SET 6 – due 4 November

Two problems inspired by the LHC: In the Standard Model the coupling of the (spinless) Higgs boson ϕ to a fermion-antifermion pair is $\mathcal{L} = (m/v)\bar{f}f\phi$ where m is the fermion's mass and v is the so called Higgs vacuum expectation value. It is is related to the Fermi coupling, in weak interactions, $G_F = 10^{-5}$ GeV^{-2}, $1/(2v^2) = G_F/\sqrt(2)$, and $v = 246$ GeV.

1) [20 points] Compute the decay rate for Higgs into an $\bar{f}f$ pair. Call the Higgs mass M. Plot your result as a function of the parameter $x = 2m/M$. The formula for the differential decay width is

$$d\Gamma = \sum_{spins} \frac{1}{2M}|\mathcal{M}|^2 \frac{d^4p_1}{(2\pi)^3}\frac{d^4p_2}{(2\pi)^3}(2\pi)^4\delta^4(P - p_1 - p_2)\delta(p_1^2 - m^2)\delta(p_2^2 - m^2) \quad (1)$$

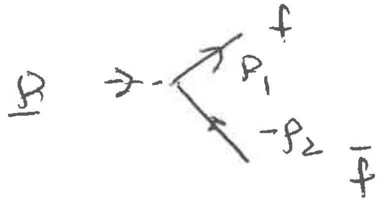
2) [25 points] Now compute the differential cross section for $\bar{f}f \to \phi\phi$. In all your calculation, except of course for the vertices, assume that the fermions are massless. (This amounts to doing the calculation at energies much greater than the fermion masses.) Use of the relativistic invariants s, t, and u will help a lot, and you can use the formula from a previous problem set,

$$d\sigma = \frac{1}{16\pi s^2}|\mathcal{M}|^2 dt. \quad (2)$$

In this case, $s + t + u = 2M^2$ (Check and confirm!) Show that the differential cross section vanishes at threshold, $s = 4M^2$. Higgs pair production has not yet been observed. This problem resembles pair annihilation into photons but is much simpler.
Note that the propagator for a spin-1/2 fermion is

$$S_F(p) = \frac{i(\not{p} + m)}{p^2 - m^2 + i\epsilon}. \quad (3)$$

1) Higgs decay into $f\bar f$, $\mathcal{L} = \frac{m}{v}\bar f f h \equiv g \bar f f h$



$\mathcal{M} = -ig\,\bar u(p_1)\, v(p_2)$

$$d\Gamma = \sum_{ss'} \frac{1}{2M} |\mathcal{M}|^2 \left[\frac{d^4p_1}{(2\pi)^3}\frac{d^4p_2}{(2\pi)^3}(2\pi)^4\delta^4(P-p_1-p_2) \times \delta(p_1^2-m^2)\,\delta(p_2^2-m^2)\right]$$

The phase space factor in [ ] is

$$\frac{d^3p_1}{4\pi^2\cdot 2E_1}\,\delta(p_2^2-m^2) \quad \text{where } p_2 = P - p_1$$

In the Higgs rest frame $P=(M,\vec 0)$,

$p_1 = (E_1,\vec p_1)$ with $p_1^2=m^2$, $p_2 = P-p_1 = (M-E_1, -\vec p_1)$

$$\frac{d^3p}{4\pi^2\cdot 2E_1}\,\delta(p_2^2-m^2) = \frac{d\Omega}{8\pi^2}\frac{p_1}{E_1}\,p_1\,dp_1\,\delta(M^2-2ME_1+E_1^2-\vec p_1^{\,2}-m^2)$$

and $p_1\,dp_1 = E_1\,dE_1$, and with $E_1^2-p_1^2=m^2$,

this is $$\frac{d\Omega}{8\pi^2}\frac{p_1}{E_1}E_1\,dE_1\,\delta(M^2-2ME_1) = \frac{d\Omega}{8\pi^2}\frac{p_1}{2M}.$$

$E_1 = M/2$ of course, $p_1^2 = E_1^2 - m^2 = \frac{M^2-4m^2}{4}$ or

$p_1 = \frac{M}{2}(1-x^2)^{1/2}$ where $x = \frac{2m}{M}$; $0<x<1$.

$$d\Gamma = \frac{1}{2M}\frac{d\Omega}{16\pi^2}\frac{p_1}{M}|\mathcal{M}|^2 = \frac{d\Omega}{32\pi^2M^2}\cdot\frac{M}{2}(1-x^2)^{1/2}\sum_{ss'}|\mathcal{M}|^2$$

$$= \frac{d\Omega}{64\pi^2}\frac{(1-x^2)^{1/2}}{M}\sum_{ss'}|\mathcal{M}|^2$$

Now for the matrix element - squared

$$\sum_{ss'} |\mathcal{M}|^2 = g^2 \, Tr \; v(p_2)\bar{v}(p_2)\, u(p_1)\bar{u}(p_1)$$

$$= g^2 \, Tr\,(\not{p}_2 - m)(\not{p}_1 + m)$$

$$= 4g^2 \, [\, p_1 \cdot p_2 - m^2 \,]$$

$$M^2 = (p_1 + p_2)^2 = 2m^2 + 2p_1 \cdot p_2 \quad \text{so}$$

$$p_1 \cdot p_2 - m^2 = \frac{M^2 - 2m^2}{2} - m^2 = \frac{M^2 - 4m^2}{2}$$

$$= \frac{M^2}{2}\left(1 - \frac{4m^2}{M^2}\right) = \frac{M^2}{2}(1 - x^2)$$

$$\sum_{ss'} |\mathcal{M}|^2 = \frac{m^2}{v^2} M^2 (1 - x^2)$$

Putting everything together

$$\frac{d\Gamma}{d\Omega} = \frac{1}{64\pi^2} \frac{(1-x^2)^{1/2}}{M} \frac{m^2}{v^2} M^2 (1 - x^2)$$

$$m^2 = \frac{x^2 M^2}{4} \quad \text{so this is}$$

$$\frac{d\Gamma}{d\Omega} = \frac{M^3}{v^2} \frac{x^2 (1-x^2)^{3/2}}{256\pi^2}$$

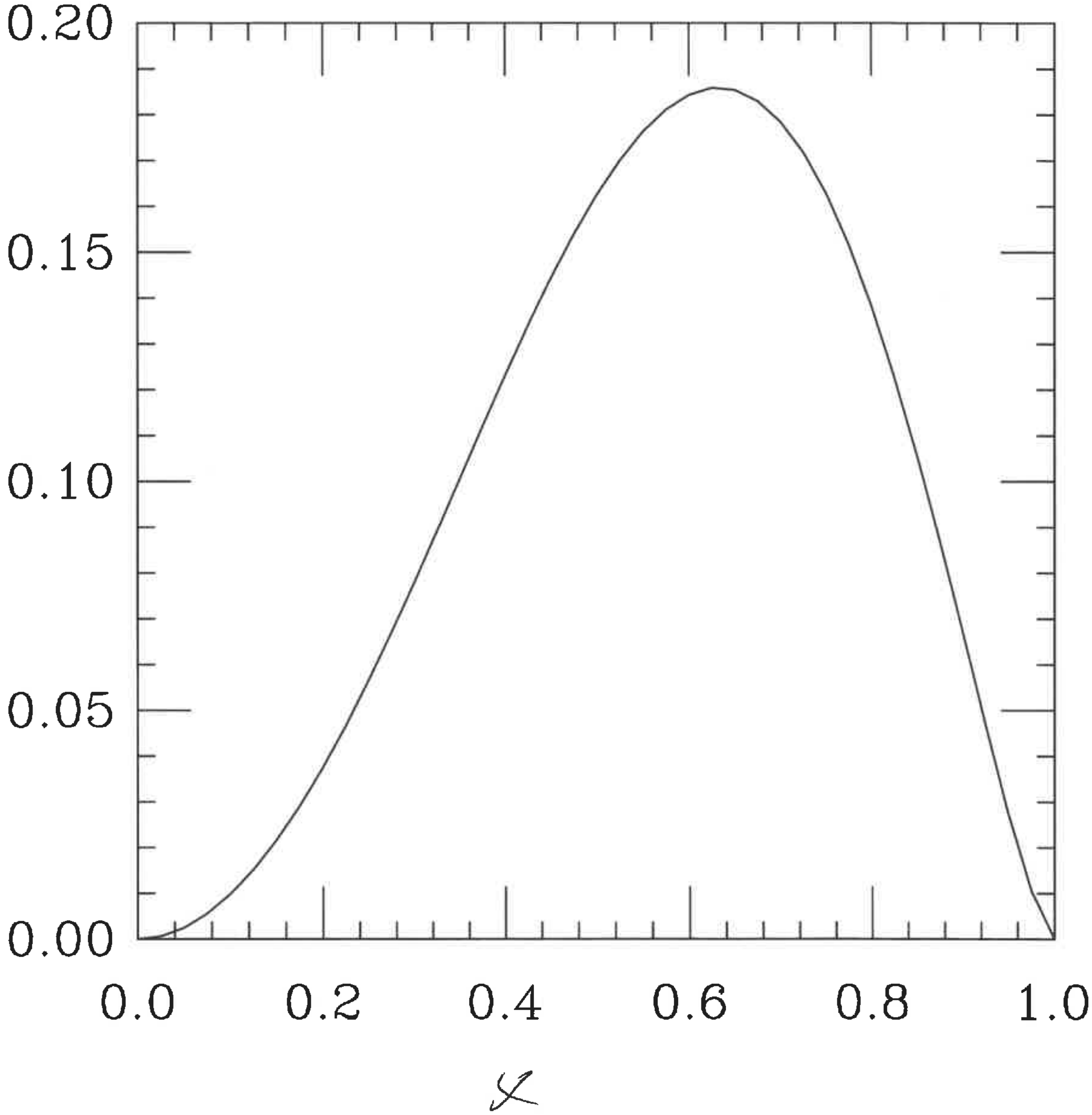
No angular dependence : $\int d\Gamma = 4\pi$

$$\Gamma = \frac{1}{64\pi} \frac{M^3}{v^2} x^2 (1-x^2)^{3/2}$$

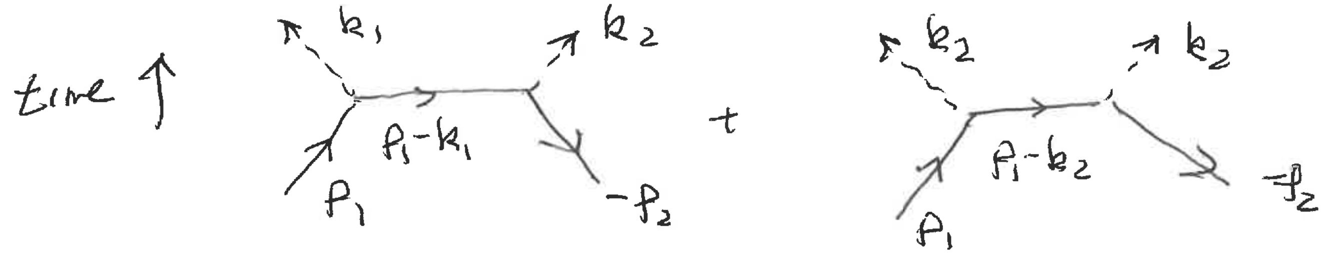
$\Gamma \sim m^2$ at small m, vanishes at $x = 1$ or $M = 2m$.

I Dont trust my 64!

$x^2 (1-x^2)^{3/2}$ vs x



2) $f\bar{f} \to hh$ mostly neglecting fermion masses



$$s = (p_1+p_2)^2 = (k_1+k_2)^2, \quad t = (p_1-k_1)^2, \quad u = (p_1-k_2)^2$$

$$\mathcal{M} = -g^2 \left[\bar{v}(p_2) \frac{i(\not{p}_1 - \not{k}_1)}{(p_1-k_1)^2} u(p_1) + \bar{v}(p_2) \frac{i(\not{p}_1 - \not{k}_2)}{(p_1-k_2)^2} u(p_1) \right]$$

It's useful to employ $\not{p}_1 u(p_1) = 0$ – then

$$\mathcal{M} = i g^2 \, \bar{v}(p_2) \left[\frac{\not{k}_1}{t} + \frac{\not{k}_2}{u} \right] u(p_1)$$

$$\frac{1}{4}\sum_{ss'} |\mathcal{M}|^2 = \frac{g^2}{4} \sum_{ss'} \Big[\bar{v}(p_2) \frac{\not{k}_1}{t} u(p_1) \bar{u}(p_1) \frac{\not{k}_1}{t} v(p_2) + \bar{v}(p_2) \frac{\not{k}_2}{u} u(p_1) \bar{u}(p_1) \frac{\not{k}_2}{u} v(p_2) + 2 \bar{v}(p_2) \frac{\not{k}_2}{u} u(p_1) \bar{u}(p_1) \frac{\not{k}_1}{t} v(p_2) \Big]$$

for massless fermions $\sum_s u(p,s)\bar{u}(p,s) = \sum_s v(p,s)\bar{v}(p,s) = \not{p}$

$$\frac{1}{4}\sum_{ss'} |\mathcal{M}|^2 = \frac{g^2}{4} \operatorname{Tr} \left[\frac{\not{p}_2 \not{k}_1 \not{p}_1 \not{k}_1}{t^2} + \frac{\not{p}_2 \not{k}_2 \not{p}_1 \not{k}_2}{u^2} + \frac{2 \not{p}_2 \not{k}_2 \not{p}_1 \not{k}_1}{tu} \right]$$

We need the formula for the trace of 4 γ's –

$$\frac{1}{4}\sum_{ss'}|\mathcal{M}|^2 = \frac{g^2}{4}\cdot 4\Big\{\Big[(p_2\cdot k_1)(p_1\cdot k_1) - (p_1\cdot p_2)(k_1^2) + (p_2\cdot k_1)(p_1\cdot k_1)\Big]\frac{1}{t^2}$$

$$+\frac{1}{u^2}\Big[(p_2\cdot k_2)(p_1\cdot k_2) - (p_1\cdot p_2)\,k_2^2 + (p_2\cdot k_2)(p_1\cdot k_2)\Big]$$

$$+\frac{2}{tu}\Big[(p_2\cdot k_2)(p_1\cdot k_1) - (p_1\cdot p_2)(k_1\cdot k_2) + (p_2\cdot k_1)(p_1\cdot k_2)\Big]\Big\}$$

This is not so bad: $k_1^2 = k_2^2 = M^2$,

$$s = (k_1+k_2)^2 = 2M^2 + 2k_1\cdot k_2 = (p_1+p_2)^2 = 2p_1\cdot p_2$$

$$t = (p_1-k_1)^2 = M^2 - 2p_1\cdot k_1 = M^2 - 2p_2\cdot k_2$$

$$u = (p_1-k_2)^2 = M^2 - 2p_1\cdot k_2 = M^2 - 2p_2\cdot k_1$$

so $$p_1\cdot k_1 = p_2\cdot k_2 = \frac{M^2-t}{2}$$

$$p_2\cdot k_1 = p_1\cdot k_2 = \frac{M^2-u}{2}$$

$$\frac{1}{4}\sum_{ss'}|\mathcal{M}|^2 = g^2\Big\{\Big(\frac{2(M^2-t)(M^2-u)}{4} - \frac{M^2 s}{2}\Big)\Big(\frac{1}{t^2}+\frac{1}{u^2}\Big)$$

$$+\frac{2}{tu}\Big(\frac{(M^2-t)^2}{4} - \frac{s(s-2M^2)}{4} + \frac{(M^2-u)^2}{4}\Big)\Big\}$$

$$= \frac{g^2}{2}\Big(\frac{1}{t^2}+\frac{1}{u^2}\Big)\Big((M^2-t)(M^2-u) - M^2 s\Big)$$

$$+\frac{g^2}{2tu}\Big((M^2-t)^2 + (M^2-u)^2 - s(s-2M^2)\Big)$$

$s+t+u = 2M^2$ so the first parenthesis is

$$M^4 - M^2(t+u+s) + tu = tu - M^4.$$

The second term is

$$(M^2-t)^2+(M^2-u)^2-s(s-2M^2)$$

$$= 2M^4-2M^2(t+u)+t^2+u^2-s^2+2M^2s$$

$$= 2M^4-2M^2(2M^2-s)+t^2+u^2-[2M^2-t-u]^2+2M^2s$$

$$= -2M^4+4M^2s+t^2+u^2-4M^4-t^2-u^2+4M^2(t+u)-2tu$$

$$= 4M^2s-6M^4+4M^2(2M^2-s)-2tu$$

$$= 2M^4-2tu$$

$$\frac{1}{4}\sum_{ss'}|\mathcal{M}|^4 = g^2(tu-M^4)\left\{\frac{1}{2}\left(\frac{1}{t^2}+\frac{1}{u^2}\right)-\frac{2}{2}\frac{1}{tu}\right\}$$

$$= g^2(tu-M^4)\frac{1}{2}\left(\frac{1}{t}-\frac{1}{u}\right)^2$$

Finally, there is an extra overall factor of $\frac{1}{2}$! because there are 2 identical bosons in the final state. It arises from symmetrizing the final state wave function $|\Psi_f\rangle = \frac{1}{\sqrt{2}}\left(|k_1 k_2\rangle + |k_2 k_1\rangle\right)$. Each component of $|\Psi_f\rangle$ picks up the 2 graphs we had, where there are $p_1 \to k_1$ and $p_1 \to k_2$ vertices. So

$$\frac{d\sigma}{dt} = \frac{g^2}{16\pi s^2}\cdot\frac{1}{2}\left(tu-M^2\right)^2\frac{1}{2}\left(\frac{1}{t}-\frac{1}{u}\right)^2.$$

$g^2 = m/v$, recall (for fermion mass m)

At threshold, $k_1 = k_2 = (M, 0)$

$$s = (k_1 + k_2)^2 = 4M^2$$

let $p_1 = M(1, 1, 0, 0)$ $p_2 = M(1, -1, 0, 0)$

(in c.m) so

$$t = (k_1 - p_1)^2 = k_1^2 - 2k_1 \cdot p_1 = M^2 - 2M^2 = -M^2$$

$$u = (k_1 - p_2)^2 = k_1^2 - 2k_1 \cdot p_2 = M^2 - 2M^2 = -M^2$$

so $tu = M^4$ and $\frac{d\sigma}{dt} = 0$ at threshold,

(also $t = u$ - even more cancellation ...)