SET 5 – due 21 October

People don't solve the Dirac equation all that often, but the first two problems show the most interesting properties of a nonrelativistic Dirac particle, its fine structure terms and its magnetic moment.

1) [10 points] Write the Dirac field for a particle in an external potential which is the fourth component of a four vector as

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}. \tag{1}$$

The Dirac equation in this case is

$$E\psi = (\alpha \cdot p + \beta m + V)\psi. \tag{2}$$

Assume $E = m + E'$ and E'/m is small. Solve for ψ_2 in terms of ψ_1. This will be an expansion in $1/m$. Obtain a Schrödinger equation for ψ_1, keeping all the terms through order $1/m^2$ which give the leading nontrivial corrections to the usual (spinless, nonrelativistic) Schrödinger equation. Identify the various terms. Let V be spherically symmetric, so

$$\vec{\nabla} V(r) = \frac{1}{r}\frac{\partial V}{\partial r}\hat{r}. \tag{3}$$

If V is a world scalar (i.e. transforms under Lorentz transformations as a scalar and appears in H as βV), what happens to the $L \cdot S$ term? (You do not need more than a sentence or two to answer this question.)

2) [10 points] Redo (1) for a free particle in an external constant magnetic field. Make the minimal substitution $\vec{p} \to \vec{p} - e\vec{A}$ where $\vec{A} = \frac{1}{2}\vec{B} \times \vec{r}$ and show that the B-dependent part of the effective Hamiltonian is $H = -\frac{e}{2m}\sigma \cdot B$ – i.e., that the electron's g-factor is 2.

3) [10 points] A Majorana fermion written as a Dirac particle is

$$\psi^M = \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L^* \end{pmatrix}. \tag{4}$$

Writing the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi = 0 \tag{5}$$

as

$$\begin{pmatrix} -m & i\sigma^\mu \partial_\mu \\ i\bar{\sigma}^\mu \partial_\mu & -m \end{pmatrix} \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L^* \end{pmatrix} = 0, \tag{6}$$

where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\sigma)$, show that the dispersion relation for a free Majorana particle is $E^2 = p^2 + m^2$.

1) $$E\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{bmatrix} m+V & \sigma\cdot p \\ \sigma\cdot p & -m+V\end{bmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$$

(Bjorken-Drell γ-convention) – this is

$$(E+m-V)\psi_2 = \sigma\cdot p\,\psi_1$$
$$(E-m-V)\psi_1 = \sigma\cdot p\,\psi_2 = \sigma\cdot p\,\frac{1}{E+m-V}\,\sigma\cdot p\,\psi_1$$

Write $E = E'+m$ where $E' \ll m$ and expand –

$$\frac{1}{E+m-V} \simeq \frac{1}{2m}\left[1 - \frac{E'-V}{2m} + \cdots\right].$$

This means $(E'-V)\psi_1 = \frac{\sigma\cdot p}{2m}\left[1-\left(\frac{E'-V}{2m}\right)\right]\sigma\cdot p\,\psi_1$

E' is the usual nonrelativistic energy eigenvalue.

To simplify the RHS, $(\sigma\cdot p)(\sigma\cdot p) = p^2$. (this uses $\sigma_i\sigma_j = \delta_{ij} + i\epsilon_{ijk}\sigma_k$, of course). The next term is

$$\sigma\cdot p\,V\,\sigma\cdot p\,\psi = -\sigma_i\frac{\partial}{\partial x_i}V\sigma_j\frac{\partial\psi}{\partial x_j}$$
$$= -\sigma_i\sigma_j\left[\left(\frac{\partial V}{\partial x_i}\right)\frac{\partial\psi}{\partial x_j} + V\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}\psi\right]$$

The δ_{ij} term in $\sigma_i\sigma_j$ gives

$$\vec\nabla V(x)\cdot\nabla\psi(x) + V(x)\nabla^2\psi$$

The crossproduct term is $i\vec\sigma\times\left[\vec\nabla V\times\vec\nabla\psi\right]$.

In addition, if we have a central potential

$$\vec\nabla V\cdot\vec\nabla = \frac{\partial V}{\partial r}\frac{\partial}{\partial r}$$
$$\vec\nabla V\times\vec\nabla = \frac{1}{r}\frac{\partial V}{\partial r}\left[\vec r\times\vec\nabla\right].\ \frac{1}{i}\vec r\times\vec\nabla = \vec r\times\vec p = \vec L$$

and $\vec S = \frac{1}{2}\vec\sigma$ so

$$E'\psi = \left[V - \frac{1}{2m}\left(1-\frac{E'-V}{2m}\right)\nabla^2 - \frac{1}{4m^2}\frac{\partial V}{\partial r}\frac{\partial\psi}{\partial r} + \frac{1}{2m^2 r}\frac{\partial V}{\partial r}\vec S\cdot\vec L\right]\psi$$

Iterate this equation: $E' - V(r) = \frac{p^2}{2m} + \cdots$.
This gives

$$E'\psi = \left[V + \frac{p^2}{2m} - \frac{p^4}{8m^3} + \frac{1}{2m^2 r}\frac{\partial V}{\partial r}\vec{L}\cdot\vec{S} - \frac{1}{4m^2}\frac{\partial V}{\partial r}\frac{\partial}{\partial r}\right]\psi$$

The first two terms are the usual NR Schrodinger equation. Then we have the first relativistic correction, then the spin-orbit coupling, and finally the "Darwin term" - it "corrects" the indeterminancy of the L-S term for s-waves ($L=0$ and $\int \frac{1}{r^2}\frac{dV}{\partial r} \times \psi^2 d^3r$ diverges) by contributing a fix-up - with $V \sim \frac{1}{r}$, it is $\Delta E \simeq \int r^2 dr \frac{1}{r^2}\psi\frac{\partial\psi}{\partial r} \propto |\psi(0)|^2$ which is nonzero only for s-waves. See, for example Condon & Shortly, "The theory of atomic spectra", p. 130 -

If V is a scalar, then it appears in the Dirac equation as βV, not V. The net result is to flip the sign of the L-S and Darwin terms. The flip of L-S has a physical explanation. The NR story is that it comes from two competing effects, $\frac{1}{2} = +1 - \frac{1}{2}$ where the +1 is from $\vec{v}\times\vec{E} \to \vec{B}$ and a $-\frac{1}{2}$ from ~~Thomas~~ Thomas precession. If $V(r)$ is a scalar, there is no $\vec{v}\times\vec{E}\to\vec{B}$, just Thomas precession

2) Now $V=0$, $E\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix} = \begin{bmatrix} m & \sigma\cdot(p-\frac{eA}{c}) \\ \sigma\cdot(p-\frac{eA}{c}) & -m \end{bmatrix}\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}$

$$(E+m)\,\psi_2 = \sigma\cdot\left(p-\frac{eA}{c}\right)\psi_1$$
$$(E-m)\,\psi_1 = \sigma\cdot\left(p-\frac{eA}{c}\right)\frac{1}{E+m}\,\sigma\cdot\left(p-\frac{eA}{c}\right)\psi_1$$

If we simply approximate $E+m = 2m$, we get

$$(E-m)\,\psi_1 = \frac{1}{2m}\sigma_i\sigma_j\left(p-\frac{eA}{c}\right)_i\left(p-\frac{eA}{c}\right)_j\psi_1$$
$$= \frac{1}{2m}\left[\left(p-\frac{eA}{c}\right)^2 + i\epsilon_{ijk}\sigma_k\left(p-\frac{eA}{c}\right)_i\left(p-\frac{eA}{c}\right)_j\right]$$

Focus on the last term - look at the parts linear $\times\,\psi$ in A:

$$\frac{ie}{2mc}\,\epsilon_{ijk}\left[-p_iA_j + A_ip_j\right]\psi. \quad (*)$$

For a pure B field, constant in direction m,

$$A_i = \frac{1}{2}\epsilon_{imn}B_mr_n.$$

eq. $(*)$ is
$$\frac{ie}{4mc}\sigma_k\epsilon_{ijk}\left[-\epsilon_{jmn}p_ir_nB_m - \epsilon_{imn}B_mr_np_j\right]\psi$$

$$= \frac{ie}{4mc}\sigma_k\left[\epsilon_{jik}\epsilon_{jmn}p_ir_nB_m - \epsilon_{ijk}\epsilon_{imn}B_mr_np_j\right]\psi$$

$$= \frac{ie}{4mc}\sigma_k\left[(\delta_{im}\delta_{kn}-\delta_{in}\delta_{km})p_ir_nB_m - (\delta_{jm}\delta_{kn}-\delta_{jn}\delta_{km})B_mr_np_j\right] \times\psi$$

look at first & 3rd term - in 3rd, j is a dummy index, rewrite with $j\to i$

First & third: $\sigma_k B_n [\delta_{\ell m}\delta_{kn}(p_\ell r_n - r_n p_\ell)]$

but $p_\ell r_n - r_n p_\ell = -i\delta_{\ell n}$

so these are $-i\sigma_k B_m \delta_{\ell m}\delta_{kn}\delta_{\ell n} = -i\vec{\sigma}\cdot\vec{B}$.

2nd and 4th terms are

$$\sigma\cdot B[-\vec{p}\cdot\vec{r} + \vec{r}\cdot\vec{p}] = 3i\,\sigma\cdot B$$

(3 for 3 dimensions) $-\ 3-1=2$

$$H = -\frac{e}{2mc}\vec{\sigma}\cdot\vec{B} = -\frac{e}{2mc}\cdot 2\cdot\vec{s}\cdot\vec{B}$$

using $\frac{1}{2}\sigma = \vec{s}$. Then

$$(E-m)\psi = \frac{1}{2m}\left(\vec{p}-\frac{e\vec{A}}{c}\right)\psi - \frac{e}{2mc}\sigma\cdot B\,\psi$$

The coefficient of $\frac{e}{2mc}$ is called g in $H = -\mu\cdot B$,

$\vec{\mu} = g\frac{e}{2mc}\vec{s}$. The electron has $g=2$.

Quicker: $H = i\frac{\sigma}{2m}\cdot\left(\left(p-\frac{eA}{c}\right)\times\left(\vec{p}-\frac{e\vec{A}}{c}\right)\right)$

$$= -\frac{ie}{2mc}(\vec{p}\times\vec{A} + \vec{A}\times\vec{p}).$$

$$(\vec{p}\times\vec{A})\psi = [\vec{p}\times\vec{A}]\psi - \vec{A}\times\vec{p}\,\psi$$

$$H = -i\frac{e}{2mc}\sigma\cdot[p\times A]$$

$$p = \frac{1}{i}\nabla,\quad \nabla\times A = B$$

$$H = -\frac{e}{2mc}\sigma\cdot B.$$

3) $\psi^M = \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L \end{pmatrix}$, $(i\gamma^\mu \partial_\mu - m)\psi = 0$

We can write the Dirac eqn using

$$\sigma^\mu = (1, \vec{\sigma}), \quad \bar{\sigma}^\mu = (1, -\vec{\sigma})$$

as
$$\begin{bmatrix} -m & i\sigma^\mu \partial_\mu \\ i\bar{\sigma}^\mu \partial_\mu & -m \end{bmatrix} \begin{pmatrix} \psi_L \\ -\sigma_2 \psi_L \end{pmatrix} = 0 \qquad (1)$$

A quick way to deal with this problem is to compute $(-i\gamma^\nu \partial_\nu - m)(i\gamma^\mu \partial_\mu - m)\psi$

$$= (\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu + m^2)\psi$$

$\gamma^\mu \gamma^\nu$ contracts to $g^{\mu\nu}$ from $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$, $\partial_\nu \partial_\mu$ is symmetric. Then we have

$$(\partial_\mu \partial^\mu + m^2)\psi = 0 = \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi$$

and if $\psi \sim e^{i(Et - \vec{p}\cdot\vec{x})}$,

$$(-E^2 + p^2 + m^2)\psi = 0.$$

A longer way is to start with (1)

$$i\bar{\sigma}^\mu \partial_\mu \psi_L + m\sigma_2 \psi_L^* = 0 \qquad (2)$$

take complex conjugate

$$-i(\bar{\sigma}^\mu)^* \partial_\mu \psi_L^* + m\sigma_2^* \psi_L = 0 \qquad (3)$$

Now $\quad \bar{\sigma}^\mu = (1, -\vec{\sigma}) = (1, -\sigma_1, -\sigma_2, -\sigma_3)$

$$(\bar{\sigma}^\mu)^* = (1, -\sigma_1, \sigma_2, -\sigma_3)$$
$$= (\sigma_2 1 \sigma_2, \sigma_2\sigma_1\sigma_2, \sigma_2\sigma_2\sigma_2, \sigma_2\sigma_3\sigma_2)$$
$$= \sigma_2 \sigma^\mu \sigma_2$$

so (3) is $\quad -i\sigma_2 \sigma^\mu \partial_\mu \sigma_2 \psi_L^* - m\sigma_2 \underset{=1}{\sigma_2\sigma_2} \psi_L = 0$

or $\quad -i\sigma^\mu \partial_\mu \sigma_2 \psi_L^* - m\psi_L = 0 \qquad (4)$

Take $\sigma^\nu \partial_\nu$ times Eq 2, use Eq 4

$$i\sigma^\nu\partial_\nu \bar{\sigma}^\mu \partial_\mu \psi_L + m \underbrace{\sigma^\nu \partial_\nu \sigma_2 \psi_L^*}_{im\psi_L} = 0$$

$$\sigma^\nu\partial_\nu \bar{\sigma}^\mu \partial_\mu \psi_L + m^2 \psi_L = 0$$

nearly back to the "quick way.."

$$\sigma^\mu \bar{\sigma}^\nu \partial_\mu \partial_\nu = \partial_0^2 - \sigma^i\sigma^j \partial_i \partial_j$$
$$= \partial_0^2 - \nabla^2$$

$(\sigma^i\sigma^j = \delta^{ij} + \text{antisymmetric})$

or $\quad \left(\frac{\partial^2}{\partial t^2} - \nabla^2 + m^2\right)\psi_L = 0$

again if $\psi_L \sim e^{i(Et - p\cdot x)}$,

$$(-E^2 + p^2 + m^2)\psi_L = 0$$

$$E = \sqrt{p^2+m^2} \quad \text{as usual.}$$