2$, $u = (p_1 - p_4)^2$. (a) [5 points] Check that $s + t + u = \sum_i m_i^2$ in the general case. (b) If energy and momenta greatly exceed masses, an often used approximation for the differntial cross section for a relativistic scattering problem is to assume that all particles are massless. Find a representation for the differential cross section for a reaction $1 + 2 \to 3 + 4$ in terms of s, t, and u in that approximation. Begin with the general formula given in the notes,

$$d\sigma = \frac{|M|^2}{2E_1 2E_2 v_{rel}}(2\pi^4)\delta^4(p_1 + p_2 - p_3 - p_4)\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3p_4}{(2\pi)^3 2E_4}. \qquad (1)$$

Your answer should look like

$$d\sigma = C\frac{1}{s}|M|^2\frac{dt\ du}{s}\delta(s + t + u) \qquad (2)$$

or

$$\frac{d\sigma}{dt} = C\frac{1}{s^2}|M|^2 \qquad (3)$$

where C is a dimensionless constant for you to find. Note that this result is valid in any reference frame; it is composed of invariants. (I evaluated the phase space integration in a convenient frame and rewrote the answer in terms of invariants. Also – distinguishable particles, assume M has no azimuthal dependence. Note $v_{rel} = 2$ in this limit.)

Keep this result for use later in the course. With massive particles, the delta function generalizes to $\delta(s + t + u - \sum_i m_i^2)$, no surprise. $d\sigma/dt$ is a relativistic generalization of $d\sigma/d\cos\theta$.

And a comment on the $2E_1 2E_2 v_{rel}$. Ryder remarks that this is equal to $4B$ where $B = [(p_1 \cdot p_2)^2 - m_1^2 m_2^2]^{1/2}$. Peskin and Schroeder write it as $4(E_2 p_1 - E_1 p_2) = 4\epsilon_{\mu x y \nu} p_1^\mu p_2^\nu$. "This is not Lorentz invariant, but it is invariant to boosts along the z-direction. In fact, this expression has exactly the transformation properties of a cross-sectional area."

3) [20 points] Evaluate the propagator for a nonrelativistic field

$$\psi(x,t) = \int \frac{d^3k}{(2\pi)^{3/2}} a(k) \exp(-i(E(k)t - \vec{k}\cdot\vec{x})) \tag{4}$$

from the formal definition of the time-ordered product. Express your answer in position-time space (this should look very familiar, it is the formula used in the spreading of a wave packet) and in momentum-frequency space (which should also be unsurprising). Here $E(k) = k^2/(2m)$, of course.

1) $\mathcal{H} = g\,\phi_1^+\phi_1\phi_2$. ϕ_2 is real - $a_c = a$. ϕ_1 is not

We want the amplitude for $\phi_1(k_1) + \phi_2(k_2) \to \phi_1(k_3) + \phi_1(k_4)$.
This will be a 2nd order process. To deal with long expressions, abbreviate

$$\phi_1(x) = \int \frac{d^3k}{(2\pi)^{3/2}} \frac{1}{\sqrt{2E(k)}} \left[a(k) e^{-ik\cdot x} + a_c^+(k) e^{ik\cdot x} \right]$$

by $\quad [a(k) + a_c^+(k)]_x$

$$\phi_2(x) = [b(k) + b^+(k)]_x$$

$$S = -\frac{g^2}{2!} \langle 0 | a(k_3)\, a(k_4) \sqrt{4E_3E_4}$$

$$\times \int d^4x\, d^4y\, T\{ [a^+(p_1) + a_c(p_1)]_x [a(p_2) + a_c^+(p_2)]_x$$

$$\times [b(p_3) + b^+(p_3)]_x$$

$$\times [a^+(p_4) + a_c(p_4)]_y [a(p_5) + a_c^+(p_5)]_y [b(p_6) + b^+(p_6)]_y \}$$

$$\times \sqrt{4E_1E_2}\; a^+(k_1)\, a^+(k_2) |0\rangle$$

There are four ways to contract fields (using, e.g.,
$a(k_d)\, a^+(k_e) = \delta^3(k_d - k_e) + a^+(k_e) a(k_d)$)

	x	y	x	y
1)	$p_1 = k_3$	$p_4 = k_4$	$p_2 = k_1$	$p_5 = k_2$
2)	$p_1 = k_3$	$p_4 = k_4$	$p_2 = k_2$	$p_5 = k_1$
3)	$p_1 = k_4$	$p_4 = k_3$	$p_2 = k_1$	$p_5 = k_2$
4)	$p_1 = k_4$	$p_4 = k_3$	$p_2 = k_2$	$p_5 = k_1$

All choices leave behind a $\langle 0|T(\phi_2(x)\phi_2(y))|0\rangle$

Consider each way in turn. Write $S = \frac{(2\pi)^4}{(2\pi)^6}\mathcal{M}$ where $\mathcal{M}$ will "lose" the δ^4 function.

First we have

$$-\frac{g^2}{2}\int \frac{d^4x\, d^4y}{(2\pi)^6}\, e^{-i(k_3-k_1)\cdot x}\, e^{-i(k_4-k_2)\cdot y}\,\langle 0|T(\phi_2(x)\phi_2(y))|0\rangle$$

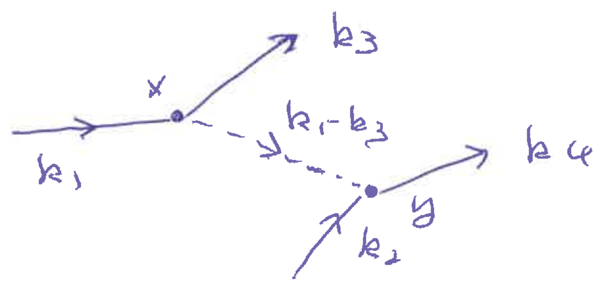
$$= -\frac{g^2}{2}\int \frac{d^4(x-y)}{(2\pi)^6}\langle 0|T(\phi_2(x)\phi_2(y))|0\rangle\, e^{-i(k_3-k_1)\cdot(x-y)} \times \int d^4y\, e^{-i(k_4+k_3-k_1-k_2)}$$

$$= -\frac{g^2}{2}\left[-i\Delta_F(k_3-k_1)\right]\frac{(2\pi)^4}{(2\pi)^6}\delta^4(k_3+k_4-k_1-k_2)$$

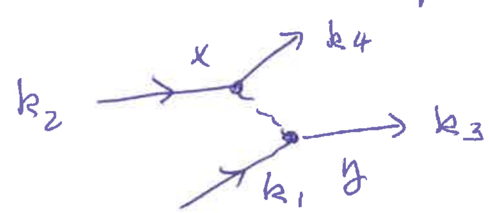
i.e. the invariant amplitude is

$$\mathcal{M}^{(1)} = -\frac{g^2}{2}\,\frac{i}{[(k_3-k_1)^2 - m_2^2 + i\epsilon]}\,.$$

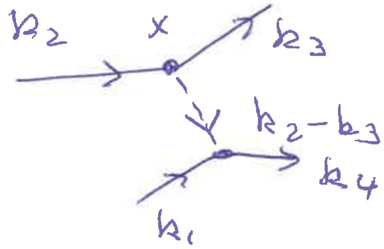
This corresponds to the graph



$\mathcal{M}^{(4)}$ is identical apart from exchanging x and y



$M^{(2)}$ is



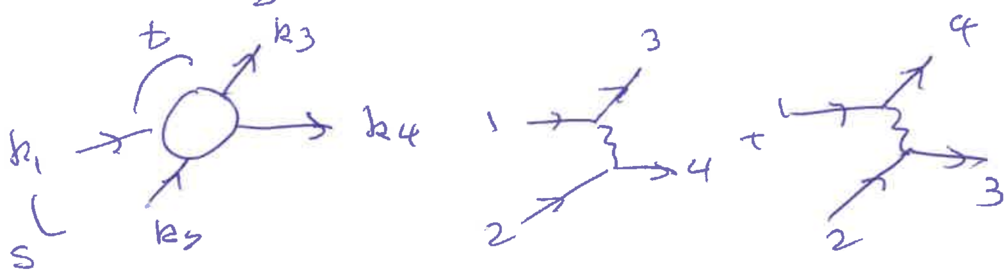
$$= -\frac{g^2}{2} \frac{i}{\left[(k_2-k_3)^2 - m_2^2 + i\epsilon\right]}$$

and $M^{(3)} = M^{(2)}$ with x and y exchanged.

Thus the invariant amplitude is

$$\mathcal{M} = -g^2 \left[\frac{i}{(k_3-k_1)^2 - m_2^2 + i\epsilon} + \frac{i}{(k_3-k_2)^2 - m_2^2 + i\epsilon} \right]$$

There is both a t-channel and a u-channel exchange.

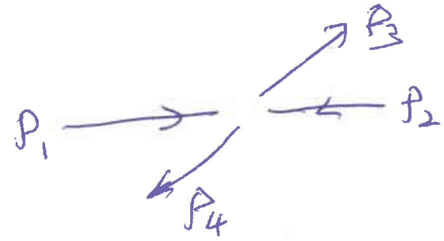


Looking ahead to problem 2, at high energy

$$\frac{d\sigma}{dt} = \frac{g^4}{16\pi s^2} \left| \frac{1}{t} + \frac{1}{u} \right|^2$$

2) Useful kinematics for $2 \to 2$ scattering

a)



$$s = (p_1+p_2)^2 = (p_3+p_4)^2$$
$$t = (p_1-p_3)^2 = (p_2-p_4)^2$$
$$u = (p_1-p_4)^2 = (p_2-p_3)^2$$

A quick way to get the part (a) identity is to square $\left[p_1+p_2-p_3-p_4\right]^2 = \sum_{i=1}^{4} m_i^2 + 2p_1\cdot p_2 - 2p_1\cdot p_3 - 2p_1\cdot p_4$
$-2p_2\cdot p_3 - 2p_2\cdot p_4 + 2p_3\cdot p_4$.

$$\begin{aligned}
0 = \sum m_i^2 &+ (p_1+p_2)^2 - m_1^2 - m_2^2 \\
&+ (p_1-p_3)^2 - m_1^2 - m_3^2 \\
&+ (p_1-p_4)^2 - m_1^2 - m_4^2 \\
&+ (p_2-p_3)^2 - m_2^2 - m_3^2 \\
&+ (p_2-p_4)^2 - m_2^2 - m_4^2 \\
&+ (p_3-p_4)^2 - m_3^2 - m_4^2
\end{aligned}$$

$$0 = 2\left[s+t+u\right] - 2\sum_{i=1}^{4} m_i^2 \quad \text{or} \quad \boxed{s+t+u = \sum m_i^2}$$

b) Begin with $d\sigma = \dfrac{|M|^2}{4E_1E_2 v_{rel}}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)$

$$\times \frac{d^3p_3}{2E_3(2\pi)^3}\,\frac{d^3p_4}{2E_4(2\pi)^3}. \qquad \times\ \cancel{\frac{d^4p_3\, d^4p_4\, \delta(p_3^2)\,\delta(p_4^2)}{(2\pi)^3\,(2\pi)^3}}$$

(~~this ... $\frac{d^3p}{2E(p)}$ ... $d^4p\,\delta(E^2-p^2-m^2)$ and setting ...~~

I will evaluate this in the CM frame and write my answer in terms of invariants, which will give a frame invariant result. To start, $v_{rel} = 2$ for 2 massless particles colliding head-on in CM frame.

I'll write my 4-vectors as $p^\mu = (E, p_z, p_x, p_y)$
and pick my axes so $p_y = 0$ (scattering in the z-x plane)
I can write $p_1 = (E, E, 0, 0)$, $p_2 = (E, -E, 0, 0)$
so $s = 4E^2$, $4E_1E_2 v_{rel} = 2s$
and we don't have energy ~~conser~~ conservation, yet)
$p_3 = (E_3, E_3\cos\theta, E_3\sin\theta, 0)$.

$p_1^2 = 0$, $p_2^2 = 0$, $p_3^2 = 0$. Now write

$$\delta^4(p_1+p_2-p_3-p_4)\,\frac{d^3p_3}{2E_3}\,\frac{d^3p_4}{2E_4} =$$

$$= \delta^4(p_1+p_2-p_3-p_4)\,\frac{d^3p}{2E_3}\,d^4p_4\,\delta(p_4^2)$$

Integrate d^4p_4 over the δ-function, leaving

$$d\sigma = \frac{1}{(2\pi)^2}\,\frac{|M|^2}{2s}\,\frac{d^3p_3}{2E_3}\,\delta((p_1+p_2-p_3)^2).$$

We write $\dfrac{d^3p_3}{2E_3} = d\cos\theta\, d\varphi\, \dfrac{E_3^2\, dE_3}{2E_3}$.

We assume ~~...~~ M has no φ dependence, so this is

$$\frac{d^3p}{2E_3} = \frac{2\pi}{2}\, d\cos\theta\, E_3\, dE_3.$$

Now $(p_1+p_2-p_3)^2 = 2p_1\cdot p_2 - 2p_1\cdot p_3 - 2p_2\cdot p_3 = s+t+u$

$$t = -2p_1\cdot p_3 = -2EE_3(1-\cos\theta)$$
$$u = -2p_1\cdot p_3 = -2EE_3(1+\cos\theta)$$

Change variables from $(E_3, \cos\theta)$ to (t, u).

The Jacobean is $\dfrac{\partial(t,u)}{\partial(E_3,\cos\theta)} = \begin{vmatrix} -2E(1-\cos\theta) & -2E(1+\cos\theta) \\ -2EE_3 & -2EE_3 \end{vmatrix}$

$$= 4E^2E_3(1-\cos\theta) - 4E^2E_3(1+\cos\theta) = 8E^2E_3$$

so $dt\,du = 8E^2 E_3\, dE_3\, d\cos\theta$

or $E_3\, dE_3\, d\cos\theta = \frac{dt\,du}{8E^2} = \frac{dt\,du}{2s}$

Putting everything together,

$$d\sigma = \frac{1}{4\pi^2}\frac{|M|^2}{2s}\frac{2\pi}{2}\frac{dt\,du}{2s}\,\delta(s+t+u)$$

$$= \frac{1}{16\pi}\frac{|M|^2}{s^2}\,dt\,du\,\delta(s+t+u).$$

More usefully,

$$\frac{d\sigma}{dt} = \frac{|M|^2}{16\pi s^2}$$

This is a generalization of $\frac{d\sigma}{d\cos\theta}$ valid in <u>any</u> frame. All the δ-functions give $E_3 = E$ ($s+t+u = 4E^2 - 4EE_3$), so in the cm,

$$t = -\cancel{4}\,2E^2(1-\cos\theta)$$

$$= -\frac{s}{2}(1-\cos\theta)$$

$$dt = \frac{s}{2}\,d\cos\theta$$

and $\frac{d\sigma}{d\cos\theta} = \frac{|M|^2}{32\pi s}$ in the c.m. frame (<u>only</u>!)

3) $\psi(x,t) = \int \frac{d^3k}{(2\pi)^{3/2}}\, a(k) \exp\left[-i(E_k t - \vec{k}\cdot\vec{x})\right]$

$E_k = \frac{k^2}{2m}$

$-i\Delta_F(x,t) = \langle 0| T\left[\psi(x,t)\,\psi^{+}(0,0)\right]|0\rangle$

$= \theta(t) \int \frac{d^3k\, d^3k'}{(2\pi)^3}\, \delta^3(k-k')\, e^{-i(E_k t - \vec{k}\cdot\vec{x})}$

(contracting $\langle 0| a(k)\, a^{+}(k')|0\rangle$)

$= \theta(t) \int \frac{d^3k}{(2\pi)^3} \exp -i(E_k t - \vec{k}\cdot\vec{x})$

The integral is done most easily in rectangular coordinates, by completing the square. In one dimension,

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \exp\left[-\frac{it}{2m}\left(k^2 - xk\frac{2m}{t} + \frac{m^2x^2}{t^2} - \frac{m^2x^2}{t^2}\right)\right]$$

$$= \frac{1}{2\pi}\sqrt{\frac{2m\pi}{it}} \exp\left[\frac{imx^2}{2t}\right] ;$$

$$-i\Delta_F(x,t) = \theta(t)\left[\frac{m}{2\pi i t}\right]^{3/2} \exp\left(\frac{im|\vec{x}|^2}{2t}\right).$$

This is the "spreading of the wave packet" formula.

To go to (k,ω) space, use $\theta(t) = \frac{i}{2\pi}\int \frac{d\omega}{\omega + i\epsilon}\, e^{-i\omega t}$

$$-i\Delta_F(\vec{k},\Omega) = \int dt\, e^{i\Omega t} \int d^3x\, e^{-ip\cdot x}\left[-i\Delta_F(x,t)\right]$$

$$= \int dt \int d^3x \; e^{i\Omega t} e^{ip\cdot x} \left[\frac{1}{2\pi} \int \frac{d\omega}{\omega + i\epsilon} e^{-i\omega t} \right]$$

$$\times \int \frac{d^3k}{(2\pi)^3} \exp\left[-i \frac{k^2}{2m} t + i \vec{k}\cdot\vec{x} \right]$$

The x-integral gives $(2\pi)^3 \delta^3(\vec{p} - \vec{k})$.

The t-integral, $2\pi\, \delta(\Omega - \omega - \frac{k^2}{2m})$

so

$$-i\Delta_F(\vec{p}, \Omega) = i \int d^3k \; \delta^3(\vec{p} - \vec{k}) \frac{d\omega}{\omega + i\epsilon} \delta(\Omega - \omega - \frac{k^2}{2m})$$

$$= \frac{i}{\Omega - \frac{k^2}{2m} + i\epsilon}$$

The place where the particle goes on-shell is $\Omega = \frac{k^2}{2m}$ (of course!)