Special relativity

Thomas DeGrand

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Chapter 1

A top-down view of relativity

"The non-mathematician is seized by a mysterious shuddering when he hears of 'four-dimensional' things, by a feeling not unlike that awakened by thoughts of the occult. And yet there is no more commonplace statement, that the world in which we live is a four-dimensional space-time continuum."–A. Einstein

1.1 What these notes are all about

I am a theorist who studies elementary particle physics. The phenomena I study take place on an energy scale where the effects of special relativity are so important that they dominate everything else. This means that, right from the beginning, everything I do has to be made consistent with the rules of special relativity, or what I compute will simply be wrong. This forces me to think in a particular way about special relativity.

In 2010 I taught sophomore modern physics for the first time ever. The course begins with an introduction to special relativity. When I read the standard textbooks for sophomore relativity, I discovered several alarming things:

- I did not know what the authors of these books were talking about
- I really didn't understand the examples in the books
- I couldn't make sense of the long story problems about rockets and twins

and so on. It also seemed to me that these books either downplayed or simply ignored most of the parts of special relativity that I thought were important, at the expense of discussions about trains, spaceships, or twins. So these notes were born, in an attempt to tell you, the sophomore reader, what I and my friends, particle theorists who use special relativity in their research, think about it. Hopefully, you will find the story interesting. Hopefully, you will also learn a few practical things (how to deal with real-world special-relativistic situations) along the way. In addition, as a by-product, I hope you will also be able to deal with all those long story problems about trains, rockets, and twins!

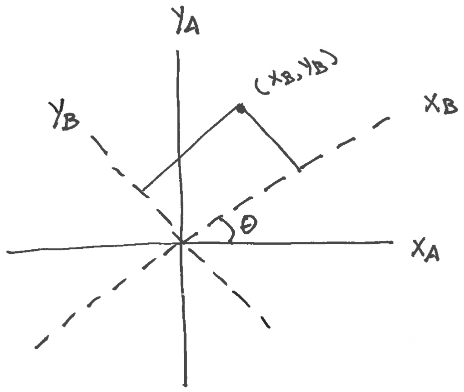

Figure 1.1: Two coordinate systems, rotated with respect to each other.

1.2 The geometry of space-time

To me, special relativity is a statement about the geometry of the world we live in, and what changes when different observers view the same set of events. Begin with an analogy, physics in a plane. Something happens at two locations in the plane, $\vec{x}_1$ and $\vec{x}_2$. Observer A lays down a set of coordinates using location $\vec{x}_1$ as its origin, and says that the coordinates of point $\vec{x}_2$ are (x_A, y_A). Observer B also puts its origin at $\vec{x}_1$, but chooses a different orientation for the coordinate axis, so the coordinates of point $\vec{x}_2$ are (x_B, y_B). Of course, $x_A \neq x_B$ and $y_A \neq y_B$. However, both A and B agree that the distance between the two points is the same:

$$R^2 = x_A^2 + y_A^2 = x_B^2 + y_B^2. \tag{1.1}$$

Furthermore, they can compute the relation between (x_A, y_A) and (x_B, y_B) – there is a linear relation between them

$$x_A = a_{11}x_B + a_{12}y_B$$

$$y_A = a_{21}x_B + a_{22}y_B. \tag{1.2}$$

The actual relation involves simple (but messy) geometry. If the axes of A and B are rotated by a relative angle θ,

$$\begin{aligned} x_A &= x_B\cos\theta - y_B\sin\theta \\ y_A &= x_B\sin\theta + y_B\cos\theta. \end{aligned} \tag{1.3}$$

And you can easily check the obvious, $x_A^2 + y_A^2 = x_B^2 + y_B^2$, by direct multiplication.

We say that x_A and y_A, or x_B and y_B, are the components of a vector $\vec{x}$. Transformations of the axes (rotations) change the individual components, but do not change the length of the vector.

Next, we notice that physical phenomena do not depend on our choice of coordinate system. We encode this observation by writing our rules for dynamics in terms of vectors, $\vec{r}$, $\vec{p}$, $\vec{F}\ldots$ with identical transformation laws

$$\begin{aligned} F'_x &= F_x\cos\theta - F_y\sin\theta \\ F'_y &= F_x\sin\theta + F_y\cos\theta. \end{aligned} \tag{1.4}$$

that is, in $\vec{F} = m\vec{a}$, both $\vec{F}$ and $\vec{a}$ are vectors. Knowing how vectors transform, we can transform our solution from one coordinate frame to another.

A particular situation might itself not be rotationally invariant (like motion constrained to a line) but we know how to transform our answer from one coordinate system to another – it's just vectors transformed into vectors. We encode this knowledge in the phrase, "the laws of physics are invariant under rotations."

Special relativity is basically the same story except that, instead of x and y (or (x, y, z), it involves (x, y, z, ct) where t is time and c is the velocity of light. Consider (to make the story easy to visualize) a world of one spatial dimension plus time. Our "two-vector" which labels an event in space and in time – or the relative separation in space and time of two events – is the pair of variables (x, ct). Nature is so assembled that all observers agree about the value of the following quantity

$$c^2\tau^2 = (ct)^2 - x^2 \tag{1.5}$$

(note the minus sign). That is, one observer sees that the two events are separated by a spatial interval x_A and a temporal interval t_A, while the second observer sees them occurring a distance x_B apart and a time t_B apart. x_A and x_B are not equal, and neither is t_A equal to t_B – the observers see the events happening at different relative times and relative positions – but they both agree, $(ct_B)^2 - x_B^2 = (ct_A)^2 - x_A^2$. The quantity they can agree on – an invariant quantity, it is called, is known as the "proper time."

And while the observers disagree about x and t, they can map one observer's observation into the others. Recall that if $x^2 + y^2$ is invariant, the transformation of bases is given by Eq. 1.3. The invariance holds because $\sin^2\theta + \cos^2\theta = 1$. We need a minus sign in the proper time. Recalling the hyperbolic trigonometric identity $\cosh^2\theta - \sinh^2\theta = 1$, we write down the transformation law of special relativity

$$\begin{aligned} ct_A &= (\cosh y)ct_B - (\sinh y)x_B \\ x_A &= -(\sinh y)ct_B + (\cosh y)x_B. \end{aligned} \tag{1.6}$$

(Pause and work this out for yourself!) This transformation is called a "Lorentz transformation."

Physically, what is transformed? Look at the second equation: x_A is equal to something times t_B plus something else times x_B. In fact. let y be tiny. You all learned in freshman calculus that $\cosh y \sim 1$ when y is small, and $\sinh y \sim y$ (just like the ordinary sine and cosine). This says that x_A equals something (y/c) times $t_B + x_B$. Brutally ignoring the difference between t_A and t_B, this says that at $t = 0$ $x_A = x_B$ and the difference between x_A and x_B grows linearly with time. Our two observers are moving with respect to each other at constant velocity. The top-down story of special relativity is about two observers of the same space time events, who are moving at a constant relative velocity with respect to each other. We speak of these observers as inhabiting different "inertial frames." They

- Measure different spatial intervals δx between the two events
- Measure different temporal intervals δt between the two events
- But agree, $(c\Delta t)^2 - (\Delta x)^2$ is the same.

Collections of variables transforming as $(ct, \vec{x})$ are called "four vectors," and our description of Nature which is constructed to encode its underlying invariance are built of them. We observe that the laws of physics do not depend on a particular reference frame (choice of $(ct_A, \vec{x}_A)$ versus $(ct_B, \vec{x}_B)$), in the sense that physical processes involve four vectors. We can describe how the physics changes as we change reference frames completely, when we describe the transformation laws for four-vectors. This tells us how observers in different inertial frames view physical processes.

Finally, when we ask that kinematics be relativistically invariant, we discover one more invariant:

All observers agree that the following combination of energy E and momentum $\vec{p}$ is the same: $E^2 - (cp)^2$ – that is, E is "like" ct and $c\vec{p}$ is "like" $\vec{x}$. One observer's E is another observer's combination of E and p. The transformation is just Eq. 1.6, (when the observers are moving with respect to each other along the $\hat{x}$ direction)

$$\begin{aligned} E_A &= (\cosh y)E_B - (\sinh y)p^x_B \\ p^x_A &= -(\sinh y)E_B + (\cosh y)p^x_B. \\ p^y_A &= p^y_B \\ p^z_A &= p^z_B. \end{aligned} \tag{1.7}$$

What is the invariant? It is the mass of the particle: $E^2 - (cp)^2 = (mc^2)^2$. Notice that if we can arrange for the particle to be at rest, $p = 0$, then the energy of a particle at rest is just its mass,

$$E = mc^2, \tag{1.8}$$

a formula you may have seen before.

But wait – all this seems fantastic!

- How do I know that all observers agree that $(c\Delta t)^2 - (\delta x)^2$ is the same? What's the evidence?
- What about $\vec{F} = m\vec{a}$? It happens, Newton's laws are inconsistent with relativity. The closest we can come is to write $F = d\vec{p}/dt$ where

$$\vec{p} = \frac{1}{\sqrt{1 - v^2/c^2}} m\vec{v} \tag{1.9}$$

Fortunately, $c = 3 \times 10^8$ meters per second. As a practical matter, $F = ma$ is often still a good approximation.

Chapter 2

Relativity for clocks and rulers

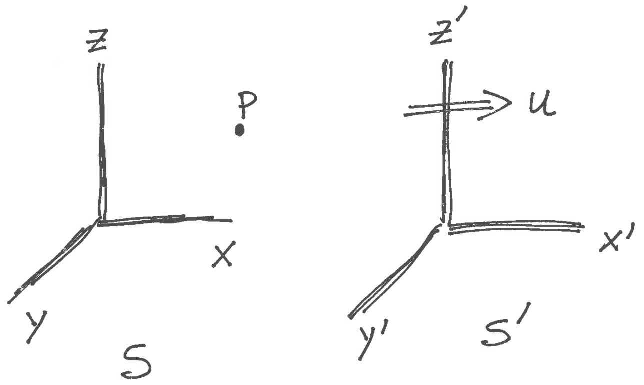

Figure 2.1: Two reference frames. The primed frame is moving to the right with velocity u with respect to the unprimed one.

2.1 Non-relativistic relativity

Newton first stated the principle of relativity: "The motions of bodies included in a given space are the same among themselves, whether that space is at rest or moves uniformly forward in a straight line."

This means that, for example, if a space ship is drifting along at a constant velocity, all experiments performed in the ship will appear the same as they would if the ship was not moving. That is the principle of relativity – very simple to state. But is it true? Only experiments can tell.

As a theoretical idea, relativity is required by Newton's laws. Suppose we have two observers, A and B. B is moving at constant velocity in the $\hat{x}$ direction with respect to A. We say, A and B have different reference frames. Call that velocity u. In a time t B has moved a distance ut from A. A labels the place in space and time where an event P occurs as (x, y, z, t), and B labels the same point as (x', y', z', t') where

$$\begin{aligned} x' &= x - ut \\ y' &= y \\ z' &= z \\ t' &= t. \end{aligned} \tag{2.1}$$

$x' = x - ut$ because everything at rest in B's frame is moving with respect to A. The last line ($t' = t$) seems very innocuous, but it is actually a deep statement.

Now suppose that P describes the location of a particle. Observer A sees that P has some velocity v and acceleration a (let's just look in the $\hat{x}$ direction), and he infers a force from

$$F = m\frac{d^2x(t)}{dt^2} = ma. \tag{2.2}$$

Observer B calculates

$$m\frac{d^2x'(t')}{dt'^2}, \tag{2.3}$$

but because $t' = t$ and $x'(t) - x(t) - ut$,

$$\begin{aligned} \frac{dx'}{dt'} &= \frac{dx'}{dt} = \frac{dx(t)}{dt} - u \\ \frac{d^2x'}{dt'^2} &= \frac{d^2x(t)}{dt^2} \end{aligned} \tag{2.4}$$

and both observers infer the same force. Newton's laws are the same in both frames and it is not possible to tell whether you are moving at a constant velocity or standing still by making mechanical measurements. The transformation Eq. 2.1 is called a Galilean transformation and we say that Newton's equations are invariant under Galilean relativity.

This was all right until about 1870, when lots of discoveries in electricity and magnetism culminated in Maxwell's equations. Maxwell's equations are not invariant under Galilean relativity; if you perform the transformation Eq. 2.1 on the coordinates, and insert this change into Maxwell's equations, they do not retain their form. If Galilean relativity were a true symmetry of Nature, electrical and optical properties would be different in a moving spaceship from those in a stationary one. Then one could determine the speed of the ship.

One consequence of Maxwell's equations is that if you shake a charged particle, it emits light. The light propagates at a velocity c which is a constant. Another consequence is that the velocity of light is independent of the velocity of the emitter. This is not $c' = c - u$!

Now we have an interesting problem. Suppose we are moving at a velocity u, and a stationary observer sends a light beam (moving at c) past us. If we measure the velocity of the light beam, would we see $c - u$? It's a logical possibility. Every experiment performed to answer this questions, however, showed that this prediction was wrong. (Most of them were designed to see the velocity of the Earth, and none of them did.) Something was wrong with the equations of physics.

When this paradox came to light, Maxwell's equations were only about twenty years old, and it was natural to assume that the problem was with them. People tried to change the laws of electrodynamics. But all the changes people tried gave rise to new effects which could be ruled out by experiment, and it gradually became apparent that the problem was not in electrodynamics; it was elsewhere.

In the meantime, Lorentz discovered a remarkable and curious fact. Making the change of variables

$$\begin{aligned} z' &= \frac{1}{\sqrt{1 - v^2/c^2}}(z - \frac{v}{c}ct) \\ ct' &= \frac{1}{\sqrt{1 - v^2/c^2}}(-\frac{v}{c}z + ct) \\ x' &= x \\ y' &= y \end{aligned} \tag{2.5}$$

left Maxwell's equations in the same form, i.e. that this is the law of relativity appropriate to electromagnetism. Eq. 2.5 is called a Lorentz transformation. (Note, parenthetically, $c^2t^2 - r^2 = c^2t'^2 - r'^2$.)

Einstein then cut the Gordian knot by proposing that all physical laws should be such that they remain unchanged under Lorentz transformations. That is, do not change electrodynamics, change mechanics. And that is what we will have to do.

2.2 The Michaelson-Morley experiment

Question: is there a special frame, in which the velocity of light is c, and in all other frames, does the velocity of light depend on the velocity of the frame with respect to the special frame?

I am not sure what people long ago were thinking about, when they asked this question, so let's ask it in a modern way: There is a "preferred rest frame" in the Universe, the frame in which an observer is at rest with respect to the Big Bang. We know that we on Earth are moving with respect to that frame, because the Earth is going around the Sun. Do we see a seasonal and directional variation in the velocity of light?

Answer: Do a test using interference of two light beams, and use the motion of the Earth around the Sun to create a changing velocity.

The physics of interference is very simple. At any fixed location in space, the electric field of a monochromatic electromagnetic wave oscillates in time as $E \propto \cos \omega t$. Now suppose we create an electromagnetic field at some point in space, split it, let part of it move along a path which takes a time t_1 to reach a detector, and let another part traverse a different path which takes a time t_2 to arrive at the same detector. The detector sees a superposition of the two beams, $E = E_1 \cos \omega t_1 + E_2 \cos \omega t_2$. If $t_1 - t_2 = \Delta t$ is such that $\omega \Delta t = 0$, 2π, $4\pi \ldots$, the amplitudes of the two beams will add and the resulting electric field – and intensity of the light – will be large. (We call this "constructive interference." However, if $\omega \Delta t = \pi$, $3\pi \ldots$, we get destructive interference. For what follows, it is more useful to compute the difference in the number of periods of light between the two paths,

$$\Delta n = \frac{\omega \Delta t}{2\pi}. \tag{2.6}$$

We have been working with the angular frequency ω, which is related to the usual (radio-dial) frequency ν through $\omega = 2\pi\nu$. Also recall the relation between velocity (c), frequency ν and wavelength λ: $\nu\lambda = c$. Putting all the pieces together, the difference in periods is

$$\Delta n = \frac{c}{\lambda} \Delta t. \tag{2.7}$$

Now to come up with two paths. This is done in a Michaelson interferometer, by splitting the beam into two beams moving at right angles to each other, letting them move away, bounce off mirrors, return, and recombine. There will be a Δt if the path lengths of the two paths are different. More to the point, if the light beam moves more slowly along one path

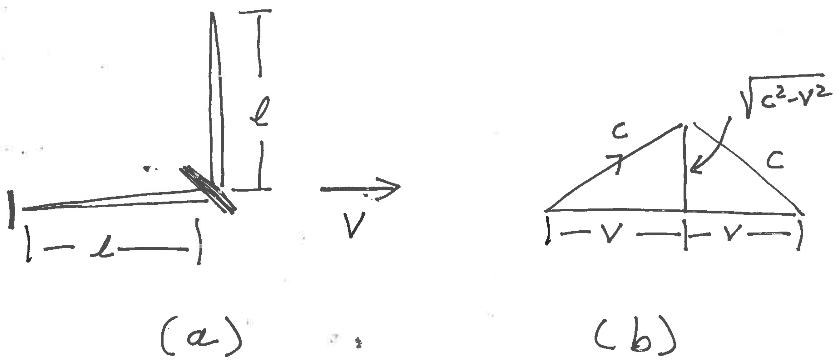

Figure 2.2: The Michaelson interferometer. Fig. (a) shows the two light beams bouncing off mirrors. The horizontal one is moving parallel to the velocity of the earth in the preferred frame; the vertical path is perpendicular to this motion. (b) Motion of light in the perpendicular direction, as seen by an observer in the preferred frame.

than along the other path, there will be a nonzero Δt even if the two paths have the same length. Let's imagine, that that is the situation. Furthermore, let's assume that one of the paths (path 1) is directed parallel to the velocity of the Earth with respect to the preferred frame, and the other path (path 2) is perpendicular to it.

It is easiest to analyze the situation in the preferred frame, where light moves at c, and then return to the other frame.

Along path 1, light moves parallel to v, so for us, on one part of its trip, light has a velocity $c - v$, and on the other part, it has $c + v$. Call the distance between source and mirror l. Then

$$t_1 = \frac{l}{c-v} + \frac{l}{c+v}. \tag{2.8}$$

Now path 2. In the preferred frame, the light moves at c, but we are moving, too. The component of the light's velocity perpendicular to the line of motion is $u = \sqrt{c^2 - v^2}$. (This is shown in the figure.) Then

$$t_2 = \frac{2l}{\sqrt{c^2 - v^2}}. \tag{2.9}$$

Putting the pieces together,

$$\Delta n = \frac{c}{\lambda}\Delta t = \frac{cl}{\lambda}(\frac{1}{c-v} + \frac{1}{c+v} - \frac{2}{\sqrt{c^2 - v^2}}). \tag{2.10}$$

As a sanity check, note $\Delta n = 0$ if $v = 0$.

To go further, we make the reasonable approximation that v/c is small. If v is the order of magnitude of the velocity of the earth around the Sun, $v/c \sim 10^{-4}$. Then we can get a perfectly adequate answer by Taylor expanding Eq. 2.10. Use

$$\begin{aligned} \frac{c}{c-v} &= \frac{1}{1-v/c} \sim 1 + v/c + (v/c)^2 + \ldots \\ \frac{c}{c+v} &= \frac{1}{1+v/c} \sim 1 - v/c + (v/c)^2 + \ldots \\ \frac{-2c}{\sqrt{c^2 - v^2}} &= \frac{-2}{\sqrt{1 - v^2/c^2}} = -2(1 + \frac{1}{2}\frac{v^2}{c^2} + \ldots) \end{aligned} \tag{2.11}$$

so

$$\Delta n = \frac{l}{\lambda}\frac{v^2}{c^2} \tag{2.12}$$

Every three months, the effect should reverse, as path 1 and 2 exchange roles with respect to v, so we are allowed to double the expected Δn. In the 1887 Michaelson-Morley experiment, $l = 11$ m, $\lambda = 600$ nm $= 6 \times 10^{-7}$ m and with $v/c \sim 10^{-4}$, $\Delta n = 0.4$. This is a huge effect, even for nineteenth century technology. They could measure an effect which was many orders of magnitude smaller. And they saw nothing.

People invented all sorts of crazy explanations. One was due to Lorentz. he suggested that material bodies contract when they are moving, so that the l of path 1 (into the wind) was changed to

$$l_{new} = l\sqrt{1 - \frac{v^2}{c^2}} < l \tag{2.13}$$

Then

$$t_1 = \frac{2l}{c}\frac{\sqrt{1 - \frac{v^2}{c^2}}}{1 - \frac{v^2}{c^2}} = \frac{2l}{c}\frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{2.14}$$

This equals t_2, so there is no effect, in agreement with experiment. But this is very ad hoc - it is as if nature is conspiring to prevent us from measuring v.

It was ultimately realized, as Poincaré pointed out, that this conspiracy is, in fact, a law of Nature. It is just not possible to determine an absolute velocity.

2.3 Introduction to relativity

Einstein's article has been translated and is on line: see

`http://www.fourmilab.ch/etexts/einstein/specrel/specrel.pdf`.

It's a fascinating read. You can probably manage most of it. Nobody ever said it better:

"... the unsuccessful attempts to discover any motion of the earth relatively to the light medium, suggest that the phenomena of electrodynamics as well as of mechanics possess no properties corresponding to the idea of absolute rest. They suggest rather that, as has already been shown to the first order of small quantities, the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We will raise this conjecture (the purport of which will hereafter be called the Principle of Relativity) to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body."

So there it is:

- In all reference frames moving with constant velocities with respect to each other, all physical phenomena occur in an identical manner. These frames are called "inertial frames."
- The velocity of light is identical in all inertial reference frames.

Now we have to think about space and time. When we though a la Galilean relativity, we assumed that everybody knew what the time was, and everybody measured the same time,

$$t' = t \tag{2.15}$$

But we have to be careful. We can synchronize watches *in one reference frame*, but generally, we will find that when we do this between two inertial frames, this synchronization will not persist. So to begin: how could we synchronize watches in one inertial frame? One way to do it would be to station observers around in the frame at various places, each with a watch, each at a known distance from a leader. The leader sends a flash of light at some predetermined time t_0. An observer a distance d away sees the flash a time $ct = d$ or $t = d/c$ later and knows that her watch should read a time $t = t_0 + d/c$.

Thus, when we speak of an inertial frame we imply that each event occurs at a unique place (x, y, z) and time t, and that everyone in that frame agrees what they are.

Now let's consider two different frames. Suppose we stand in frame S which we believe to be motionless (better to say, we are at rest in this frame) and we have another frame S' moving at a velocity v (in the $\hat{x}$ direction, say). How do measurements of time compare in the two frames?

In order to do this, we have to watch the machinery of the clock to see how it might be viewed in different frames. Real clocks are complicated, and so we will take a simple clock. It is really kind of silly, but it works in principle. It is a rod with a mirror at one end and a light and a photocell at the other end. A flash of light travels from the bulb to the mirror and back to the photocell, which makes a tick. We build two such clocks, make them the same length, and synchronize them so they start together, and then they will always agree thereafter.

Now the person in frame S' (in the train, spaceship, whatever) puts her clock in the window so that we can see it as she flies by at her constant velocity v. She measures a time between ticks of

$$\Delta t' = \frac{2l}{c}. \tag{2.16}$$

What do we see? Let's call our time between ticks Δt. Frame S' has a velocity v. The light moves at c, but in our frame it takes a zigzag path. When the frame moves a distance $d = v\frac{\Delta t}{2}$, the light must traverse a distance $\sqrt{l^2 + d^2}$. Thus the light moves a distance $2\sqrt{l^2 + d^2}$ in a time Δt.

We also know that the light's velocity is c. Thus

$$c\Delta t = 2\sqrt{l^2 + d^2} = 2\sqrt{l^2 + v^2\frac{(\Delta t)^2}{4}} \tag{2.17}$$

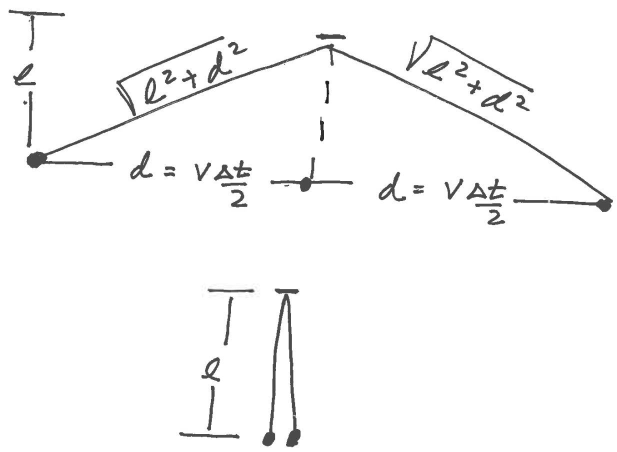

Figure 2.3: (The path the light ray in the clock takes. In the lower picture, as seen in a frame where the clock is stationary, and what an observer watching the clock fly by will see.

or

$$\frac{c^2(\Delta t)^2}{4} = l^2 + \frac{v^2(\Delta t)^2}{4} \tag{2.18}$$

so

$$\Delta t = \frac{2l}{\sqrt{c^2 - v^2}} = \frac{2l}{c}\frac{1}{\sqrt{1 - v^2/c^2}}. \tag{2.19}$$

But this is

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - v^2/c^2}}. \tag{2.20}$$

For any $v < c$, $1/\sqrt{1 - v^2/c^2} \geq 1$ so $\Delta t \geq \Delta t'$. The apparent time between ticks is longer for the moving clock. The moving clock appears to the stationary observer to be running slowly. This phenomenon is called *time dilation.*

At first thought this seems to be asymmetric – that S' is a special frame since in any frame other than S' we'd say that time for S' is running slowly. But in fact, S' is a special frame, it is the frame where the flash and the tick occurred at the same place. This is not true in any other frame. Times measured in the special frame, where two events occur at the same place, are called "proper times."

While we have argued that time dilation will occur for our particular stick - and - mirror clock, the result must be independent of the kind of clock. To see this, get an ordinary clock, which ticks at the same rate as our stick clock. Now put it in the rocket. Perhaps this clock will not run slow? No, if it kept different time, the person in the rocket would notice, and know that she was moving.

Note that if all clocks run slower, we have to say that in a certain sense, time itself runs slower. The pilot's pulse rate, his thoughts, how long it takes to age, all run slower, because he can't tell he is moving.

An interesting example of this slowing down is furnished by an elementary particle called the mu meson. (It is basically a heavy electron.) It decays spontaneously (in its rest frame) with an average lifetime of about 2.2×10^{-6} sec. It's clear that even if it had $v \sim c$ it could travel no farther than $ct = 3 \times 10^8$ m/sec $\times 2.2 \times 10^{-6}$ sec $= 660$ m. And yet, mu mesons are created tens of kilometers high in the atmosphere and penetrate to the earth's surface. (They are a background to high energy physics experiments.) How can this be? Time dilation! In their rest frame, mu mesons live $\Delta t_0 = 2$ microseconds, while to us, who see them moving, they last a dilated time

$$\frac{\Delta t_0}{\sqrt{1 - v^2/c^2}}. \tag{2.21}$$

If $1/\sqrt{1-v^2/c^2} = 100$, they would live 100 times longer and move 100 times farther before decaying.

Notice that we do not know how the mu meson decayed, all we know is that its motion satisfies the principle of relativity. We can still predict, of $v/c = 0.9$, then the apparent lifetime is

$$\frac{2.2 \times 10^{-6}}{\sqrt{1-(0.9)^2}} sec. \tag{2.22}$$

Time dilation is actually used in modern high energy experiments to measure lifetimes of particles. The idea is to produce the particles at some known velocity, and measure the distance they go before they decay. (One can measure v by, e.g. the bend of the track in a magnetic field,)

$$d = v\Delta t \tag{2.23}$$

and

$$\Delta t \frac{\Delta t_0}{\sqrt{1-v^2/c^2}} \tag{2.24}$$

If v/c is near 1 we can get a large d even though Δt_0 might be short.

Finally, we showed that to the stationary S, the moving S' had time running slow. We could turn it around. The moving observer sees a stationary clock. However, the observer sees the clock moving with respect to him, and will see it running slow.

Various paradoxes immediately present themselves. The most famous is the "twin paradox." one of a pair of identical twins goes off on a long relativistic journey and returns home still youthful, to marry his great-grandniece (R. Heinlein, "Time for the Stars"). This is not a paradox. Time dilation arises between inertial frames, frames which move at a constant velocity with respect to each other. If you and someone else are in inertial frames, you fly past each other, and you are never going to meet again. The youthful twin, unlike his old brother, did not live in an inertial frame.

Before we continue, some common definitions:

$$\begin{aligned} \beta &= \frac{v}{c} \\ \gamma &= \frac{1}{\sqrt{1-v^2/c^2}}. \end{aligned} \tag{2.25}$$

Note

$$\beta^2 = 1 - \frac{1}{\gamma^2} \tag{2.26}$$

and so we say $\Delta t = \gamma \Delta t_0$. And do you recall our original definition of the Larentz transformation, Eq. 1.6? Its y is related to our no-standard form by $\tanh y = \beta$.

So we see that the rules of relativity show us that the passage of time depends on the frame in which it is measured. The same thing is true of distances, as we now see. Let's look at our moving observer S', who flies along moving to the right in our frame, carrying a meter stick (oriented along his direction of motion). . He is in the rest frame of the stick, and for him it is a meter long. Call the length of the stick in his frame l' or l_0.

Now he can also measure the length of the stick in a crazy way. He sees the observer S moving to the left at a velocity v, and can find l' from a time measurement $\Delta t'$ which is the elapsed time beginning when the observer is at the front of the stick, then at its back., $l' = l_0 = v\Delta t'$. Note $\Delta t'$ is not a proper time because the two events (front of the stick next to observer, back of the stick next to observer) happen at different locations in S'.

This would be sort of silly if it were not for the fact that the observer in S cannot measure the length of the meter stick directly, because it is moving. He has to use the second method, following the rod as it flies by, converting a time into a distance. Then $l = v\Delta t$.

This time is a proper time. The two events (front of the stick next to observer, back of the stick next to observer) happen at the same location in S. Thus we can relate the two measurements using time dilation, $\Delta t' = \gamma \Delta t$ – the non - proper - time is longer. But then

$$l = v\frac{\Delta t'}{\gamma} = \frac{l_0}{\gamma} = l_0\sqrt{1 - v^2/c^2}. \tag{2.27}$$

l_0 is the length of the object measured in the object's rest frame, its "proper length." Observe that $l < l_0$. This is the phenomenon of "length contraction."

Length contraction allows us to understand the long lifetime of the mu meson from the point of view of an observer traveling along with it. Recall that to us, the meson moves a distance

$$d = \frac{vt_0}{\sqrt{1 - v^2/c^2}} \tag{2.28}$$

before decaying, because it lives a time $t_0/\sqrt{1 - v^2/c^2}$. The muon sees a distance $d\sqrt{1 - v^2/c^2}$ of the Earth's atmosphere (shortened by length contraction) rush by, at a velocity v, in its brief time of existence, 2 microseconds, that is $t_0 = (d/v)\sqrt{1 - v^2/c^2}$.

Let's quit beating around the bush and derive the Lorentz transformations. Again, define a stationary frame S and a frame moving to the right (that is, with velocity v in the

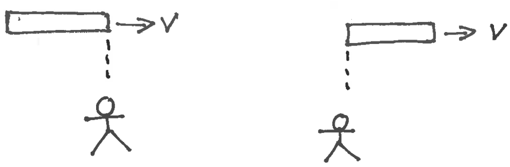

Figure 2.4: The stick moving past the observer.

$\hat{x}$ direction), called S'. Let's consider a point P in space, labeled by coordinates (x, y, z, t) in S and (x', y', z', t') in S'. How are (x, y, z, t) and (x', y', z', t') related?

For motion along x, $y' = y$ and $z' = z$. (Einstein in his first article considers $y' = \lambda y$ for some scale change λ and concludes $\lambda = 1$, so maybe just asserting $y' = y$ and $z' = z$ is not thorough enough. It works, though.)

Let's suppose the point P is stationary in S' and try to see how it would appear in S. Nonrelativitically, we would say $t' = t$ and $x = x' + vt$. But we know that the distance x' is Lorentz contracted. We also know that vt is just vt, since x and t are in the same frame, so

$$x = vt + \frac{x'}{\gamma} \tag{2.29}$$

and thus

$$x' = \gamma(x - vt). \tag{2.30}$$

Can you convince yourself that

$$x = \gamma(x' + vt')? \tag{2.31}$$

(That's easy – let P be stationary in S and ask what S' sees, $v \to -v$.) Now we solve $x' + vt' = x/\gamma$ for t' in terms of x and t.

$$\begin{aligned} vt' &= \frac{x}{\gamma} - \gamma(x - vt) \\ &= \frac{1}{\gamma}(x - \gamma^2 x + \gamma^2 vt). \end{aligned} \tag{2.32}$$

Now

$$\gamma vt' = (1 - \gamma^2)x + \gamma^2 vt \tag{2.33}$$

and

$$1 - \gamma^2 = 1 - \frac{1}{1 - v^2/c^2} = \frac{1 - v^2/c^2 - 1}{1 - v^2/c^2} = \gamma^2 \frac{v^2}{c^2} \tag{2.34}$$

so

$$\gamma vt' = \gamma^2 vt - \gamma^2 \frac{v^2}{c^2} x \tag{2.35}$$

giving us the Lorentz transformations

$$\begin{aligned} t' &= \gamma(t - \frac{v}{c^2}x) \\ x' &= \gamma(x - vt). \end{aligned} \tag{2.36}$$

Let's stare at these equations for a minute. The overall γ factor is just time dilation; set $x = 0$ to see this. The vx/c^2 is unexpected. What is going on? It says that two events which happen at the same time in S (set $t = t_1 - t_2 = 0$) but not at the same place (set $x = x_1 - x_2$) do not occur simultaneously to an observer in S', The elapsed time between the two events is

$$t'_2 - t'_1 = \frac{\gamma v}{c^2}(x_1 - x_2). \tag{2.37}$$

This phenomenon is called the "failure of simultaneity at a distance," and to make it clearer, imagine an experiment. A person in a space ship synchronizes two clocks, located at the front and back of the space ship, by setting a light bulb halfway in between. She flashes the light to start the clocks, and they start at the same time. However, an outside observer, watching the space ship move to the right, sees the left hand clock rushing toward the light and the right hand clock moving away from the light. In both frames, the light comes out of the light bulb moving at c, so the light reaches the left hand clock (the clock at the rear of the space ship) first, and reaches the right hand clock later. So the person in the ship knows that the times of the two events, the starting of the two clocks, are simultaneous, in her rest frame. But events with equal values of t' with different spatial locations must correspond to different values of t in the other frame.

Indeed, that is encoded in the Lorentz transformations

$$\begin{aligned} \Delta x' &= \gamma(\Delta x - v\Delta t) \\ \Delta t' &= \gamma(\Delta t - \frac{v}{c^2}\Delta x). \end{aligned} \tag{2.38}$$

While the two observers disagree about the lengths of space separations and of time interval, they agree the the following property is an invariant:

$$(\Delta x')^2 - c^2(\Delta t')^2 = \gamma^2((\Delta x)^2 - 2v\Delta x\Delta t + v^2(\Delta t)^2 - c^2((\Delta t)^2 - 2\frac{v}{c^2}\Delta x\Delta t + \frac{v^2}{c^4}(\Delta x)^2)) \tag{2.39}$$

and canceling the cross terms,

$$\begin{aligned} (\Delta x')^2 - c^2(\Delta t')^2 &= \gamma^2((\Delta x)^2 - c^2(\Delta t)^2)(1 - \frac{v^2}{c^2}) \\ &= (\Delta x)^2 - c^2(\Delta t)^2. \end{aligned} \tag{2.40}$$

Well, of course! This is the invariance of the proper time between the two space-time events.

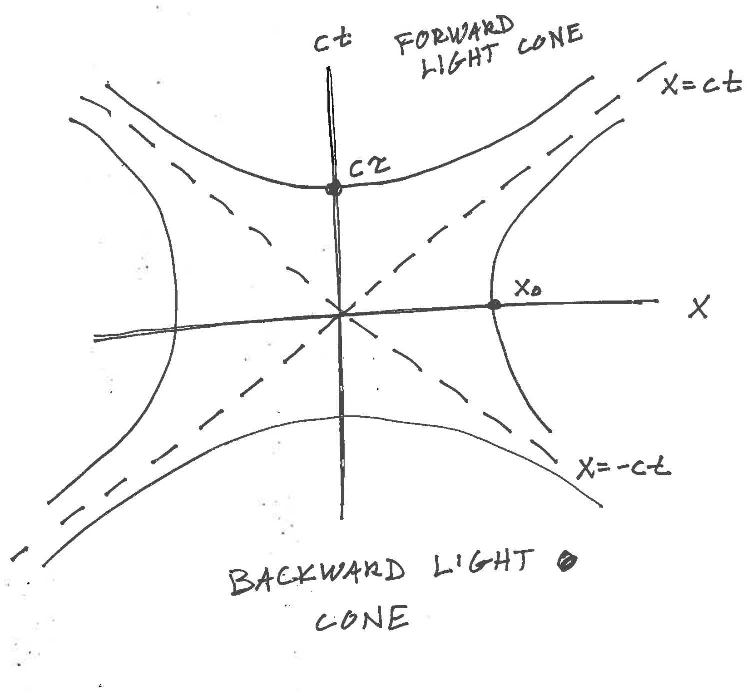

Figure 2.5: Space-time map, showing the "light cone" (lines $x - \pm ct$), the forward and backward light cones ($(ct)^2 > x^2$), and the regions of space-like separation. Lorentz transformations carry us along hyperbolas in the picture.

2.4 Space-time diagrams

Space-time diagrams are a useful set of pictures for visualizing Lorentz transformations. Pick a coordinate system whose x axis is distance and whose y axis is time, actually ct. Imagine that one event occurs at the origin of the coordinate system, and the other event has space-time coordinates (x, ct). Under a Lorentz transformation, x and t change, but $x^2 - c^2t^2$ is unchanged. What is the curve in $x, ct)$ space, where $x^2 - c^2t^2$ is a constant? That is easy, it is a hyperbola. Under a Lorentz transformation, a space-time point moves from one location on a hyperbola to another point, on the same hyperbola. And there are two possibilities for the hyperbola, depending on whether the squared proper time $c^2\tau^2 = c^2t^2 - x^2$ is positive or negative. Look at Fig. 2.5.

The first case is $c^2\tau^2 > 0$. The hyperbolas contain the points $x = 0$, $ct = \pm c\tau$. Asymptotically, the two hyperbolas approach the lines $x = \pm ct$. These lines make up the "light cone" (add an extra spatial dimension to create the cone), the path in space a light ray emitted from the origin would take. The "forward light cone" is the region between $x = ct$ and $x = -ct$ with $t > 0$ and the "backward light cone" is the region between $x = ct$ and $x = -ct$ with $t < 0$. Since the hyperbolas contain the points $x = 0, ct = \pm c\tau$, there is some frame in which $x'' = 0$, $ct'' = c\tau$ – a frame where the two space-time events occur at the same place, but different times. In that frame, the elapsed time between the two events is $t = \tau$. It is easy to see that in all other frames, $t > \tau$. This is time dilation, again.

Notice that observers in all frames agree on which space-time event happened earlier in time. If, in the special frame where $x'' = 0$, $t'' > 0$, then in all frames, $t > 0$.

The other case is $c^2\tau^2 < 0$. Her there is a special frame in which events occur simultaneously, the frame where $t'' = 0$, $\Delta x'' = x_0$ where $c^2\tau^2 = -x_0^2$. In all other frames, events occur at different times. Also note, in all other frames, the distance between the points is greater than x_0.

Furthermore, notice that different observers can disagree on which event occurred earlier. In one frame, one can have $t' < 0$, in another frame $t' > 0$. The Lorentz transformation just moves us around on the hyperbola.

We call events whose space-time separation puts them at positive $c^2\tau^2$ "time-like" and events for which $c^2\tau^2 < 0$ "space-like." It is clear that these regions are disjoint. In fact, there is something else to think about – the "world line" of a particle. See Fig. 2.6 In some frame, plot it position $x(t)$ as a function of its time. Assume that $x(0) = 0$ (so it begins at the origin) and assume that its velocity v is always less than c. The world line of the particle will be nearly vertical, if its velocity is small. Put dots on the space-time diagram for different times, $x(t) = \frac{v}{c}(ct)$.

Now perform a Lorentz transformation, to see the particle's world line in another frame. Each dot moves on its own hyperbola. We could work out the location as a function of time, but let's not do that, instead, let's do something more profound: notice that the particle always remains in the forward light cone. If the particle is emitted and later (in some frame) hits a wall, then in another frame the (Lorentz-transformed) location of the wall also remains in the forward light cone. Projecting roughly along the axis, $v(t) < c$ for all observers, and cause (emission of the particle) precedes effect (hitting the wall) for all observers. Nature will be causal.

Superluminal motion would involve trajectories like $x(t) = \frac{v}{c}(ct)$ with $v/c > 1$. Imagine

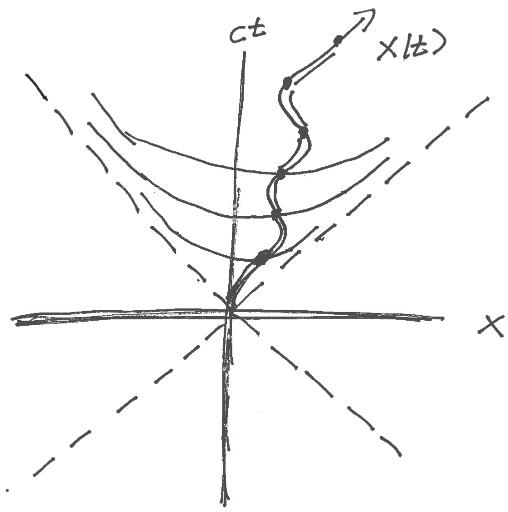

Figure 2.6: Space-time map, showing the world line of a letter. The hyperbolas show how the world line would deform under a Lorentz transformation.

the two space-time points for the beginning and the end of a trajectory. They would be space-like-separated. Under a Lorentz transformation, they would continue to be space-like-separated, and in fact, there will be frames in which the arrival time of the particle occurs before the emission time. Effect would precede cause. It is a good thing for us physicists that Nature does not work this way. In fact, Nature apparently works very hard to avoid acausal effects. But that is a story for another day.

To summarize this section, "Letters travel inside the light cone."

2.5 Revisiting some examples

Time dilation: Suppose frame S' moves to the right compared to frames S. We know

$$\Delta x = \gamma(\Delta x' + v\Delta t'$$

$$c\Delta t \quad = \quad c\gamma(\Delta t' + \frac{v\Delta x'}{c^2}) \tag{2.41}$$

(the plus sign because x' moves away from x). If the primed frame measures a proper time, $\Delta x' = 0$ and so $\Delta t = \gamma \Delta t'$. The time interval is longer in the unprimed frame. Also, $\Delta x = v\gamma\Delta t' = v\Delta t$, so the events (start of clock) and (stop of clock) occur at different places. Go back and look at muon decay in the atmosphere!

If the interval between events in the primed frame is zero, so $\Delta t' = 0$, $\Delta x' = L'$, $\Delta x = \gamma L'$ is greater than L', but this is not a pure length. The two events occur at different unprimed times, indeed $\Delta t = \gamma v L'/c^2$. There is a loss of simultaneity.

For pure length contraction, we must work harder. Recall the story, the observer sees the head and the tail of the spaceship cross a common location. Again we have two space-time points. The first is at a (common) origin, $(0, 0)$. In the unprimed frame, the second point, where the tail of the ship passes the marker, is at $(x, t) = (0, t)$. In the primed frame, the second point is at $(x', t') = (l_0, l_0/v)$ since the marker moves by at a velocity v. In the unprimed frame, the observer infers a length $l = vt$, so that $(x, t) = (0, l/v)$. Now we check the Lorentz transformations. The unprimed frame records $x = \gamma(x' - vt') = \gamma(l_0 - v(l_0/v)) = 0$ for the second spatial point. This is what we expected, but it serves to check our signs. The time for the second event is $t = \gamma(t' - vx'/c^2)$ or

$$t = \frac{l}{v} = \gamma(\frac{l_0}{v} - \frac{vl_0}{c^2}); \tag{2.42}$$

This is

$$l = l_0\gamma(1 - \frac{v^2}{c^2}) = l_0\frac{\gamma}{\gamma^2} = \frac{l_0}{\gamma} \tag{2.43}$$

which is length contraction.

Another way to see this is via the proper time: $c^2t^2 - x^2 = c^2l^2/v^2$ and $c^2t'^2 - x'^2 = c^2l_0^2/v^2 - l_0^2$, Then

$$l^2 = l_0^2(1 - \frac{v^2}{c^2}) \tag{2.44}$$

or again, $l = l_0/\gamma$.

To me, length contraction is a peculiar corner of special relativity, with lengths defined very particularly. The “natural” way (in my opinion) to deal with special relativity is just to say that the spatial interval between two events, Δx, changes, the temporal interval Δt also changes, and the Lorentz transformations tell us the change.

As our final example, we have the famous problem of the Martian in the flying saucer, the farmer and the barn. The saucer has length L_S in its rest frame. The barn has two doors for the saucer to fly through, but it has a length L_B in its frame, and $L_B < L_S$. The saucer moves to the right with velocity v in the barn's frame. The farmer wants to trap the saucer in the barn by closing both doors simultaneously (and then quickly re-opening them to prevent damage to the barn.)

The farmer thinks that he will have no problem doing this feat, because the saucer is length-contracted. He will be successful if $L_B > L_S/\gamma$.

You might think that there is a paradox at work, because in the Martian's frame, the barn is also length contracted. But, in fact, in the Martian's frame, there is a loss of simultaneity in the opening and closing of the doors. To work it out, in the farmer's frame the ends of the barn are at $(x_L, t_L) = (0, 0)$ and $(x_R, t_R) = (L_B, 0)$. This is our example from above, so $x = \gamma(x' + vt')$ and so on. $(x_L, t_L) = (0, 0)$ means that $(x'_L, t'_L) = (0, 0)$, the zero of our coordinate axis is on the left end of the barn in both frames. Now $x_R = L_B = \gamma(x'_R + vt'_R)$ and $t_r = 0 = \gamma(t'_R + vx'_r/c^2)$ or $t'_R = -vx'_r/c^2$. The right door closes and opens at an earlier time than the left door, to the Martian. Where is the Martian? $L_B = \gamma(x'_R - v^2x'_R/c^2)$, plugging in t'_R, which is

$$L_B = \gamma(1 - \frac{v^2}{c^2})x'_R = \frac{x'_R}{\gamma} \tag{2.45}$$

So $x'_R = \gamma L_B$ and if this is greater than L_s, nothing will happen. The front of the ship is at γL_B, the back is at $\gamma L_B - L_S$ and if this is greater than zero, the tail is clear of the left door.. This agrees with the farmer's calculation, no collision if $L_B > L_S/\gamma$.

2.6 Velocity addition

An observer in a frame S sees a particle moving with a uniform velocity $\vec{u}$. What velocity does an observer in another inertial frame (S') see? Assume that S' has velocity $\vec{v}$ with respect to S. Non-relativistically,

$$\vec{u}' = \vec{u} - \vec{v} \tag{2.46}$$

but we expect something more interesting to happen!

If the velocities are all constant, we can handle them like you did in high school:

$$u_x = \frac{\Delta x}{\Delta t} \qquad u_y = \frac{\Delta y}{\Delta t} \tag{2.47}$$

and so on. Let's suppose that $\vec{v}$ points in the $\hat{x}$ direction. Then

$$\begin{aligned}\Delta x' &= \gamma(\Delta x - v\Delta t)\\ \Delta t' &= \gamma(\Delta t - \frac{v}{c^2}\Delta x)\\ \Delta y' &= \Delta y\end{aligned} \tag{2.48}$$

and

$$\begin{aligned} u'_x = \frac{\Delta x'}{\Delta t'} \qquad u_x = \frac{\Delta x}{\Delta t}\\ u'_y = \frac{\Delta y'}{\Delta t'} \qquad u_y = \frac{\Delta y}{\Delta t}.\end{aligned} \tag{2.49}$$

Then

$$\begin{aligned} u'_x &= \frac{\gamma(\Delta x - v\Delta t)}{\gamma(\Delta t - \frac{v}{c^2}\Delta x)}\\ &= \frac{\frac{\Delta x}{\Delta t} - v}{1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t}}\\ &= \frac{u_x - v}{1 - \frac{vu_x}{c^2}}\end{aligned} \tag{2.50}$$

The velocities perpendicular to the boost direction are different because of time dilation:

$$\begin{aligned} u'_y &= \frac{\Delta y}{\gamma(\Delta t - \frac{v}{c^2}\Delta x)}\\ &= \frac{\frac{\Delta y}{\Delta t}}{\gamma(1 - \frac{v}{c^2}\frac{\Delta x}{\Delta t})}\\ &= \frac{u_y}{\gamma(1 - \frac{vu_x}{c^2})}\end{aligned} \tag{2.51}$$

To go from $\vec{u}'$ to $\vec{u}$, just flip the sign of v, of course.

For two examples of this result, consider first the case where $u'_x = c$: we sit in a rocket ship and fire a laser along our direction of motion. An observer sees that the photons in the laser have a velocity

$$u_x = \frac{u' + v}{1 + \frac{u'v}{c^2}} = \frac{c + v}{1 + \frac{v}{c}} = c \tag{2.52}$$

The speed of light is the same in all reference frames, as it should be.

As our next example, fire a bullet moving at $u = c/2$ in a rocket moving at $v = c/2$. An observer on the ground sees

$$u' = \frac{c/2 + c/2}{1 + (\frac{c}{2})(\frac{c}{2})\frac{1}{c^2}} = \frac{c}{\frac{5}{4}} = \frac{4}{5}c. \tag{2.53}$$

So $\frac{1}{2} + \frac{1}{2} = \frac{4}{5}$ in special relativity!

Of course, sufficiently low velocities just add. If two velocities are $v/c = 10^{-6}$ ($v = 300$ meters per second) the deviation from addition is $v^2/c^2 = 10^{-12}$, invisible for all practical purposes.

2.7 Doppler shift

While the speed of light is the same in all inertial reference frames, this does not mean that a light beam would look the same to observers in different frames. The reason is that a light beam is a wave, and the motion of the source or the observer expands or compresses the wave train. Let's consider for simplicity the case where an emitting object moves directly away from or toward a receiver. The source emits light of frequency f and wavelength λ; recall that

$$\lambda f = c. \tag{2.54}$$

Now suppose that the wave train is emitted by a source in frame S' at time $t = 0$. After one period (call this time interval Δt), the front of the wave train (viewed in frame S) has moved a distance $c\Delta t$ to the right. At the same time the source has moved a distance $v\Delta t$ to the right. So the distance between crests (or nodes) of the wave is $\lambda_{observed} = c\Delta t - v\Delta t$. The observed frequency is then

$$f_{observed} = \frac{c}{\lambda_{observed}} = \frac{c}{(c-v)\Delta t} = \frac{1}{\Delta t(1 - v/c}. \tag{2.55}$$

Δt is the time measured in S. It is not the period in S', it has been time-dilated. The elapsed time in S', $\Delta t'$, is the proper time because both ends of the wave were emitted from the same location, so $\Delta t = \gamma \Delta t'$. And $\Delta t' = 1/f_{source}$, the inverse frequency of the radiation in the source frame. Putting all the pieces together,

$$f_{observed} = \frac{f_{source}}{\gamma(1 - v/c)}$$

$$
\begin{aligned}
&= f_{source}\frac{\sqrt{1-\frac{v^2}{c^2}}}{1-\frac{v}{c}} \\
&= f_{source}\sqrt{\frac{1+v/c}{1-v/c}}.
\end{aligned}
\tag{2.56}
$$

This is for a source going *towards* the observer: $f_{observed} > f_{source}$. If the source is receding from the observer, v flips its sign and $f_{observed} < f_{source}$. It is easy to remember the sign if you think about the expansion of the universe: distant objects are receding from us, their light is red-shifted, so the frequency is lowered.

There are two "classical" Doppler effects, both involving spectral lines. We have already mentioned the expansion of the Universe. One can measure the velocity of recession of distant objects from the red-shift of their spectral emission lines. Some of them are as large as $f_{source}/f_{observed} \sim 3-4$.

An extreme version of a Doppler shift is a background to the primordial black body radiation from the Big Bang. You probably know the story: the Universe began in with the Bang and expanded and cooled. When it got cold enough (roughly) that atoms formed, the Universe became transparent to radiation. It has been expanding and cooling ever since, and the temperature of the radiation has fallen. A hot object emits broad spectrum "black body" radiation, and the intensity of the radiation peaks at a frequency which is proportional to the temperature. We on Earth are moving with respect to the Bang, so the radiation we see is red shifted on one side and blue shifted on the other. The pictures you see of the black body spectrum have had this effect removed during the analysis.

More mundanely, in a hot vapor the atoms can have fairly large velocities due to thermal motion, and their spectral emission lines are broadened. A typical velocity is about 300 m/sec ($\frac{1}{2}mv^2 \sim \frac{3}{2}kT$). This is so small that $\gamma \sim 1$ and

$$
f_{observed} = \frac{f_{source}}{\gamma(1 \pm v/c)} \sim \frac{f_{source}}{1 \pm v/c} \sim f_{source}(1 \mp \frac{v}{c}). \tag{2.57}
$$

Thus $\Delta f/f \sim v/c \sim 10^{-6}$. For yellow light (600 nm) this is 6×10^{-4} nm, which is small, but observable. In fact, it is a serious problem for precision spectroscopy: suppose you are looking for part-per-million effects. This is a background which is a hundred times bigger than what you are looking for.

We can use the Doppler shift to show that the twin paradox is not a paradox. Our traveling twin goes to a star a distance L away, taking a time T (both as seen in our frame), by moving at a velocity v. Her ship has a radio broadcasting at a frequency f_0 in her frame (this can serve as an avatar for her heart beat). Her last broadcast, when she arrives at the star, takes an additional time $c\Delta t = L$ to return to us. T and L are related through $L = vT$, so we receive her signal at a time

$$T_{last} = (1 + \frac{v}{c})T. \tag{2.58}$$

Since she is moving away from us, her signal is red-shifted, so we receive it at a frequency

$$f' = f_0\sqrt{\frac{1 - v/c}{1 + v/c}} \tag{2.59}$$

We hear a total of $f'T_{last}$ beeps. This is

$$\begin{aligned} f_0\sqrt{\frac{1-v/c}{1+v/c}}(1+\frac{v}{c})T &= f_0\sqrt{\frac{(1-v/c)(1+v/c)^2}{1+v/c}}T \\ &= f_0\sqrt{(1-v/c)(1+v/c)}T \\ &= f_0\sqrt{1-v^2/c^2}T \\ &= f_0\frac{T}{\gamma}. \end{aligned} \tag{2.60}$$

She sends $f_0 t'$ beeps, so

$$f_0\frac{T}{\gamma} = f_0 t'; \tag{2.61}$$

or $T = \gamma t'$. She is younger than us. During the return, her signal is blue shifted, but because the signal and antenna are moving closer together, there is another cancellation. The effect remains.

Chapter 3

Relativistic kinematics

3.1 Energy and momentum in special relativity

Most textbook derivations of the relativistic versions of energy and momentum are a bit awkward:

- Guess the answer
- Show that if momentum is conserved in one frame, then it is conserved in another frame, usually by explicitly performing a Lorentz transformation

Instead, let's back off, and ask, "What is the conserved momentum?" In non-relativistic physics, we know that $\vec{p} = m\vec{v}$ because in a collision, when forces are short range (such as the collision of billiard balls), that quantity is what we experimentally observe to be conserved.

What do we want from relativity? We want some quantity $\hat{p}$ which transforms like $\vec{x}$ and ct (i.e., a four-vector), whose vector part looks like $m\vec{v}$ when the velocity is small. This does not mean that we have invented a useful $\hat{p}$, in the sense that its conservation encodes some useful dynamics. That is a matter for experiment to settle. But it does mean that we have something which is consistent with relativity.

It might be, that what we want is not unique – that there are many possible choices for relativistic momentum. Then we would be in trouble; without extra knowledge about how real dynamical systems evolve,we would not know which choice is "really" momentum. But, it happens, there is only one choice for the momentum, and it involves a quantity called the "velocity four-vector."

Recall that $\hat{x} = (ct, \vec{r})$ is the coordinate four-vector. Imagine a velocity four vector $v = (u_0, \vec{u})$, and if it is a four vector, it will transform exactly like $\hat{x}$,

$$\begin{aligned} r'_z &= \gamma r_z - \beta\gamma ct \\ ct' &= -\beta\gamma r_z + \gamma ct \end{aligned} \tag{3.1}$$

(and so on), that is

$$\begin{aligned} u'_z &= \gamma u_z - \beta\gamma u_0 \\ u'_0 &= -\beta\gamma u_z + \gamma u_0 \\ u'_x &= u_x \end{aligned}$$

$$u'_y = u_y. \tag{3.2}$$

(In both cases we are boosting along the $\hat{z}$ axis).

Hold that thought and return to momentum conservation. Imagine that we have a set of colliding particles (think of very speedy billiard balls). Each of them carries a four vector momentum $\hat{p}_i = (p_{0i}, \vec{p}_i)$ (the subscript labels the ith billiard ball). Ordinary momentum conservation in a collision of two billiards is

$$\vec{p}_1 + \vec{p}_2 = \vec{p}_3 + \vec{p}_4. \tag{3.3}$$

We have to figure out what p_{0i} is, but until we do, let's assume that it is conserved, too:

$$p_{01} + p_{02} = p_{03} + p_{04} \tag{3.4}$$

Then if the four vectors $\hat{p}_i$ and $\hat{u}_i$ are proportional, Eq. 3.2 means that momentum conservation in one frame automatically insures momentum conservation in all frames. For example (flip the signs of vectors 3 and 4 so that the sum of all the p's adds to zero)

$$\sum_i p'_{iz} = \gamma(\sum_i p_{zi}) - \beta\gamma(\sum_i p_{0i}) \tag{3.5}$$

and from Eqs. 3.3-3.4, the objects in parentheses are zero so the left hand side is zero, too.

So what is $\hat{u}$? All we have are $\vec{x}$ and ct, (which form a four-vector) and we know that the proper time $c^2\tau^2 = (ct)^2 - x^2$ is an invariant. So we play a trick: an invariant is the same in all frames. We guess

$$\hat{u} = \frac{d\hat{x}}{d\tau} \tag{3.6}$$

(the ratio of a four vector to an invariant is a four vector). In a frame where the particle is at rest, proper time is clock time, so the spatial components of $\hat{u}$ will just be proportional to the ordinary velocity dx/dt.

Let's invoke the chain rule a few times. Recalling $x_0 = ct$ and $t = \gamma\tau$,

$$u_0 = \frac{dx_0}{d\tau} = \frac{dx_0}{dt}\frac{dt}{d\tau} = c \times \gamma = \frac{c}{\sqrt{1 - u^2/c^2}} \tag{3.7}$$

while

$$\vec{u} = \frac{d\vec{x}}{d\tau} = \frac{d\vec{x}}{dt}\frac{dt}{d\tau} = \vec{u} \times \gamma = \frac{\vec{u}}{\sqrt{1 - u^2/c^2}} \tag{3.8}$$

or our four vector is $\hat{u} = \gamma(c, \vec{u})$ where $\gamma = 1/\sqrt{1 - u^2/c^2}$.

The length of a four vector is supposed to be an invariant. Is it?

$$\hat{u}^2 = \gamma^2 c^2 - \gamma^2 u^2 = (\frac{1}{1 - u^2/c^2})(c^2 - u^2) = c^2 \tag{3.9}$$

This is kind of a dull invariant, but it is an invariant, none the less – c, like any constant, is the same in all frames.

(An aside: showing that $\hat{u}$ transforms as a four vector is quite annoying, but it works. By this I mean, suppose that in one frame the billiard carries four velocity $\hat{u}$, but in a frame moving with respect to this frame with a velocity v, is $\hat{u}'$ given by Eq. 3.2? It works; note that in Eq. 3.2, $\gamma = 1/\sqrt{1 - v^2/c^2}$.)

So, what have we just done? Suppose that the particle's velocity u is small. Then $\gamma = 1\sqrt{1 - u^2/c^2}$ is nearly $\gamma = 1$ and the four vector is $\hat{u} = (c, \vec{u})$. Its vector components are just the usual velocity. This says that the obvious choice for a four momentum is just $\hat{p} = m\hat{u}$ or in components (still in the non-relativistic limit) $\hat{p} = (mc, m\vec{u})$.

Now go back to the exact formula for $\hat{u}$. The quantity which behaves like the vector part of a four vector (or, which IS the vector part of a four vector), is

$$\vec{p} = \gamma m \vec{u} = \frac{m}{\sqrt{1 - u^2/c^2}} \vec{u}. \tag{3.10}$$

The time-like part of $\hat{p}$ is

$$p_0 = m u_0 = \gamma m c. \tag{3.11}$$

So, what is u_0? Again, take the non-relativistic limit, but more carefully:

$$u_0 = \gamma c = \frac{c}{\sqrt{1 - u^2/c^2}} = c[1 + \frac{1}{2}\frac{u^2}{c^2} + \ldots] \tag{3.12}$$

and

$$p_0 = mc + \frac{1}{2}\frac{mu^2}{c} + \ldots \tag{3.13}$$

Hmm... $\frac{1}{2}mu^2$ – this looks like the kinetic energy! What is conservation of p_0 (in the low velocity limit)? Write down $p_{01} + p_{02} = p_{03} + p_{04}$

$$\frac{1}{c}[m_1 c^2 + m_2 c^2 + \frac{1}{2}m_1 u_1^2 + \frac{1}{2}m_2 u_2^2 = m_3 c^2 + m_4 c^2 + \frac{1}{2}m_3 u_3^2 + \frac{1}{2}m_4 u_4^2]. \tag{3.14}$$

In a classical billiard ball collision, each of the input particles is identical to one of the outgoing particles. Let's suppose that particles 1 and 3 are identical, and 2 and 4 are identical. The Eq. 3.14, the equation for the conservation of p_0, is the equation for the conservation of kinetic energy. Our conservation laws make perfect sense!

Notice that because $\hat{p}$ is a four vector, its conservation is guaranteed in all frames.

So we have arrived at our desired object: the relativistic four momentum is $\hat{p} = (E/c, \vec{p})$ where $\vec{p} = \gamma m \vec{u}$ and E is the relativistic generalization of the usual kinetic energy. I'll just call it the energy, no adjectives. What is it? $E = \gamma mc^2$. What is the energy associated with a particle at rest? If the velocity is zero, $\gamma = 1$ and

$$E = mc^2. \tag{3.15}$$

A famous formula – the energy of a massive object is not just its kinetic energy, the energy includes its rest mass. The kinetic energy, per se, has become a not-so-useful derived quantity, $KE = E - mc^2$. We'll return to this point below.

Said differently, and more profoundly, the squared length of the momentum four-vector is

$$c^2\hat{p}^2 = E^2 - c^2p^2 = (mc^2)^2. \tag{3.16}$$

In words: the mass is an invariant; it is the same in all reference frames. Observers in different frames may disagree about a particle's energy and its momentum (although they know how to connect their disagreements, just by performing Lorentz transformations) but the mass is the mass to everyone.

3.2 Summarizing results, and some useful formulas

Let's sum up what we have found. First, the energy of a moving particle of mass m is $E = \gamma mc^2$. This is a useful fact, for if we know m and E, we know $\gamma = E/(mc^2)$. An example of the use of this fact is a restatement of the problem of time dilation for the muon's lifetime, in a much more useful way: Suppose that the muon is produced high in the atmosphere with an energy E, how long does it live? Answer: Its lifetime is dilated by a factor $E/(mc^2)$. Simple!

The momentum is

$$p = \frac{mv}{\sqrt{1 - v^2/c^2}} \tag{3.17}$$

which obviously becomes very large as the particle's velocity approaches c.

Next, notice that the ratio of momentum to energy is proportional to the velocity:

$$\frac{cp}{E} = \frac{\gamma muc}{\gamma mc^2} = \frac{u}{c} = \beta. \tag{3.18}$$

Knowing a relativistic particle's momentum and its energy, we can find its velocity, basically just the ratio.

Finally, notice that if E is known (and the mass), then the momentum is known, or vice versa

$$\begin{aligned} E &= \sqrt{c^2p^2 + (mc^2)^2} \\ cp &= \sqrt{E^2 - (mc^2)^2} \end{aligned} \tag{3.19}$$

Note that if a particle is massless, then the energy-momentum relation is simple: $E = cp$ (from Eq. 3.19) and the particle has a velocity $\beta = cp/E = 1$ or $v = c$.

"Massless particles travel at the velocity of light."

You knew this already! Think about the photon! And moreover, massive particles always travel at velocities less than c. This is because

$$\beta = \frac{cp}{E} = \frac{\sqrt{E^2 - (mc^2)^2}}{E} = \sqrt{1 - (\frac{mc^2}{E})^2} < 1. \tag{3.20}$$

Do not make the shameful mistake of thinking that energy is $1/2mv^2$ and hence a massless particle has no energy! $1/2mv^2$ is an approximate formula for the change in energy of a massive particle when its velocity changes from zero to v. Massless particles are stuck at $v = c$.

3.3 Simple examples

Particle physicists are very cavalier about the distinction between mass and energy. We say, "The mass of the proton is 938 MeV." Of course, we mean that M_pc^2, an energy is 938 million electron volts.

Why electron volts? This is the typical energy for an electron bound to an atom. Recall that the MKS charge of an electron is $e_Q = 1.6 \times 10^{-19}$ Coulombs. A typical atomic potential is about a volt (the usual MKS volt). You know that is true if you have ever purchased a battery; batteries store chemical energy. But we can derive this number: an atom is about $r \sim 10^{-8}$ cm or 10^{-10} meters. The electrostatic potential this distance from a unit-electron-charge particle is

$$V = \frac{1}{4\pi\epsilon_0}\frac{e}{r} = [9 \times 10^9][\frac{1.6 \times 10^{-19}}{10^{-10}}] \sim 1 \text{ V} \tag{3.21}$$

using $1/(4\pi\epsilon_0) = 10^{-7}c^2$. It is inconvenient to carry around a big exponent, so we just define 1 eV = 1.6×10^{-19} Coul $\times 1$ volt = 1.6×10^{-19} Joules. An MeV is 10^6 eV, a GeV is 10^9 eV. This is 1000 MeV, roughly the mass of a proton. The energy of a proton beam at the large Hadron Collider is a nominal 10 TeV (tera – 10^{12}), so how fast is the proton? $\gamma = E/(m_P c^2) \sim (10^{13}$ eV $/10^9$ eV is about 10^4.

I already said that there were many types of energy. Adjective-free energy is $E = \gamma mc^2$. It is the sum of the mass and the kinetic energy (somewhat recursively, $E_K = E - mc^2$.) Because $\gamma - 1 \sim (1/2)v^2/c^2$ for small v/c, $E_K = (1/2)mv^2$ for a slowly moving massive particle. This is a useful quantity n billiard ball collisions, where nothing happens to change the masses of the particles. Then energy conservation is just kinetic energy conservation.

However, there are many cases where this is not a useful way to think. Here are some examples.

First, the mass of a bound state will be less than the masses of the constituents. Consider the hydrogen atom, where the electron is bound to the proton by a Coulomb potential, $V = -(1/(4\pi\epsilon_0))e^2/r$. The electron cannot scape to infinite distance without help because the energy of the bound state is less than the masses of the bound states. Mass is energy: what is the mass of the hydrogen atom? $M_H c^2 = m_p c^2 + m_e c^2 - \epsilon$ where ϵ is the binding energy.

Some nuclei are radioactive. They can decay in various ways. One possibility is that they can emit a gamma ray. Assuming that the parent and daughter are so heavy that they do not recoil, the masses are related to the gamma ray energy by $M_{parent}c^2 = M_{daughter}c^2 + E_\gamma$. For long lived isotopes, one can measure the masses directly, compare to the energy of the photon, and confirm this result.

Most elementary particles decay. The neutral pi meson or π^0, a bound state of a quark and an anti-quark, can decay into two photons. If the pi meson is at rest, the photons must come out back to back with equal and opposite momentum. For a photon, $E = pc$ so the photons have equal energy, $2E_\gamma = E_{initial} = m_\pi c^2$, i.e. each photon has energy $(1/2)m_\pi c^2$.

Fission: A nuclear reactor is a source of low energy neutrons. Some isotopes of uranium are unstable against fission, $n + U \to 2$ nuclei of mass m_1 and m_2 plus energy. The energy conservation formula is $E(n) + M_U c^2 = E(m_1) + E(m_2)$ where $E(m_i) = \sqrt{(cp_i)^2 + (m_i c^2)^2}$. "Exothermic" chemical reactions are similar, except that the particles involved are atoms or molecules. In elementary school you might have said, "energy is released." Energy is conserved, so what this means is that the decay products have energies greater than their rest masses, hence they have kinetic energy, which can be given to the environment when the decay products again scatter inelastically off it.

3.4 Simple homework-style examples

Consider first the decay of a an unstable (radioactive?) particle of mass M. For simplicity we assume it fragments into two equal mass particles of mass m, and for more simplicity we assume that the original particle is at rest. Then the decay products emerge back to back with equal momentum $\vec{p}_1 = -\vec{p}_2$. We will call the magnitude of the momentum p. Energy conservation says that

$$Mc^2 = \sqrt{(cp)^2 + (m_1c^2)^2} + \sqrt{(cp)^2 + (m_2c^2)^2}. \tag{3.22}$$

All this is true even if the two decay products have different mass. We can use this equation to find p. The algebra is less annoying if $m_1 = m_2$. Then

$$Mc^2 = 2\sqrt{(cp)^2 + (mc^2)^2} \tag{3.23}$$

or

$$\frac{(Mc^2)^2}{4} = (cp)^2 + (mc^2) \tag{3.24}$$

or

$$cp = \sqrt{\frac{(Mc^2)^2}{4} - (mc^2)^2} \tag{3.25}$$

This is a unique value, if M and m are known. Notice that $p \to 0$ as $M \to 2m$: all the energy goes into the masses of the decay products, there is nothing left for kinetic energy. The total energy of each decay fragment is $Mc^2/2$, of course – this follows from momentum conservation.

Now suppose that the decaying particle is moving. What are the energies and momenta of the decay products?

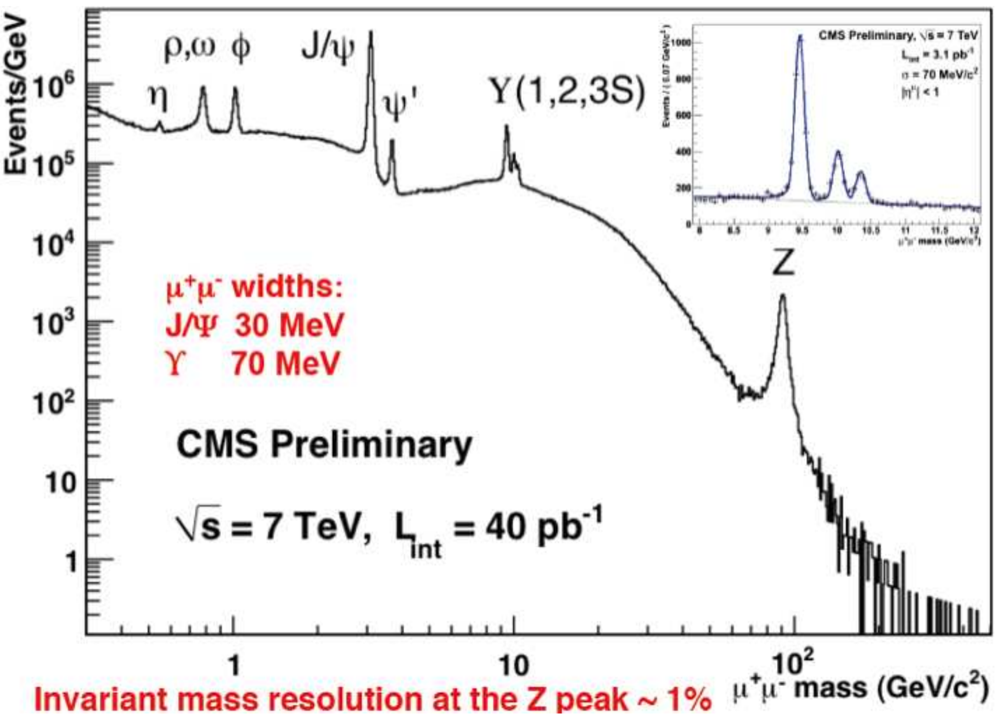

Figure 3.1: Events produced at one of the Large Hadron Collider's detectors, as a function of the invariant mass of a $\mu^+\mu^-$ pair.

Finally, we have touched modern research. In high energy physics experiments, most of the produced particles are unstable; they decay before they reach the detector. How are they discovered, then? Basically, through a generalization of this result. Four-vector energy and momentum are conserved in the decay, so if the decaying particle has momentum $\vec{P}$, energy E and mass M, we know that

$$(Mc^2)^2 = E^2 - c^2|\vec{P}|^2 \tag{3.26}$$

Energy conservation says that $E = \sum_i E_i$ (summing over the energies of the decay products) and $\vec{P} = \sum_i \vec{p}_i$ summing over their momenta. Then one computes

$$(\sum_i E_i)^2 - c^2|\sum_i \vec{p}_i|^2 \tag{3.27}$$

and for all events where the decay products came from the decaying parent, this combination of invariants is equal to $(Mc^2)^2$ and is often referred to as the "invariant mass."

To dig in deeper gets complicated. Let's return to our two-body decay example. in the parent particle's rest frame, the decay products come off back to back with equal and opposite momentum. Suppose that the parent is moving, and suppose that it is moving perpendicular to the line the decay products would take, in its rest frame. What are the energies and momenta of the decay products in the frame where the parent is moving.

The general answer is also simple: just perform a Lorentz transformation. Assume that the parent is moving in the $\hat{z}$ direction, and the decay occurs in the particle's rest frame (we will call this the primed frame) in direction $\hat{y}$. Now we recall our dictionary: $(E, p_z c, p_y c) \leftrightarrow (ct, z, y)$ or

$$\begin{aligned} ct = \gamma(ct' + \frac{v}{c}z') \quad &\rightarrow \quad E = \gamma(E' + \frac{v}{c}cp'_z) \\ z = \gamma(z' + \frac{v}{c}ct') \quad &\rightarrow \quad cp_z = \gamma(cp'_z + \frac{v}{c}E') \\ y = y' \quad &\rightarrow \quad p'_y = p_y \end{aligned} \tag{3.28}$$

Let's check this for the initial particle, whose four momentum is $((E', p'_z c, p'_y c) = (Mc^2, 0, 0)$: $E = \gamma Mc^2$ and $cp_z = \gamma\beta Mc^2$. Quite familiar. For one of the decay products, the four momentum in the parent's rest frame is $E' = Mc^2/2$, $p'_y c = \sqrt{(Mc^2/2)^2 - (mc^2)^2}$, $p'_z = 0$. Then $E = \gamma Mc^2/2$ (the two particles have the same energy, by symmetry, and it is half the original energy), $cp_z = \beta\gamma Mc^2/2$ (equal again, by symmetry), and $p_y = p'_y$. Can you check that $E^2 - c^2p^2 = (mc^2)^2$? The answer is rather messy. If the decay products had had components of their momentum along the boost direction, life would have been much messier!

Notice some physics, however. If the parent particle moves faster and faster, p_z for the decay product becomes bigger and bigger, while p_y remains unchanged. This means that the opening angle between the decay products becomes smaller and smaller: $\tan\theta = p_y/p_z \sim 1/\gamma$. For small angles, $\tan\theta \sim \theta$ so the opening angle goes to zero like $1/\gamma = Mc^2/E$.

The cyclotron

Relativistic analogs of $F = ma$ problems can become quite complicated, but there is one simple example we can work through,and it even has a relativistic moral: that is the cyclotron. The Lorentz force law for a particle in a magnetic field B is

$$\frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B} \tag{3.29}$$

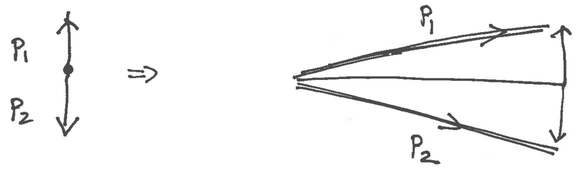

Figure 3.2: A resonance decays into two particles, back to back in its rest frame. In the moving frame, both decay products are thrown forward.

For motion in a plane perpendicular to the applied field, $\vec{v} \times \vec{B}$ is directed radially inward and we have uniform circular motion. You hopefully recall the non-relativistic version of this problem: $p = mv$, $dv/dt = v^2/r$ is the usual centripetal acceleration, and the equation of motion is

$$m\frac{v^2}{r} = qvB. \tag{3.30}$$

This says that $r = mv/(qB)$. For circular motion the angular velocity ω and the adjective-free velocity v are related by $v = \omega r$, so $m\omega^2 r = q\omega r B$ and because "r cancels r" (in the words of E. O. Lawrence, inventor of the cyclotron), $\omega = qB/m$. This is the physical principle for the cyclotron. The classical cyclotron (circa 1930's) has two "dees," half-cylindrical hollow metal shells aligned perpendicular to the field. An alternating voltage of frequency $\omega = qB/m$ (the "cyclotron frequency") between the dees accelerates the particles as they cross the gap. The radius of the beam increases as it is accelerated, $r = mv/qB$, but the frequency of the orbit is unchanged so that regardless of the particle's energy, it is still accelerated.

The relativistic version of this story is simply told. Eq. 3.29 is unchanged. However, now the momentum is $\vec{p} = \gamma m\vec{v}$. Nothing else changes; the direction of the force is as it was there is still uniform circular motion. The only difference is the stray factor of γ in $r = p/(qB) = \gamma mv/(qB)$ and the cyclotron frequency is $\omega = qB/(\gamma m)$. Now we are in trouble, because ω depends on the velocity of the moving particle, through γ.

For the first working cyclotron, in 1932, this was no big deal. Lawrence's machine accelerated protons to a kinetic energy of 10 MeV (huge at the time), but

$$\gamma = \frac{E}{mc^2} = \frac{mc^2 + E_K}{mc^2} \sim 1.01 \tag{3.31}$$

for 10 MeV protons. This is too small an effect to cause a problem. However, Lawrence's next machine was designed to produce 100 MeV protons. Now $\gamma = 1.1$ and there was going to be trouble, a constant modulating frequency did not correspond to the cyclotron frequency at all energies. (The great physicist Hans Bethe noticed this fact and told the Berkeley scientists abut it, but as good American experimentalists they did not care to hear about special relativity...) Anyway, the cure was to exploit the fact that the more energetic the particle, the bigger its orbit, $r = \gamma mv/(qB)$ and one had to modify the magnetic field so that it varied radially, $B(r) \sim 1/\gamma$ so that the cyclotron frequency would remain effectively energy-independent.

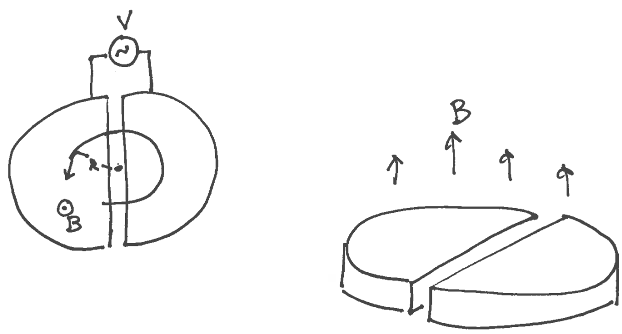

Figure 3.3: Two views, top and side, of a cyclotron. The metallic semicircles are the "dees." Particles are accelerated across the gap by an oscillating electric field. A magnetic field normal to the plane of the dees confines the particles.

Chapter 4

The end of special relativity

4.1 Summary of the important concepts and results

The story of relativity is essentially a story of geometry: we live in a four-dimensional world whose coordinates are the three spatial directions and time. Observers disagree on the values of the time intervals or the spatial separations of events, but they all agree that the proper time between events, $c^2\tau^2 = c^2(t_1 - t_2)^2 - (\vec{x}_1 - \vec{x}_2)^2$ is the same. The conversion of time intervals and spatial intervals is encoded in the Lorentz transformations, which for a boost along the $\hat{z}$ direction is

$$\begin{aligned}
\Delta z' &= \gamma\Delta z - \beta\gamma\Delta ct \\
\Delta ct' &= -\beta\gamma\Delta z + \gamma\Delta ct \\
\Delta x' &= \Delta x \\
\Delta y' &= \Delta y
\end{aligned} \tag{4.1}$$

and of course $\beta = v/c$, $\gamma = 1/\sqrt{1 - v^2/c^2}$.

There is an identical story for kinematics. Observers moving with respect to each other disagree as to the energy and momentum of moving objects, but they agree on the value for the mass, $(mc^2)^2 = E^2 - c^2p^2$. E and $c\vec{p}$ form a four vector just as ct and $\vec{r}$ do, and the transformation of energy and momentum are given again by the Lorentz transformations, with the substitution of E for ct and $c\vec{p}$ for $\vec{r}$.

Everything else can be derived from these relations.

4.2 What's left

Finally, I ought to at least mention aspects of relativity which have been left out in these brief notes.

The biggest ghost at the table is what happens when the energy density which is present is strong enough to create observable gravitational fields. This is the realm of general relativity. Special relativity becomes "special" – a special case where we say "space is flat" as opposed to the case of "curved space" where particles – and reference frames – accelerate.

That was a big reach, so let's consider some more approachable extensions.

First, we could ask, how do moving objects really look? It turns out that they are not just squashed into pancakes by length contraction; it is a much more interesting story. To begin the story, how do you see a moving object? By scattering light off it. And the light you scatter is moving at a finite velocity, while the moving object is also moving. I'll stop here and let you discover the answer on your own. (The source, J. Terrell, Physical Review 116, 1041 (1959), is a bit dense.)

Second, we have only considered (when we considered this possibility at all) a pair of successive Lorentz transformations which are chosen to be along the same axis. In this special case the product of the two Lorentz transformations is itself a Lorentz transformation. It is easy to work out the mathematics in this case: you can show that the resulting boost angle is the sum of the two boost angles. The case of successive Lorentz transformations which are not parallel is much more interesting: they are equivalent to a combination of a boost plus a rotation. The rotation can be –and was – observed in atomic spectroscopy, in a famous factor of 1/2.

In words, what happens is this. As an electron orbits a nucleus, it observes the nuclear electric field converted (by an analog of a Lorentz transformation) into a mixture of an electric and a magnetic field. Electrons possess spin, an intrinsic angular momentum, and a resulting magnetic dipole moment. This dipole moment interacts with the induced magnetic field and slightly shifts the energy of the electron. The shift is of course dependent on the orientation of the spin, and amounts to the statement that the energy of the electron is slightly spin-dependent. This shift can be observed in the spectral lines when light is emitted by atoms, as their electrons decay from higher energy states to lower energy ones. This relativistic effect is called "spin-orbit" coupling. But it turns out that when you calculate the size of the energy shift by considering the $E \to B$, magnetic dipole interaction, you will get an answer which disagrees with experiment by a factor of two. The missing reduction – an overall reduction in the result by 1/2– is called the "Thomas precession" after the person who first explained it – in 1924. (Apparently even Einstein was surprised by it.) You will meet the effect in second semester quantum mechanics, but its full derivation is for some reason usually postponed to graduate school.

An obvious place to re-encounter relativity is in classical electricity and magnetism. It turns out that the ordinary electrostatic potential ϕ is part of a four-vector, bundled with a quantity called the "vector potential" $\vec{A}$: ϕ and $\vec{A}$ occupy the same places as the energy E and momentum p. As you move from one inertial frame to another, they mix (equivalently, moving observers see different ϕ's and $\vec{A}$'s.)

The electric and magnetic fields also mix with each other as one moves fro frame to frame. Their transformation law is more complicated; they are NOT four vectors. (How could they be? There are three components of $\vec{E}$ and three of $\vec{B}$. Six isn't four!). Finally, there is the Lorentz force law

$$\vec{F} = q(\vec{E} + \frac{\vec{v}}{c} \times \vec{B}). \tag{4.2}$$

It has an interesting relativity story attached to it, too. And beyond that, lots of relativistic physics associated with the motion of charged particles and how they emit radiation.

The interplay of relativity and quantum physics is the most fascinating arena of all. The physics we create to describe Nature at its most fundamental level has to be consistent both with special relativity and with the laws of quantum physics. While this sounds esoteric, I can give three examples of fairly important consequences of this consistency.

The first is the existence of antiparticles. Without them, quantum mechanics and relativity would be mutually incompatible.

The second is the existence of particles with "spin," like the electron. (You may have heard – in fact I just told you – that the electron is a "spin -1/2" particle; it carries an intrinsic angular momentum whose vector value projected on any axis is restricted to be plus or minus $1/2\hbar$; $\hbar$ is Planck's constant h divided by 2π. That all fundamental particles – and hence all matter – have angular momentum which is either a half integer or an integer multiple of $\hbar$, is a consequence of special relativity.

Finally, there is the spin-statistics theorem. Particles of half-integer spin, like electrons, cannot all be put in the same state. All of chemistry (and all of biology and geology) is a consequence of this fact, called the Pauli exclusion principle. The rules you learned in high school for building the periodic table encode this principle. There, the simple story was, "only two electrons in every state;" the real story is that any state can only be occupied by zero or one electron. And as the state is labeled by all its quantum numbers, including spin, each high school state is really two states, one for each spin value, and can contain at most two electrons. However, particles with integer spin like to be in the same state. Lasers employ this property of the photon. And in Colorado, "Bose-Einstein condensation" of cold trapped atom gases arises because of this fact. There is a sign outside the physics building... The explanation of the Pauli exclusion principle, and the fact that integer spin particles like to clump, can only be found in relativistic quantum field theory, the union of quantum mechanics and special relativity.

Okay, to be fair, in atomic and molecular physics, relativity is a small effect. Recall

$$\sqrt{p^2c^2 + m^2c^4} \sim mc^2 + \frac{p^2}{2m} - \frac{1}{8}\frac{(pc)^2}{(mc^2)^3} + \dots \tag{4.3}$$

The second term is

$$\frac{p^2}{2m} = \frac{1}{2}mc^2(\frac{v}{c})^2 = \frac{1}{2}mv^2 \tag{4.4}$$

and the third term is

$$\frac{p^4}{m^3c^2} \sim mc^2(\frac{v}{c})^4. \tag{4.5}$$

In atoms, we'll find that $v/c \sim 1/137$, so the ratio of the usual kinetic energy to the leading relativistic correction is only

$$\frac{\frac{p^4}{m^3c^2}}{\frac{p^2}{2m}} \sim (\frac{v}{c})^2 \sim 10^{-4} \tag{4.6}$$

This gives small but important effects. As a first pass, we can neglect them. Even later, when we want to be precise, we can treat special relativity as a small correction to a non-relativistic system, and just think of the kinetic energy as $(1/2)mv^2$ and the potential energy as some $V(r)$, and proceed – quite a way – ignoring relativity.