# 4

# Natural Deduction in Sentential Logic

## 1 The concept of proof

We have at least partly achieved the goal we set ourselves in Chapter 1, which was to develop a technique for evaluating English arguments for validity. However, there is a respect in which our approach to arguments differs from that of the typical person involved in a debate. The typical debater does not proceed by stating her premises, then her conclusion, and then defying the opposition to describe how it would be possible for the premises to be true and the conclusion to be false. Rather, one tries to *reason from* one's premises *to* one's conclusion. In other words, to show that the conclusion follows from the premises, one tries to *deduce* it *from* the premises. And there is nothing in the techniques of Chapter 3 which captures the idea of arguing for a conclusion by deducing it from premises.

Deducing the conclusion of an argument from the premises is ordinarily called *giving* an argument. By 'giving an argument' we mean to connote an activity which stands in contrast to dogmatic assertion. However, we have already reserved the term 'argument' for the listing of the premises and the conclusion, so we will call the new component of deducing the conclusion from the premises *giving a proof* of the argument. Here is an informal illustration of what is involved in giving a proof. In §3 of Chapter 1 we presented the following argument:

> If the safe was opened, it must have been opened by Smith, with the assistance of Brown or Robinson. None of these three could have been involved unless he was absent from the meeting. But we know that either Smith or Brown was present at the meeting. So since the safe was opened, it must have been Robinson who helped open it.

As remarked earlier, this is a valid argument, as we could now show by translating it into LSL and demonstrating the validity of its form. But it is also possible to show that the English argument is valid using a less formal method, by exhibiting how its conclusion can be deduced from its premises. To do this, we begin by listing the premises individually:

(1) If the safe was opened, it must have been opened by Smith, with the assistance of Brown or Robinson.
(2) None of these three could have been involved unless he was absent from the meeting.
(3) Either Smith or Brown was present at the meeting.
(4) The safe was opened.

From this we have to deduce that Robinson helped open the safe. Here is how we might proceed:

(5) Smith opened the safe and either Brown or Robinson helped (from (4) and (1)).
(6) Smith was absent from the meeting (from (5) and (2)).
(7) Brown was present at the meeting (from (6) and (3)).
(8) Brown did not help to open the safe (from (7) and (2)).
(9) Therefore Robinson did (from (8) and (5)).

This is set out more carefully than in ordinary conversation, but the mechanics are the same. We arrive at the conclusion by making a succession of small steps from what we have already established, or from what we are given in the premises. Each step is easily seen to be correctly inferred from previously established lines, and the earlier lines appealed to at each step are noted. Since the individual steps are unobjectionable, we have demonstrated that the conclusion follows from the premises, but we have done it by giving a proof, not by constructing interpretations.

We are now going to articulate the process of giving a proof. To do this we have to be completely explicit about what principles can be appealed to in making individual steps like (5)–(9) above. These principles are called *rules of inference* and the process of using them to deduce a conclusion from premises is called *natural deduction*. A *system of natural deduction* is simply a collection of rules of inference. Systems of natural deduction were first described and investigated by Gerhard Gentzen, and the system we shall present here is Gentzen's system NK (German: *Natürliche Kalkül*).

We formulate the rules of inference not for English but for LSL. The rationale for abstracting from English is the same as before: whether or not a conclusion can be deduced from some premises depends not on the subject-matter of the premises and conclusion but on their logical form. For example, one could replace the argument about who helped open the safe with a parallel argument about, say, which country will have the biggest reduction in inflation next year, and which the next biggest (see the exercise following). So long as the premises have the same logical forms as the premises of the argument about who Smith's accomplice was, a five-step proof of the conclusion of the new argument could be given in parallel with our five-step proof of 'Robinson helped open it', each step justified by the same rules applied to the same previous line numbers. That is, whether or not a conclusion is deducible from certain premises depends just on their forms. Since we exhibit the forms by translating into LSL, we may as well construct the proofs in LSL from the outset.

One final introductory remark. The argument displayed above is the sort of thing one might find in a Sherlock Holmes novel, but of course precise proof is most commonly found in mathematics. A mathematician once related being asked why we insist on proving things in mathematics—“Why can’t we just trust each other?” the student wondered. Proof is important in mathematics primarily because it is the only way to extend mathematical knowledge, but this aspect of its importance affects only those few individuals capable of making mathematical discoveries. However, for the rest of us, *following* a proof has an effect that is not obtained if we simply trust professors of mathematics not to make mistakes: following a proof helps us to understand *why* a particular proposition is true. For example, one might just accept that there are infinitely many prime numbers, but to see why this is so, we need to grasp one of the (many) proofs of this result. This makes the nature of proof in mathematics a subject of universal interest in its own right, and the historical motivation for developing the systems of natural deduction to be presented in this book was simply to study the kinds of reasoning accepted in standard mathematics. Indeed, classical logic is sometimes defined as the logic used in classical mathematics. Still, mathematical reasoning is really only a more rigorous version of reasoning of the type of the detective’s about who opened the safe, and so the applicability of the study of deduction is not restricted to mathematics.

### ❑ Exercise

The following argument has the same form as the example of this section:

> If some countries reduce inflation next year, Brazil will reduce it the most, with either Argentina or Britain next. None of these three will reduce inflation unless they increase interest rates. But either Brazil or Argentina will hold interest rates down. So since some countries will reduce inflation next year, Britain will be the second most successful.

Deduce the conclusion of this argument from the premises in a way that exactly parallels the deduction of ‘Robinson helped open the safe’ from the premises of the example in this section.

## 2 Rules for conjunction and the conditional

In order to begin a proof, one needs to have some lines to which rules of inference can be applied. These lines are usually the premises of the argument to be proved, and we think of the process of listing the premises, as in (1)–(4) of the example of the previous section, as itself involving the application of a rule of inference. For the moment, we will take this rule to say that we may begin a proof by writing down one or more lines and labeling them as premises. This rule is sometimes called the Rule of Assumptions.

In the system NK, each of our five connectives is governed by two rules of inference. For each connective *c*, there is a rule which says when a statement can be inferred from some already established statements, one of which has *c* as its main connective. This rule is called the *elimination rule* for *c*, since the formula which is inferred will have fewer occurrences of *c* than the formula of which *c* is the main connective. For example, the elimination rule for ‘&’ allows us to infer ‘A’ from ‘A & B’, while the elimination rule for ‘→’ allows us to infer ‘B’ from ‘A’ and ‘A → B’ (inspection of line 5 in the example of §1 indicates that it is justified by the elimination rule for ‘→’, since line 1 is a conditional and line 4 is its antecedent). We can express the content of these two rules in words as: from a conjunction we can infer either conjunct, and from a conditional and its antecedent we can infer its consequent. Each connective *c* also has an *introduction rule*, which says when a formula with *c* as its main connective may be inferred from already established formulae. For example, the introduction rule for ‘&’ allows us to infer ‘A & B’ from ‘A’ and ‘B’; more generally, from any two formulae we may infer their conjunction. Since these three rules are the simplest, we begin our exposition of NK with them.

The following is a valid LSL argument-form:

*Example 1:* A & B
C & D
(A & D) → H
∴ H

Here is what an NK-proof of the conclusion of Example 1 from its premises looks like. We abbreviate ‘&-Elimination’ and ‘&-Introduction’ by ‘&E’ and ‘&I’, and ‘→-Elimination’ by ‘→E’.

| | | | | |
|---|---|---|---|---|
| 1 | (1) | A & B | Premise | |
| 2 | (2) | C & D | Premise | |
| 3 | (3) | (A & D) → H | Premise | |
| 1 | (4) | A | 1, &E | |
| 2 | (5) | D | 2, &E | |
| 1,2 | (6) | A & D | 4,5 &I | |
| 1,2,3 | (7) | H | 3,6 →E | ◆ |

The intuitive idea behind the proof is straightforward. In order to obtain ‘H’, the consequent of (3), we should apply →E to (3), but to use →E we need a conditional *and* its antecedent, so we must first obtain the antecedent ‘A & D’ of (3). This formula is a conjunction, so we should try to obtain it by &I, which requires that we first derive its two conjuncts. The conjuncts occur separately in (1) and (2), so we obtain each on its own by using &E. On the right of the proof we have written the justification of each line: (1)–(3) are premises and (4)–(7) are justified in the way explained.

Our explanation of the proof of Example 1 also illustrates an important technique in constructing proofs. It is rarely efficient to begin by blindly applying rules to the premises to see what happens: it is much better to formulate a *strategy*. A strategy can be worked out by looking at the conclusion and asking what rule might deliver it at the last line of the proof. *If the conclusion contains connectives, the rule is likely to be the introduction rule for the main connective;* if it does not, as in our case, the relevant rule can often be gleaned by inspecting the premises. In Example 1 it seems plausible that at the last line, we will be applying →E using (3). This tells us that at some previous line we have to establish the antecedent of (3), 'A & D'. This latter formula has '&' as its main connective and so will likely be inferred by &-Introduction. This means in turn that we have to obtain 'A' by itself and 'D' by itself, and looking at the premises, it is clear how this should be done. Our procedure, therefore, is to start by setting ourselves the overall 'goal' of proving the conclusion of the argument-form, and then break down this task into a sequence of subgoals until we arrive at subgoals whose execution is obvious. We then put all the steps together in the appropriate order, marking the finish of the proof with '◆' or some similar symbol.

We have left an important feature of our sample proof unexplained, the line numbers on the left. *We use these numbers to keep track of which premises any given line 'depends' upon.* The premises themselves are not inferred from any other lines, so each premise has its own line number written on the left. At line 4 we apply a rule of inference to line 1 to obtain 'A', so (4) depends on (1), and similarly (5) depends on (2). At line 6 we conjoin a line that depends on (1) and a line that depends on (2), so the result depends on both (1) and (2); finally, at line 7 we use line 3 and line 6; (6) depends on (1) and (2), and (3) depends on (3), so (7) depends on (1), (2) and (3). In general, then, *the premise numbers which should be listed on the left are the premise numbers that the lines cited on the right depend upon.* Intuitively, the numbers on the left at a line j are the premises which are needed, or at least which have been used, to derive the formula at j. There may be other premises listed at the start of the proof, but if their numbers are not on the left at j, they were not used in deriving the formula at j.

One advantage of this 'bookkeeping' device is that it makes it easy to check at the end of a proof that we have indeed proved what was asked. In order to show that the conclusion of Example 1 follows from its three premises, we require not merely a proof which begins with those premises and whose last line is the desired conclusion, but a proof in which *the last line depends upon no lines other than premises of Example 1*. Had there been other numbers on the left of line 7, this would show that we had not derived 'H' from the right premises. In a correct proof of an argument-form, therefore, the formula at the last line is the conclusion of the argument-form, and the lines on which it depends are all premises which are listed at the start. Note that we do not say that the lines on which the conclusion depends are *all of* the premises which are listed: it is sufficient that the conclusion depend on a subset of the listed premises. For example, if we had somehow been able to derive 'H' without using, say, premise 2, then only the numbers 1 and 3 would be on the left at line 7. But we would still have a proof of Example 1; after all, if 'H' follows from (1) and (3), it follows from (1), (3) and any other statements whatsoever—the extra statements would simply be redundant.

Having seen the way in which proofs are written down in NK, we can give more exact statements of the four rules of inference we have introduced so far.

> ***Rule of Assumptions*** (preliminary version): The premises of an argument-form are listed at the start of a proof in the order in which they are given, each labeled 'Premise' on the right and numbered with its own line number on the left. Schematically
>
> $$\begin{array}{llll} \mathrm{j} & (\mathrm{j}) & p & \text{Premise} \end{array}$$
>
> ***Rule of &-Elimination:*** If a conjunction $\ulcorner p \,\&\, q \urcorner$ occurs at a line j then at any later line k one may infer either conjunct, labeling the line 'j &E' and writing on the left all the numbers which appear on the left of line j. Schematically:
>
> $$\begin{array}{llll} \mathrm{a}_1,\ldots,\mathrm{a}_n & (\mathrm{j}) & p \,\&\, q & \\ & \vdots & & \\ \mathrm{a}_1,\ldots,\mathrm{a}_n & (\mathrm{k}) & p & \mathrm{j}\ \&\mathrm{E} \end{array} \qquad \text{OR} \qquad \begin{array}{llll} \mathrm{a}_1,\ldots,\mathrm{a}_n & (\mathrm{j}) & p \,\&\, q & \\ & \vdots & & \\ \mathrm{a}_1,\ldots,\mathrm{a}_n & (\mathrm{k}) & q & \mathrm{j}\ \&\mathrm{E} \end{array}$$

Thus the rule of &E is really two rules, one which allows us to infer the first conjunct of a conjunction, the other of which allows us to infer the second. We choose which to apply depending on what we want to infer.

> ***Rule of &-Introduction:*** For any formulae $p$ and $q$, if $p$ occurs at line j and $q$ occurs at line k then the formula $\ulcorner p \,\&\, q \urcorner$ may be inferred at line m, labeling the line 'j,k &I' and writing on the left all numbers which appear on the left of line j and all which appear on the left of line k. Here we may have j < k, j = k or j > k. Schematically:
>
> $$\begin{array}{rcll} \mathrm{a}_1,\ldots,\mathrm{a}_n & (\mathrm{j}) & p & \\ & \vdots & & \\ \mathrm{b}_1,\ldots,\mathrm{b}_u & (\mathrm{k}) & q & \\ & \vdots & & \\ \mathrm{a}_1,\ldots,\mathrm{a}_n,\mathrm{b}_1,\ldots,\mathrm{b}_u & (\mathrm{m}) & p \,\&\, q & \mathrm{j,k}\ \&\mathrm{I} \end{array}$$

It is important to master how schemata are used to explain the application of a rule. Compare the schema for &I with line 6 in the proof of Example 1. The formula $p$ is 'A', the formula $q$ is 'D', line j is line 4 and line k is line 5. Because the proof is very simple, lines 4 and 5 each depend on just one premise. So the application of the schema to lines 4 and 5 is for the case where $n = u = 1$; $\mathrm{a}_1$ is the number 1 and $\mathrm{b}_1$ is the number 2. Line m is line 6, and as the schema requires, the numbers 1 and 2 are mentioned on its left.

However, there are respects in which the application of a rule may be more flexible than its schema seems to indicate. In the schema for &I, the line with the first conjunct $p$ of $\ulcorner p \,\&\, q \urcorner$, line j, is a *separate* line occurring *before* the line with the second conjunct *q*, but this number and order of occurrence of lines are not necessary: the order of the conjuncts at line m need not reflect the order in which they occur earlier in the proof, and &I may be applied to a single line to yield a conjunction with the same formula as its two conjuncts; for example, if 'A ∨ B' is the formula at line r, &I applied to r and r again produces '(A ∨ B) & (A ∨ B)', a step we would label 'r,r &I'. This is what is meant by saying that we may have j < k, j = k or j > k.

> ***Rule of →-Elimination:*** For any formulae *p* and *q*, if $\ulcorner p \to q \urcorner$ occurs at a line j and *p* occurs at a line k then *q* may be inferred at line m, labeling the line 'j,k →E' and writing on the left all numbers which appear on the left of line j and all which appear on the left of line k. We may have j < k or j > k. Schematically:

| | | | |
|---|---|---|---|
| $a_1,\ldots,a_n$ | (j) | $p \to q$ | |
| | ⋮ | | |
| $b_1,\ldots,b_u$ | (k) | $p$ | |
| | ⋮ | | |
| $a_1,\ldots,a_n,b_1,\ldots,b_u$ | (m) | $q$ | j,k →E |

Here are two further examples of proofs in NK using these rules. First we prove the valid LSL argument

*Example 2:* A & (B & C)
∴ C & (B & A)

The conclusion is a conjunction, so we will likely obtain it at the last line of our proof by an application of &I to two previous lines where the conjuncts appear. Our subgoals, therefore, are (i) to derive 'C' on a line by itself, and (ii) to derive 'B & A' on a line by itself. 'B & A' is another conjunction, so the subgoal for (ii) is to derive its two conjuncts 'B' and 'A' separately. In sum, then, we want to derive the individual sentence-letters 'A', 'B' and 'C' by themselves and then put them together in the right order using &I. To obtain the letters by themselves, we can apply &E to the premise.

| | | | |
|---|---|---|---|
| 1 | (1) | A & (B & C) | Premise |
| 1 | (2) | A | 1 &E |
| 1 | (3) | B & C | 1 &E |
| 1 | (4) | B | 3 &E |
| 1 | (5) | C | 3 &E |
| 1 | (6) | B & A | 4,2 &I |
| 1 | (7) | C & (B & A) | 5,6 &I ◆ |

This proof illustrates the important point that an elimination rule for a connective *c can only be applied to a line if that line has an occurrence of c as its main connective, and the rule must be applied to that occurrence of the con-*

*nective.* So at line 2, for example, we cannot apply &E to the second occurrence of '&' in line 1. In this respect the rules of inference follow the syntactic structure of the formula. If you are applying an elimination rule for a connective *c* to a formula on a line and it is not the main connective of the formula which you are eliminating, then you have made a mistake (a very common one for beginners!).

Another example:

*Example 3:* A → (B → (C → D))
C & (A & B)
∴ D

Since the conclusion has no connectives, it will not be inferred by an introduction rule. Inspecting the premises, we see that the conclusion is the consequent of a conditional which is a subformula of the first premise, so we can obtain the conclusion if we can obtain that conditional, 'C → D', on a line by itself, and also its antecedent on a line by itself. The antecedent of 'C → D' comes from premise 2, so the main subgoal is to extract 'C → D' from premise 1. We can do this with two further applications of →E.

| | | | |
|---|---|---|---|
| 1 | (1) | A → (B → (C → D)) | Premise |
| 2 | (2) | C & (A & B) | Premise |
| 2 | (3) | A & B | 2 &E |
| 2 | (4) | A | 3 &E |
| 1,2 | (5) | B → (C → D) | 1,4 →E |
| 2 | (6) | B | 3 &E |
| 1,2 | (7) | C → D | 5,6 →E |
| 1 | (8) | C | 2 &E |
| 1,2 | (9) | D | 7,8 →E ◆ |

Again, note that it is not possible to apply →E directly to (8) and (1) to obtain (9). In other words, the following four-line proof is *wrong:*

| | | | |
|---|---|---|---|
| 1 | (1) | A → (B → (C → D)) | Premise |
| 2 | (2) | C & (A & B) | Premise |
| 1 | (3) | C | 2 &E |
| 1,2 | (4) | D | 1,3 →E (NO!) |

The main connective of (1) is the *leftmost* arrow in it, with antecedent 'A' and consequent 'B → (C → D)'. *It is only to the main connective of a line that →E can be applied* (*similarly for every other elimination rule*), so we cannot apply →E to line 1 until we have 'A' on a line by itself, which we manage at line 4 in the correct proof above.

We said earlier that every connective has both an introduction rule and an elimination rule, but so far we have only stated an elimination rule for the conditional. What should the introduction rule be? In other words, what kind of reasoning do we use to *establish* a conditional? Suppose we want to prove that *if* our currency loses value, *then* our trade deficit will narrow. A natural way of arguing for this conditional would be the following:

> Suppose our currency loses value. Then foreign imports will cost more in this country and our exports will cost less abroad. So our demand for imports will fall while demand in other parts of the world for our goods will increase. Therefore our trade deficit will narrow.

This little argument appeals to many suppressed premises, for example, that if the price of a good increases, demand for it slackens ('perfect elasticity' in economists' jargon) and that our manufacturers can increase production to meet increased demand. These might be regarded as other premises from which the conditional 'if our currency loses value, then our trade deficit will narrow' is to be deduced. As can be seen, the argument works in the following way: to derive the conditional, we begin by supposing that its antecedent is true ('Suppose our currency loses value'). Then using the other implicit premises and various rules of inference, we derive the consequent ('Therefore our trade deficit will narrow'). This shows that the consequent follows from the antecedent and the other premises: the consequent is true *if* the antecedent and the other premises are true. Hence the conditional 'if our currency loses value, then our trade deficit will narrow' is true if those other premises by themselves are true. This last point is important: the truth of the whole conditional depends not on the truth of the other premises *and* the antecedent, but only on the other premises; one can certainly convince oneself of the conditional 'if our currency loses value then our trade deficit will narrow' by an argument such as the one displayed without holding that its antecedent 'our currency will lose value' is in fact true.

The rule of →-Introduction formalizes this procedure. In a formal proof, all premises are listed explicitly and we can appeal to them in the normal way; the strategy for deriving a conditional, therefore, is just to add an extra assumption after the premises, this assumption being the antecedent of the conditional which we want to derive. Once we have deduced the consequent of the conditional from the premises together with the antecedent of the conditional, we are entitled to assert that the conditional itself follows from the premises without the extra assumption.

Here is a simple illustration of the strategy, in a proof of the following LSL argument-form:

*Example 4:* $R \rightarrow (S \rightarrow T)$
$S$
$\therefore R \rightarrow T$

We have to derive the conditional 'R → T', so we should add its antecedent 'R' to the premises as an extra assumption (like supposing our currency loses value). The goal is then to derive its consequent 'T' from the two premises and the extra assumption, after which we may assert that the conditional 'R → T' follows from the two premises alone. The NK-proof of Example 4 is:

| | | | |
|---|---|---|---|
| 1 | (1) | $R \rightarrow (S \rightarrow T)$ | Premise |
| 2 | (2) | $S$ | Premise |
| 3 | (3) | $R$ | Assumption |
| 1,3 | (4) | $S \rightarrow T$ | 1,3 →E |
| 1,2,3 | (5) | $T$ | 4,2 →E |
| 1,2 | (6) | $R \rightarrow T$ | 3,5 →I ◆ |

There are a number of features of this proof to which close attention should be paid. First, assumptions, like premises, depend on themselves, so at line 3 we put ‘3’ on the left. Second, line 3 is justified by essentially the same rule as lines 1 and 2. What allows us to write the formulae at (1) and (2) is not that they are premises, though that is why we write these particular formulae. Lines like (1) and (2), and also (3), are unobjectionable because, in general, a line in a proof may be read as claiming that the formula on the line holds ‘according to’ the formulae whose numbers are on the left of the line. And, of course, any formula holds according to *itself*. The Rule of Assumptions, which relies on this trivial fact, is simply the rule that any formula may be written at any line in a proof if it is numbered to depend on itself. So it is not strictly obligatory to distinguish between ‘assumption’ and ‘premise’. However, we shall continue to use ‘premise’ for a formula which is a premise of the LSL argument-form which is being proved and ‘assumption’ for an extra hypothesis made in the course of giving a proof.

Third, the numbering on the left of line 6 should be noted: the assumption number ‘3’, which appears on the left of (5), has been dropped, and the assumption itself has become the antecedent of the conditional at line 6. We said earlier that once we have derived the consequent of a conditional from its antecedent together with other premises, we are ‘entitled to assert that’ the conditional follows from the premises by themselves. This is implemented in NK by dropping the assumption number and absorbing the assumption by making it the antecedent of the conditional. That is, at line 5 in the proof we show that ‘T’ follows from lines 1, 2 and 3, where ‘R’ is the formula at line 3, so at line 6 we say that ‘$R \rightarrow T$’ follows just from 1 and 2, which is what was to be proved. Dropping the assumption number and making the assumption the antecedent of a conditional is called *canceling* or *discharging* the assumption. That is why the correct version of the Rule of Assumptions is not as liberal as it appears. To complete a proof we may make as many extra assumptions as we like, but if we use these assumptions they will eventually have to be discharged. For the moment, then, *the only situation in which we make an assumption is when we want to derive a conditional, and the assumption formula is always the antecedent of the conditional we want to derive.* Note that if we do not drop 3 from the left at 6, then the last line would be represented as still depending on 3, which is an assumption, not a premise of Example 4. Therefore the proof would not be a proof of Example 4, since not all numbers on the left of the last line would be numbers of premises of Example 4. It would instead be a proof of the easier problem that has the extra premise ‘R’.

We can iterate applications of →I, as we must to prove the following argument-form:

*Example 5:* ((A & B) & C) → D
∴ C → (B → (A → D))

To derive 'C → (B → (A → D))' we should assume 'C' with the goal of deducing 'B → (A → D)'; in turn, to derive 'B → (A → D)' we should assume 'B' with the goal of deducing 'A → D'; in turn again, to derive 'A → D' we should assume 'A' with the goal of deducing 'D'. To obtain 'D', we use our assumptions to construct the antecedent 'A & (B & C)' of the premise, and then apply →E. The proof is:

| | | | |
|---|---|---|---|
| 1 | (1) | ((A & B) & C) → D | Premise |
| 2 | (2) | C | Assumption |
| 3 | (3) | B | Assumption |
| 4 | (4) | A | Assumption |
| 3,4 | (5) | A & B | 4,3 &I |
| 2,3,4 | (6) | (A & B) & C | 5,2 &I |
| 1,2,3,4 | (7) | D | 1,6 →E |
| 1,2,3 | (8) | A → D | 4,7 →I |
| 1,2 | (9) | B → (A → D) | 3,8 →I |
| 1 | (10) | C → (B → (A → D)) | 2,9 →I ◆ |

When a proof has many assumptions, it is important to check that the last line depends only on premises: the numbers on the left at the end should not include any of the assumptions. We made three assumptions in this proof, and all are used, but they are successively discharged at lines 8, 9 and 10, so that the last line of the proof depends only on line 1, which is how we want it.

We now state the new rule of →I and the final version of Assumptions.

> ***Rule of Assumptions:*** At any line j in a proof, any formula $p$ may be entered and labeled as an assumption (or premise, where appropriate). The number j should then be written on the left. Schematically:

| | | | |
|---|---|---|---|
| j | (j) | $p$ | Assumption (or: Premise) |

> ***Rule of →-Introduction:*** For any formulae $p$ and $q$, if $q$ has been inferred at a line k in a proof and $p$ is an assumption or premise occurring at a line j, then at line m we may infer $\ulcorner p \to q \urcorner$, labeling the line 'j,k →I', and writing on the left the same assumption numbers as occur on the left of line k, except that we delete j if it is one of these numbers. Here we may have j < k, j = k or j > k. Schematically:

| | | | |
|---|---|---|---|
| j | (j) | $p$ | Assumption (or: Premise) |
| | ⋮ | | |
| $a_1,\ldots,a_n$ | (k) | $q$ | |
| | ⋮ | | |
| $\{a_1,\ldots,a_n\}/j$ | (m) | $p \to q$ | j,k →I |

In this statement of the rule, the notation '$\{a_1,\ldots,a_n\}/j$' indicates the set of numbers $\{a_1,\ldots,a_n\}$ with j removed if it is one of them. Applications of →I when $j = k$ yield conditionals with the same formula as antecedent and consequent. Notice that it is not a requirement of applying →I that j be one of the premise or assumption numbers on which line k depends, nor is it a requirement that line j occur before line k. Here is a correct proof in which $k < j$ (the consequent appears in the proof before the antecedent!) and j does not appear on the left of line k, a proof of a valid LSL argument-form:

*Example 6:* $A$
$\therefore B \to A$

The validity of this argument-form reflects the fact that a conditional with a true consequent is true. The proof is:

| | | | |
|---|---|---|---|
| 1 | (1) | $A$ | Premise |
| 2 | (2) | $B$ | Assumption |
| 1 | (3) | $B \to A$ | 2,1 →I ◆ |

In terms of our schema for →I, $j = 2$ and $k = 1$. So the consequent of (3) is at (1) and the antecedent at (2), and the antecedent is not used to derive the consequent. In §1 and §7 of Chapter 3 we noted that the function-table for '→' has some odd-looking consequences for the truth-values of English conditionals. However, it can be formally proved that the function-table for '→' is the correct account of the conditional *if* the rules of →E and →I are correct: the rules and the table go hand in hand. Someone who rejects the table as an account of the meaning of 'if...then...' therefore owes us a new account of 'if...then...'-introduction, presumably involving a requirement that the antecedent be *non-redundantly used* in deriving the consequent. There is a branch of logic known as *relevance logic* which develops this approach (see Read), but it turns out that to implement the idea of non-redundant use in a reasonably satisfactory way leads to systems that are significantly more complicated than classical logic.

Our last example is a proof of the valid argument-form

*Example 7:* $A \to (B \to C)$
$\therefore (A \to B) \to (A \to C)$

which is straightforward provided we pay attention to the form of the conclusion. The conclusion is a conditional with '$A \to B$' as antecedent, so in our proof we should assume '$A \to B$' and try to derive '$A \to C$'. '$A \to C$' is another conditional, so to derive it we use the same method over again of assuming the antecedent and deriving the consequent. That is, we assume 'A' and try to derive 'C'. We then discharge our assumptions on a 'last in, first out' basis. This strategy yields the proof set out over the page. And more generally, in a logically organized proof where there are a number of assumptions made with a view to applying →I, any given application of →I will discharge the undischarged assumption most recently made.

| | | | |
|---|---|---|---|
| 1 | (1) | A → (B → C) | Premise |
| 2 | (2) | A → B | Assumption |
| 3 | (3) | A | Assumption |
| 2,3 | (4) | B | 2,3 →E |
| 1,3 | (5) | B → C | 1,3 →E |
| 1,2,3 | (6) | C | 4,5 →E |
| 1,2 | (7) | A → C | 3,6 →I |
| 1 | (8) | (A → B) → (A → C) | 2,7 →I ◆ |

Here are important points to remember about →I. In doing the exercises which follow, *the reader should make frequent reference to this list.*

- Use →I only when you wish to *derive* a conditional $\ulcorner p \rightarrow q \urcorner$ (if one of your premises or assumptions is a conditional, this is *not* a reason to assume its antecedent, since premises or assumptions are not things which you are trying to *derive*).
- To derive $\ulcorner p \rightarrow q \urcorner$ using →I, assume the antecedent $p$ and try to derive the consequent $q$. Do *not* assume anything other than $p$, and assume the *whole* of $p$. For example, if $p$ is a conjunction, do not assume just one of the conjuncts.
- When a conditional $\ulcorner p \rightarrow q \urcorner$ is derived by →I, the antecedent $p$ must always be a formula which you have assumed at a previous line; it cannot be a formula which you have derived from other premises and assumptions.
- When you apply →I, remember to discharge the assumption by dropping the assumption number on the left.
- Check that the last line of your proof does not depend on any extra assumptions you have made over and above your premises.

## ❑ Exercises

Give proofs of the following LSL argument-forms:

(1) A → B
B → C
∴ A → C

*(2) A → B
A → C
∴ A → (B & C)

(3) A
∴ B → (A & B)

(4) A → B
∴ (A & C) → (B & C)

*(5) (A & B) → C
∴ A → (B → C)

(6) A → (B → C)
∴ (A & B) → C

(7) (A & B) → C
A
∴ B → C

(8) A → B
∴ (C → A) → (C → B)

(9) A & (A → (A & B))
∴ B

(10) A → (A → B)
∴ A → B

(11) (A & B) → (C & D)
∴ [(A & B) → C] & [(A & B) → D]

(12) C → A
C → (A → B)
∴ C → (A & B)

(13) A → (B → C)
(A & D) → E
C → D
∴ (A & B) → E

(14) A → (B → C)
∴ B → (A → C)

(15) A → (B → C)
D → B
∴ A → (D → C)

*(16) A → B
∴ A → (C → B)

(17) A & (B & C)
∴ A → (B → C)

(18) B & C
∴ (A → B) & (A → C)

## 3 Sequents and theorems

When the conclusion of an LSL argument-form can be deduced from its premises in the system NK, the argument-form is said to be *NK-provable*, its conclusion is said to be a *deductive consequence in NK* of its premises, and its premises are said to *deductively entail* its conclusion *in NK* (we often omit the qualifier 'in NK'). At the end of §4 of Chapter 3 we introduced the notion of *semantic* consequence and the double-turnstile symbol '$\vDash$' which expresses it (page 66). The notion of *deductive* consequence is a companion concept, in the sense that it provides another way of explaining the idea of a conclusion 'following from' premises, and we introduce the *single turnstile* '$\vdash_{NK}$' to express it. When a conclusion $q$ has been deduced from premises $p_1,\ldots,p_n$ in the system NK, we write '$p_1,\ldots,p_n \vdash_{NK} q$', which is read '$p_1,\ldots,p_n$ deductively entail $q$ in NK' or '$q$ is a deductive consequence of $p_1,\ldots,p_n$ in NK'. An expression '$p_1,\ldots,p_n \vdash_{NK} q$' is often called a *syntactic sequent*, so in the previous section we established the following syntactic sequents:

(1) $A \,\&\, B, C \,\&\, D, (A \,\&\, D) \to H \vdash_{NK} H$
(2) $A \,\&\, (B \,\&\, C) \vdash_{NK} C \,\&\, (B \,\&\, A)$
(3) $A \to (B \to (C \to D)), C \,\&\, (A \,\&\, B) \vdash_{NK} D$
(4) $R \to (S \to T), S \vdash_{NK} R \to T$
(5) $((A \,\&\, B) \,\&\, C) \to D \vdash_{NK} C \to (B \to (A \to D))$
(6) $A \vdash_{NK} B \to A$
(7) $A \to (B \to C) \vdash_{NK} (A \to B) \to (A \to C)$

Note that any particular sequent itself asserts that its conclusion can be derived in NK from its premises: that is the effect of '$\vdash_{NK}$'. So, strictly speaking, syntactic sequents are true or false rather than provable or unprovable (*arguments* are provable or unprovable). But since showing a syntactic sequent to be true is done just by giving a proof, we will allow ourselves to speak of syntactic sequents as provable or unprovable. In general we omit the qualifiers 'syntactic' and 'semantic' where it is clear what kind of sequent is being discussed.

We make the meaning of '$\vdash_{NK}$' absolutely precise by the following definition, which should be compared with the definition of '$\vDash$' on page 66:

> $p_1,\ldots,p_n \vdash_{NK} q$ if and only if there is a numbered sequence of lines whose last line is $q$, each line in the sequence is either one of the premises $p_1,\ldots,p_n$, or an assumption (in both cases with appropriate numbering on the left), or is correctly inferred from previous lines by some rule of NK, and every number on the left of the last line is the number of a line where one of $p_1,\ldots,p_n$ appears as a premise.

Notice that since we do not require that the last line of a proof have on its left numbers of lines where *all* of $p_1,\ldots,p_n$ appear as premises, every proof is a proof of infinitely many sequents, since there are infinitely many different ways of augmenting the premises actually used in a proof with redundant premises.

In defining semantic consequence we singled out the special case in which $n = 0$, that is, the case where there are no $p_1,\ldots,p_n$. This left us with '$\vDash q$', which means that $q$ is a tautology. With deductive consequence we may also consider the case where $n = 0$. We read '$\vdash_{NK} q$' as: $q$ is a *theorem* of NK. Inspecting the definition of '$\vdash_{NK}$' and deleting material having to do with $p_1,\ldots,p_n$, we see that '$\vdash_{NK} q$' means that $q$ can be derived from no premises. In other words, $q$ can be derived just from assumptions, all of which are discharged so that the last line of the proof depends on nothing at all. Here is the simplest possible example—we show $\vdash_{NK} A \rightarrow A$:

*Example 1:* Show $\vdash_{NK} A \rightarrow A$.

| | | | |
|---|---|---|---|
| 1 | (1) | A | Assumption |
| | (2) | $A \rightarrow A$ | 1,1 →I ◆ |

Referring to our statement of the rule of →I on page 96, Example 1 illustrates the case where j = k. When the assumption 1 is discharged at line 2, nothing remains on the left.

It should be clear from the way in which →I works that for every provable sequent there is a corresponding theorem with embedded conditionals that have the various premises of the sequent as antecedents; in general, if we have constructed a proof of $p_1,\ldots,p_n \vdash_{NK} q$ then it is straightforward to give a proof of $\vdash_{NK} p_1 \rightarrow (p_2 \rightarrow \ldots (p_n \rightarrow q)\ldots)$, since we can append successive applications of →I to our proof of $p_1,\ldots,p_n \vdash_{NK} q$. For example, a solution to Exercise 2.1, $A \rightarrow B, B \rightarrow C \vdash_{NK} A \rightarrow C$, can be easily converted into a proof which establishes that $\vdash_{NK} (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ by making the premises assumptions and appending enough applications of →I. Here $n = 2$ and $p_1$ = '$A \rightarrow B$', $p_2$ = '$B \rightarrow C$' and $q$ = '$A \rightarrow C$' so

$$\vdash_{NK} p_1 \rightarrow (p_2 \rightarrow q)) \quad = \quad \vdash_{NK} (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C)).$$

*Example 2:* Show $\vdash_{NK} (A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$.

| | | | |
|---|---|---|---|
| 1 | (1) | $A \rightarrow B$ | Assumption |
| 2 | (2) | $B \rightarrow C$ | Assumption |
| 3 | (3) | A | Assumption |
| 1,3 | (4) | B | 1,3 →E |
| 1,2,3 | (5) | C | 2,4 →E |
| 1,2 | (6) | $A \rightarrow C$ | 3,5 →I |
| 1 | (7) | $(B \rightarrow C) \rightarrow (A \rightarrow C)$ | 2,6 →I |
| | (8) | $(A \rightarrow B) \rightarrow ((B \rightarrow C) \rightarrow (A \rightarrow C))$ | 1,7 →I ◆ |

The premises of Exercise 2.1 on page 98 are listed here as assumptions, since in the current problem they are antecedents of conditionals which we are trying to deduce. The assumptions we make are discharged in reverse order of listing by repeated uses of →I. Note again that discharging an assumption *always* creates a conditional with that assumption as antecedent.

## ❑ Exercises

Show the following:

(1) $\vdash_{NK} (A \,\&\, B) \rightarrow (B \,\&\, A)$
(2) $\vdash_{NK} (A \rightarrow B) \rightarrow (A \rightarrow B)$
(3) $\vdash_{NK} A \rightarrow (B \rightarrow A)$
*(4) $\vdash_{NK} [A \rightarrow (B \,\&\, C)] \rightarrow [(A \rightarrow B) \,\&\, (A \rightarrow C)]$
(5) $\vdash_{NK} [(A \rightarrow A) \rightarrow B] \rightarrow B$

## 4 Rules for negation

A very common strategy in ordinary reasoning and especially in mathematics is known as *reductio ad absurdum*. Say that two formulae are *explicitly contradictory* if and only if one is of the form $q$ and the other of the form $\ulcorner{\sim}q\urcorner$, that is, if one is the negation of the other. Then the principle upon which *reductio* depends is the following: if explicitly contradictory formulae follow from a collection of premises and assumptions, then one of the collection may be rejected. There are many famous illustrations of the strategy in mathematics, but here is a very simple one:

***Theorem:*** For any number $x > 1$, the least factor of $x$ other than 1 is a prime number.

*Proof:*

(a) Let $j$ be the least factor of $x$ other than 1.
(b) Suppose for *reductio* that $j$ is composite.
(c) Then $j = yz$, $1 < y \leqslant z < j$.
(d) So $x = jk = (yz)k = y(zk)$.
(e) Thus $y$ is a factor of $x$ less than $j$ and other than 1.
(f) (e) contradicts (a).
(g) The contradiction is generated by (b), so (b) should be rejected.
(h) Hence $j$ is not composite (which means it is prime).

The explicitly contradictory formulae at which we arrive in this argument are (a) and (e). The assumption we made that led to this contradiction is '$j$ is composite'. Of course, the proof also makes unacknowledged use of other principles of arithmetic as premises, for example, in deriving (d). But if these other principles are things we know, the 'blame' for the contradiction about $j$ rests on the assumption (b) that $j$ is composite. So we can reject this assumption and conclude that $j$ is not composite.

How might this strategy be implemented in NK? Our proof above has two stages. In the first stage we make an assumption, (b), and then reason so that we end up with explicitly contradictory statements, (a) and (e), as lines in the proof. In the second stage we infer from the fact that we have obtained explicitly contradictory statements that the assumption must be wrong and so we reject it. These two stages will correspond in NK to ~-Elimination and ~-Introduction respectively. The second stage involves ~-Introduction because *rejecting* an assumption is denying or negating it, so a 'not' is introduced. The first stage is labeled in NK as ~-Elimination for the following reason. When we infer the explicitly contradictory formulae $q$ and $\ulcorner{\sim}q\urcorner$, we know that the collection of premises and assumptions which we have used cannot all be correct, since they lead to something absurd, such as (e) and (a) in the example. Using the symbol '⋏' to stand for the idea of something absurd, we make this explicit in the proof by writing down '⋏' on a line by itself, putting on its left all the numbers of the premises and assumptions on which $q$ and $\ulcorner{\sim}q\urcorner$ depend; so the line 'says' that these premises and assumptions lead to an absurdity (this roughly corresponds to line f in the example). We label the line 'j,k ~E', where j and k are the lines where the mutually contradictory statements appear. The step is called ~-Elimination since line j contains a '~' as its main connective, while '⋏' contains no occurrence of '~'. After we have written down '⋏', we can go ahead and apply ~I, inferring the negation of one of the premises or assumptions in the proof. '⋏' is called the *absurdity symbol*.

Here is an illustration of these two rules:

*Example 1:* Show $A \rightarrow B, {\sim}B \vdash_{NK} {\sim}A$.

| | | | |
|---|---|---|---|
| 1 | (1) | $A \rightarrow B$ | Premise |
| 2 | (2) | ${\sim}B$ | Premise |
| 3 | (3) | $A$ | Assumption |
| 1,3 | (4) | $B$ | 1,3 →E |
| 1,2,3 | (5) | ⋏ | 2,4 ~E |
| 1,2 | (6) | ${\sim}A$ | 3,5 ~I ◆ |

This proof should be studied carefully. The conclusion is '~A', with main connective '~', hence we try to obtain it by ~I. This requires us to assume 'A' with the subgoal of deriving '⋏'. Deriving '⋏' requires deriving the two halves of an explicit contradiction, and looking at our premises and assumption, we see how to obtain two such formulae. Once we have obtained them at lines 2 and 4, we can 'eliminate' the negation symbol in line 2 by writing '⋏' at line 5. The line where we write '⋏' depends on whatever the explicitly contradictory lines depend on, so (5) depends on lines 1, 2 and 3. And now that '⋏' has been inferred, we can reject one of the premises or assumptions on which it depends. To reject a formula means writing it down *prefixed by a negation symbol*. Thus we arrive at (6); on the right we label our inference '~I', attaching the line numbers of the assumption being rejected and of the relevant occurrence of '⋏', while on the left we drop the line number of the assumption we are rejecting, in this case 3. The justification for dropping the line number is that the assumption is, after all, being *rejected:* in deriving absurdity, we assume (3)'s truth, but we do not assume its truth when rejecting it. Note that at (6) we can write down the negation of *any* of the premises and assumptions on which (5) depends; the one we choose is dictated by the formula we are trying to infer.

The negation rules are therefore as follows:

***Rule of ~-Elimination:*** For any formula $q$, if $\ulcorner {\sim}q \urcorner$ has been inferred at a line j in a proof and $q$ at a line k, $j < k$ or $k < j$, then we may infer '$\curlywedge$' at line m, labeling the line 'j,k ~E' and writing on its left the numbers on the left at j and on the left at k. Schematically (with $j < k$):

$$\begin{array}{rcll}
a_1,\ldots,a_n & (j) & {\sim}q & \\
 & \vdots & & \\
b_1,\ldots,b_u & (k) & q & \\
 & \vdots & & \\
a_1,\ldots,a_n,b_1,\ldots,b_u & (m) & \curlywedge & j,k\ {\sim}E
\end{array}$$

***Rule of ~-Introduction:*** If '$\curlywedge$' has been inferred at line k in a proof and $\{a_1,\ldots,a_n\}$ are the premise and assumption numbers '$\curlywedge$' depends upon, then if $p$ is an assumption (or premise) at line j, $\ulcorner {\sim}p \urcorner$ may be inferred at line m, labeling the line 'j,k ~I' and writing on its left the numbers in $\{a_1,\ldots,a_n\}/j$. We may have $j < k$ or $k < j$ or $j = k$. Schematically (with $j < k$):

$$\begin{array}{rcll}
j & (j) & p & \text{Assumption} \\
 & \vdots & & \text{(or: Premise)} \\
a_1,\ldots,a_n & (k) & \curlywedge & \\
 & \vdots & & \\
\{a_1,\ldots,a_n\}/j & (m) & {\sim}p & j,k\ {\sim}I
\end{array}$$

Recall that $\{a_1,\ldots,a_n\}/j$ is $\{a_1,\ldots,a_n\}$ with j removed if it is one of them (as with →I, j does not *have* to be in $\{a_1,\ldots,a_n\}$).

It must be admitted that the sense in which ~E *eliminates* a negation is rather notational. On some other approaches, rather than a ~-E rule, we would use &I to infer $\ulcorner q \mathbin{\&} {\sim}q \urcorner$ when we have $q$ and $\ulcorner {\sim}q \urcorner$ on separate lines, and then reject a premise or assumption using a rule of *reductio*. Here the explicit contradiction $\ulcorner q \mathbin{\&} {\sim}q \urcorner$ plays the role of '$\curlywedge$', and we do not need a new rule to obtain it from its conjuncts, since we already have &I. But there are certain technical and philosophical reasons to formulate the negation rules using '$\curlywedge$'. A technical consideration is that some sequents require roundabout proofs if explicit contradictions are used in place of '$\curlywedge$'. For instance, in the next section of this chapter, we give a ten-line proof of the sequent ${\sim}A \mathbin{\&} {\sim}B \vdash_{NK} {\sim}(A \vee B)$. The reader who, after mastering the ∨-rules, attempts to prove the sequent using explicit contradictions in place of '$\curlywedge$', will find that the inability to derive the *same* explicit contradiction from 'A' and 'B' without extra steps complicates the proof. A philosophical consideration arises from the perspective that rules of inference, not truth-tables, constitute the primary way of explaining the meanings of connectives. The version of ~I which uses explicit contradictions is unsatisfactory from this point of view, since negation occurs in the formula $\ulcorner q \mathbin{\&} {\sim}q \urcorner$ to which the introduction rule for negation is applied, and so we would be explaining negation in terms of itself. Of course, whether the formulation of ~I in terms of '$\curlywedge$' is an improvement depends on whether absurdity can be understood independently of understanding negation.

Since the combination of ~E and ~I always produces a formula whose main connective is '~', these rules are used to derive negative formulae, as in this example:

*Example 2:* Show ~B $\vdash_{\mathrm{NK}}$ ~(A & B).

| | | | |
|---|---|---|---|
| 1 | (1) | ~B | Premise |
| 2 | (2) | A & B | Assumption |
| 2 | (3) | B | 2 &E |
| 1,2 | (4) | ⋏ | 1,3 ~E |
| 1 | (5) | ~(A & B) | 2,4 ~I ◆ |

Since we wish to derive '~(A & B)' we expect to conclude the proof with an application of ~I, which yields the subgoal of proving '⋏'. To derive '⋏' we make an assumption which will lead to explicitly contradictory formulae, an assumption from whose rejection (negation) we can obtain '~(A & B)'. The simplest such assumption is of course 'A & B', whose rejection *is* '~(A & B)'.

~I is like →I in that it allows us to discharge assumptions. Here is an example in which both rules are used.

*Example 3:* Show ~(A & B) $\vdash_{\mathrm{NK}}$ A → ~B.

In order to derive 'A → ~B' we will use →I, which means we assume 'A' with the subgoal of proving '~B'. '~B' has '~' as main connective, so we can likely obtain it by ~I, which means we should assume 'B', as at line 3 below, with the subgoal of proving '⋏'. And with our two assumptions in place, it is easy to see how to generate explicitly contradictory formulae.

| | | | |
|---|---|---|---|
| 1 | (1) | ~(A & B) | Premise |
| 2 | (2) | A | Assumption |
| 3 | (3) | B | Assumption |
| 2,3 | (4) | A & B | 2,3 &I |
| 1,2,3 | (5) | ⋏ | 1,4 ~E |
| 1,2 | (6) | ~B | 3,5 ~I |
| 1 | (7) | A → ~B | 2,6 →I ◆ |

But suppose that we had been asked instead to prove the sequent ~(A & ~B) $\vdash_{\mathrm{NK}}$ A → B.

*Example 4:* Show ~(A & ~B) $\vdash_{\mathrm{NK}}$ A → B.

The first two lines are clear:

| | | | |
|---|---|---|---|
| 1 | (1) | ~(A & ~B) | Premise |
| 2 | (2) | A | Assumption |

The problem is now to deduce 'B' so that →I gives us the conclusion formula

'A → B'. But here we face a difficulty, for 'B' has no main connective, so our usual guide as to what I-rule should be used to infer the goal formula is not applicable. In such a case we turn to the premises, to see if there is any obvious way of applying E-rules to them. But this time there is not. So it is unclear how 'B' is to be obtained. However, there is a formula which is intuitively equivalent to 'B' and which does have a main connective, so that an I-rule is suggested by which to infer it. This formula is '~~B', the *double negation* of 'B'. If we could derive '~~B' and if we also have a rule which allows us to *cancel* a double negative '~~', we would thereby arrive at 'B' and could then finish with →I to get 'A → B'. As far as deriving '~~B' is concerned, its main connective is '~', so we would expect to get it by ~I. Hence we should assume '~B' and aim to deduce '⋏' by ~E. We get immediate confirmation that this strategy is on the right lines by looking at the premise and assumption already listed. The premise has '~' as main connective and so its role will presumably be to figure in an application of ~E. Such an application can be made if we can derive 'A & ~B', which is simple if we are going to assume '~B'. Let us use 'Double Negation', 'DN' for short, for the rule which allows canceling of '~~'. Then the proof is:

| | | | |
|---|---|---|---|
| 1 | (1) | ~(A & ~B) | Premise |
| 2 | (2) | A | Assumption |
| 3 | (3) | ~B | Assumption |
| 2,3 | (4) | A & ~B | 2,3 &I |
| 1,2,3 | (5) | ⋏ | 1,4 ~E |
| 1,2 | (6) | ~~B | 3,5 ~I |
| 1,2 | (7) | B | 6 DN |
| 1 | (8) | A → B | 2,7 →I ◆ |

We state the new rule as follows:

> ***Rule of Double Negation:*** For any formula *p*, if ⌜~~*p*⌝ has been inferred at a line j in a proof, then at line k we may infer *p*, labeling the line 'j DN' and writing on its left the same numbers as on the left of line j. Schematically:

| | | | |
|---|---|---|---|
| $a_1,\ldots,a_n$ | (j) | ~~*p* | |
| | ⋮ | | |
| $a_1,\ldots,a_n$ | (k) | *p* | j DN |

If there is a rule that allows canceling '~~', then, since this looks rather like an elimination rule, it may seem that there ought to be a companion rule allowing addition of two '~~'s. But DN has a technical character that is different from elimination rules, and we will see later that, anyway, a rule for adding '~~' would be redundant. However, the rule of DN is certainly not redundant. Without it, there is no way of proving the previous sequent just using the other rules we have currently introduced. In fact, the rule of DN, or some rule or group of rules which has the same effect, is in a sense *characteristic* of classical logic, in that certain nonclassical alternatives to classical logic differ from classical logic precisely in rejecting this rule and its equivalents. For example, the main nonclassical alternative is *intuitionistic logic*, in which there is no way in general of canceling a double negation. Intuitionistic logic is described in some detail in Chapter 10; see also Van Dalen.

One aspect of the rule of ~I is worth elaborating upon. It may seem strange that we can appeal to '⅄' to reject an assumption which is not used in deriving '⅄' (referring to the schema for ~I on page 104, this is the case where j is not in $\{a_1,\ldots,a_n\}$). However, the following proof is technically correct:

*Example 5:* Show $B, {\sim}B \vdash_{NK} A$.

| | | | |
|---|---|---|---|
| 1 | (1) | B | Premise |
| 2 | (2) | ~B | Premise |
| 3 | (3) | ~A | Assumption |
| 1,2 | (4) | ⅄ | 1,2 ~E |
| 1,2 | (5) | ~~A | 3,4 ~I |
| 1,2 | (6) | A | 5 DN ◆ |

What this means is that from contradictory premises, any formula at all may be deduced; for though we derive 'A' in this example, the same sequence of steps would allow us to derive any other wff $p$ of LSL, beginning with the assumption $\ulcorner {\sim}p \urcorner$ at line 3. Intuitively, this corresponds to the thought that, if 'B' and '~B' were *both* true, then *anything at all* would be true.[1]

A final comment concerns the new symbol '⅄' which has appeared in this section. What kind of an item is it? Syntactically, it behaves like an object language sentence-letter, since other connectives attach to it to form compound formulae like '~~⅄' and 'A → ⅄'. Semantically, however, it is more like a truth-functional connective. It is governed by the stipulation that every interpretation assigns ⊥ to '⅄', which means that we can think of it as expressing a *constant zero-place* truth-function, the one which takes an empty input and outputs ⊥. So in the future we will include it among sentence-letters when describing the syntax of a language, but give it its own evaluation clause, like a connective, when describing a language's semantics.

Here are some useful points to bear in mind when using the negation rules:

- If you are trying to derive a formula with '~' as its main connective, use ~I to obtain it. This means that you should assume the formula within the scope of the main '~' and try to derive '⅄' by ~E.
- When you apply ~I, the formula which you infer must be the negation of a premise or assumption. It cannot be the negation of a formula which has been inferred from other lines by an I or E rule for a connective.

---

[1] The sequent $B, {\sim}B \vdash_{NK} A$ is sometimes embodied in classical logic as a rule of inference, from contradictory lines (equivalently, from '⅄') to infer any formula, pooling premises and assumptions. In systems with DN, this rule, known as *ex falso quodlibet*, is redundant, but there are other ways of formulating classical logic in which it is required. See further §10 of this chapter.

- If one of your premises or assumptions has ‘~’ as its main connective, it is likely that its role in the proof will be to be one of a pair of contradictory formulae in an application of ~E. You should therefore consider trying to derive the formula within the scope of the ‘~’ to get the other member of the contradictory pair.
- If you are trying to deduce a sentence-letter and there is no obvious way to do it, consider trying to derive its double negation and then use DN. Any double negation has ‘~’ as its main connective. Now refer to the first point in this list.
- At this point in the development of NK, *you should only assume a formula p if you are trying to deduce its negation or trying to deduce a conditional with p as antecedent.* A common beginner’s error is to make assumptions just for the sake of being able to apply an E-rule (particularly →E). At this stage, *only* make an assumption when you have reasoned how you will use ~I or →I to discharge it.

## ❑ Exercises

Show the following (‘$p \dashv\vdash_{NK} q$’ abbreviates ‘$p \vdash_{NK} q$ and $q \vdash_{NK} p$’):

*(1) $A \rightarrow \sim B \vdash_{NK} B \rightarrow \sim A$
(2) $\vdash_{NK} \sim(A \,\&\, \sim A)$
(3) $A \vdash_{NK} \sim\sim A$
(4) $\sim\sim A \rightarrow B, \sim B \vdash_{NK} \sim\sim\sim A$
(5) $A \rightarrow B \vdash_{NK} \sim(A \,\&\, \sim B)$
(6) $\sim(A \,\&\, B), A \vdash_{NK} \sim B$
(7) $A \vdash_{NK} \sim(B \,\&\, \sim(A \,\&\, B))$
*(8) $\vdash_{NK} (A \,\&\, \sim A) \rightarrow B$
(9) $B \rightarrow (A \,\&\, \sim A) \vdash_{NK} \sim B$
(10) $A \rightarrow B, A \rightarrow \sim B \vdash_{NK} \sim A$
(11) $(A \,\&\, B) \rightarrow \sim A \vdash_{NK} A \rightarrow \sim B$
(12) $(A \,\&\, \sim B) \rightarrow B \vdash_{NK} A \rightarrow B$
*(13) $A \rightarrow B, B \rightarrow \sim A \vdash_{NK} \sim A$
(14) $A \rightarrow (B \rightarrow C) \vdash_{NK} \sim C \rightarrow \sim(A \,\&\, B)$
(15) $A \rightarrow \sim(B \,\&\, C), B \rightarrow C \vdash_{NK} A \rightarrow \sim B$
(16) $A, A \rightarrow C, \sim B \rightarrow \sim C \vdash_{NK} \sim(A \rightarrow \sim B)$
(17) $\vdash_{NK} (\sim A \rightarrow A) \rightarrow A$
*(18) $(A \,\&\, \sim B) \rightarrow \sim A \vdash_{NK} A \rightarrow B$
(19) $\vdash_{NK} \sim\curlywedge$
(20) $\sim A \dashv\vdash_{NK} A \rightarrow \curlywedge$ (do *both* directions)
(21) $\sim A \vdash_{NK} A \rightarrow B$
(22) $\sim(A \rightarrow B) \dashv\vdash_{NK} A \,\&\, \sim B$ (do *both* directions)
(23) $\sim(A \,\&\, B), \sim(A \,\&\, \sim B) \vdash_{NK} \sim A$
(24) $\sim(A \rightarrow \sim A), \sim(B \rightarrow \sim B), \sim(A \,\&\, B) \vdash_{NK} \curlywedge$
(25) $A \rightarrow \sim(B \rightarrow C) \vdash_{NK} (A \rightarrow B) \,\&\, (A \rightarrow \sim C)$
(26) $\vdash_{NK} \sim\{[\sim(A \,\&\, B) \,\&\, \sim(A \,\&\, \sim B)] \,\&\, [\sim(\sim A \,\&\, B) \,\&\, \sim(\sim A \,\&\, \sim B)]\}$

## 5 Rules for disjunction

The remaining connectives for which we require rules are disjunction and the biconditional. The introduction rule for '$\vee$' is extremely simple: if a formula $p$ occurs on a line then we may infer the disjunction $\ulcorner p \vee q \urcorner$ or the disjunction $\ulcorner q \vee p \urcorner$ at a new line, depending on the same premises and assumptions. For example, if we have established that water is $H_2O$, then we can infer the weaker statement that either water is $H_2O$ or water is $H_4O_2$ (this would be useful if we knew of something else that followed given that water is one or the other). Indeed, the new disjunct need have no relation to the one which has been established: if we have established that water is $H_2O$, then we can infer that either water is $H_2O$ or Moses was the author of the Pentateuch—though it is less clear how *this* could be useful!

Here is a simple example of $\vee$I:

*Example 1:* Show (A $\vee$ (D & E)) → B $\vdash_{NK}$ A → B.

Our strategy is to use →I to derive 'A → B', so we assume 'A' and pursue the subgoal of deducing 'B'. 'B' is the consequent of (1), so we can obtain it by →E if we can obtain the antecedent of (1), which we accomplish at (3)

| | | | |
|---|---|---|---|
| 1 | (1) | (A ∨ (D & E)) → B | Premise |
| 2 | (2) | A | Assumption |
| 2 | (3) | A ∨ (D & E) | 2 ∨I |
| 1,2 | (4) | B | 1,3 →E |
| 1 | (5) | A → B | 2,4 →I ◆ |

The intuitive justification for line 3 is that we already have 'A' at (2), and if 'A' holds according to the premises and assumptions on which line 2 depends, then by the truth-table for '$\vee$', *any* disjunction with 'A' as a disjunct will hold according to those premises and assumptions. We state the rule as follows:

> ***Rule of $\vee$-Introduction:*** For any formula $p$, if $p$ has been inferred at line j, then for any formula $q$, either $\ulcorner p \vee q \urcorner$ or $\ulcorner q \vee p \urcorner$ may be inferred at line k, labeling the line 'j, $\vee$I' and writing on its left the same premise and assumption numbers as on the left of j. Schematically:
>
> | | | | | | | | | |
> |---|---|---|---|---|---|---|---|---|
> | $a_1,\ldots,a_n$ | (j) | $p$ | | | $a_1,\ldots,a_n$ | (j) | $p$ | |
> | | ⋮ | | | OR | | ⋮ | | |
> | $a_1,\ldots,a_n$ | (k) | $p \vee q$ | j $\vee$I | | $a_1,\ldots,a_n$ | (k) | $q \vee p$ | j $\vee$I |

The rule of $\vee$-Elimination is somewhat more complex. What can we infer from a disjunction $\ulcorner p \vee q \urcorner$? Certainly not $p$ by itself, nor $q$ by itself. To see what is possible, let $\ulcorner p \vee q \urcorner$ be the sentence 'either water is $H_2O$ or water is $H_4O_2$', and suppose that we are in the early days of chemistry and do not know which disjunct is true, though (somehow) we know the disjunction. One thing which follows from the disjunction is that water contains hydrogen, and the reason it follows is that it follows from *both* disjuncts, so that it does not matter which one of the two is true: assuming that it is the first disjunct which is true, then water contains hydrogen, and assuming that it is the second which is true, then water still contains hydrogen. We could set this out semi-formally as follows:

| | | | |
|---|---|---|---|
| 1 | (1) | Water is $H_2O$ or water is $H_4O_2$ | Premise |
| 2 | (2) | Water is $H_2O$ | Assumption |
| 2 | (3) | Water contains hydrogen | From (2) |
| 4 | (4) | Water is $H_4O_2$ | Assumption |
| 4 | (5) | Water contains hydrogen | From (4) |
| 1 | (6) | Water contains hydrogen | From (1) |

Lines (2)–(5) are numbered on the left as would be expected, but at line 6 we revert to whatever (1) depends on. The reason is that once we have shown that 'water contains hydrogen' follows from each disjunct separately, we have shown that it follows from the disjunction itself, so we repeat the statement, as at (6), asserting that it depends only on whatever the disjunction depends upon. Therefore we discharge the assumptions of the individual disjuncts: the statement 'water contains hydrogen' at line 6 no longer depends on assuming the truth of (2) or (4).

On the other hand, if all we know is 'either water is $H_2O$ or water is some other substance with oxygen in its chemical composition', then we cannot infer 'water contains hydrogen' since we cannot infer it from the second disjunct, 'water is some other substance with oxygen in its chemical composition'. This example is paradigmatic of when something cannot be deduced from a disjunction, so along with the previous example, we can make a first approximation to a rule of $\vee$-Elimination, that a statement $r$ follows from a disjunction $\ulcorner p \vee q \urcorner$ if and only if $r$ follows from $p$ and $r$ follows from $q$. Consequently, if we are trying to derive a statement $r$ from a disjunction $\ulcorner p \vee q \urcorner$ ($r$ is called the *target* formula) we have to show that $r$ follows from $p$ and that it follows from $q$. We do this by assuming $p$ and deriving $r$ and then by assuming $q$ and deriving $r$; this done, we can say that $r$ follows from $\ulcorner p \vee q \urcorner$. At the end of the process, $r$ depends on whatever $\ulcorner p \vee q \urcorner$ depends on. In particular, though we assumed $p$ in order to show that $r$ follows from it, and likewise assumed $q$, at the end of the process where we claim to have shown that $r$ follows from the disjunction in virtue of following from each disjunct separately, $r$ does not depend on either of the assumptions $p$ and $q$, but only on the disjunction (or whatever it depends upon) itself.

However, our example of a successful inference from a disjunction is rather special, in that no other information is needed to infer $r$ ('water contains hydrogen') from each disjunct ('water is $H_2O$' and 'water is $H_4O_2$'), above and beyond the information in that disjunct. A different example might involve an $r$ which follows from each disjunct only with the aid of further premises from chemistry, for example, $r$ might be 'water contains the simplest of all the elements'. In that case, deducing $r$ would require the extra premise 'hydrogen is the simplest of all elements' so $r$ would depend both on the disjunction and on this premise. Thus the correct account of what $r$ depends on is that when $r$ is shown to follow from $p$ and to follow from $q$, this means that it follows from $\ulcorner p \vee q \urcorner$ and depends on whatever premises and assumptions $\ulcorner p \vee q \urcorner$ depends upon, but also on whatever other premises and assumptions are used in obtaining $r$ from $p$, except $p$ itself, and in obtaining $r$ from $q$, except $q$ itself.

Here is a simple proof which illustrates reasoning from a disjunction:

*Example 2:* Show (A & B) ∨ (A & C) $\vdash_{\text{NK}}$ A.

To derive the target formula 'A' from the disjunction '(A & B) ∨ (A & C)', we show that 'A' follows from each disjunct separately. This involves assuming the first disjunct, deriving 'A' from it, and then assuming the second disjunct and deriving 'A' from it. Once we have done that we repeat 'A' and adjust premise and assumption numbers to indicate that 'A' has been shown to follow from the disjunction itself; it is this step that is labeled ∨-Elimination, '∨E' for short.

| | | | | |
|---|---|---|---|---|
| 1 | (1) | (A & B) ∨ (A & C) | Premise | |
| 2 | (2) | A & B | Assumption | |
| 2 | (3) | A | 2 &E | |
| 4 | (4) | A & C | Assumption | |
| 4 | (5) | A | 4 &E | |
| 1 | (6) | A | 1,2,3,4,5 ∨E | ◆ |

This proof exhibits the five-number format for labeling lines inferred by ∨E. The first number is the number of the disjunction itself, 1; the second is the number of the line, 2, where the first disjunct is assumed; the third is the number of the line, 3, where the target formula is inferred from the first disjunct (perhaps with the aid of other premises and assumptions, though not in this example); the fourth number is the number of the line, 4, where the second disjunct is assumed; and the fifth number is the number of the line, 5, where the target formula is derived from the second disjunct (perhaps with the aid of other premises and assumptions, though not in this example). The line labeled by '∨E' is the line where we assert the target formula a third time, altering premise and assumption numbers to indicate that we are now claiming that the target formula follows from the disjunction itself. The rule for writing numbers on the left of a line labeled '∨E' is this: first, write in the numbers from the left of the line where the disjunction itself occurs (in our example, this means we write the number 1 on the left of (6)); second, write in all numbers from the left of the line where the target formula is derived from the first disjunct of the disjunction, but *excluding* the number of the line where that disjunct is assumed (in our example, this step adds no numbers on the left of (6)); third, write in all numbers from the left of the line where the target formula is derived from the second disjunct of the disjunction, but *excluding* the number of the line where that disjunct is assumed (in our example, this step also adds no numbers on the left of (6)). Following this procedure in our example means that only the number 1 appears on the left of (6).

Generalizing this example, we may state the rule of ∨E as follows:

***Rule of*** **∨*-Elimination:*** If a disjunction $\ulcorner p \lor q \urcorner$ occurs at a line g, *p* is assumed at line h, *r* is derived at line i, *q* is assumed at line j and *r* is derived at line k, then at line m we may infer *r*, labeling the line 'g,h,i,j,k ∨E' and writing on its left every number on the left at line g, and at line i (except h), and at line k (except j). Schematically:

| | | | |
|---|---|---|---|
| $a_1,\ldots,a_n$ | (g) | $p \lor q$ | |
| | ⋮ | | |
| h | (h) | $p$ | Assumption |
| | ⋮ | | |
| $b_1,\ldots,b_u$ | (i) | $r$ | |
| | ⋮ | | |
| j | (j) | $q$ | Assumption |
| | ⋮ | | |
| $c_1,\ldots,c_w$ | (k) | $r$ | |
| | ⋮ | | |
| X | (m) | $r$ | g,h,i,j,k ∨E |

where X is the set $\{a_1,\ldots,a_n\} \cup \{b_1,\ldots,b_u\}/h \cup \{c_1,\ldots,c_w\}/j$.

In this statement of the rule, the symbol '∪' stands for set-theoretic *union.* Y ∪ Z is the set consisting in all the members of Y and all the members of Z, so $\{a_1,\ldots,a_n\} \cup \{b_1,\ldots,b_u\}/h \cup \{c_1,\ldots,c_w\}/j$ is the set of all numbers on the left at line g, together with all numbers on the left at line i except h if it is one of them, together with all numbers on the left at line k except j if it is one of them. h need not be one of $b_1,\ldots,b_u$ nor j one of $c_1,\ldots,c_w$; this occurs when the assumed disjunct is not used in deriving the target formula.

The rule allows h = i or j = k, which occurs when the target formula is itself one of the disjuncts. A case where h = i is illustrated by the shortest proof of the sequent A ∨ B, ~B $\vdash_{NK}$ A, which embodies a familiar way in which disjunction and negation interact in inference—if a disjunction is true and one of its disjuncts is false, it must be the other disjunct which is true (recall Example K, page 8).

*Example 3:* Show A ∨ B, ~B $\vdash_{NK}$ A.

| | | | |
|---|---|---|---|
| 1 | (1) | A ∨ B | Premise |
| 2 | (2) | ~B | Premise |
| 3 | (3) | A | Assumption |
| 4 | (4) | B | Assumption |
| 5 | (5) | ~A | Assumption |
| 2,4 | (6) | ⋏ | 2,4 ~E |
| 2,4 | (7) | ~~A | 5,6 ~I |
| 2,4 | (8) | A | 7 DN |
| 1,2 | (9) | A | 1,3,3,4,8 ∨E ◆ |

The proof works by using ∨E on line 1, taking ‘A’ as target formula. That is, the idea is to infer ‘A’ from the first disjunct of (1) and then again from the second. Since ‘A’ *is* the first disjunct of (1), that part of the process goes quickly. Inferring ‘A’ from the second disjunct ‘B’ requires that we use the other premise of the sequent, ‘~B’. The reasoning in this part of the ∨E is the same as that in Example 4.5 (page 107).

Here is a slightly more complex example, which additionally illustrates a common exception to our rule of thumb that the last line of a proof will be inferred by the introduction rule for the main connective of the conclusion.

*Example 4:* Show $A \,\&\, (B \vee C) \vdash_{NK} (A \,\&\, B) \vee (A \,\&\, C)$.

A first thought about strategy might be that we should try to derive one or other of the conclusion-disjuncts ‘A & B’ or ‘A & C’ from the premise, and then use ∨I to obtain the target formula ‘(A & B) ∨ (A& C)’. If this is to work, then the first line of the proof is

1 (1) A & (B ∨ C) Premise

and the second-last line of the proof has to be either (*) or (**) below:

1 (n–1) A & B (*)

1 (n–1) A & C (**)

if ∨I is to produce ‘(A & B) ∨ (A& C)’ on the last line, depending just on the premise of the sequent. But this will not work. The reason is that our rules of inference are deliberately chosen to ensure that *we cannot use them to prove a sequent unless the conclusion of the sequent is a semantic consequence of the premises of the sequent*. In our example, therefore, the suggested ∨I strategy will work only if either ‘A & B’ or ‘A & C’ is a semantic consequence of ‘A & (B ∨ C)’, that is, only if (†) or (††) holds:

$A \,\&\, (B \vee C) \vDash (A \,\&\, B)$ (†)

$A \,\&\, (B \vee C) \vDash (A \,\&\, C)$ (††)

But by considering the assignment of ⊤ to ‘A’ and ‘C’, ⊥ to ‘B’, we see that $A \,\&\, (B \vee C) \nvDash (A \,\&\, B)$, and by considering the assignment ⊤ to ‘A’ and ‘B’, ⊥ to ‘C’, we see that $A \,\&\, (B \vee C) \nvDash (A \,\&\, C)$. Consequently, in a correct proof of Example 4, the last line will not be obtained by ∨I.

However, one of the conjuncts of the premise is a disjunction, ‘B ∨ C’, on which we can use ∨E, and ∨I will often give us the target formula *within* an application of ∨E. That is, we should try to derive the target formula from ‘B’, and then from ‘C’, using the first conjunct ‘A’ of the premise where necessary, and this will show that the target formula follows from ‘B ∨ C’, together with ‘A’. Here is the proof:

| | | | |
|---|---|---|---|
| 1 | (1) | A & (B ∨ C) | Premise |
| 1 | (2) | A | 1 &E |
| 1 | (3) | B ∨ C | 1 &E |
| 4 | (4) | B | Assumption |
| 1,4 | (5) | A & B | 2,4 &I |
| 1,4 | (6) | (A & B) ∨ (A & C) | 5 ∨I |
| 7 | (7) | C | Assumption |
| 1,7 | (8) | A & C | 2,7 &I |
| 1,7 | (9) | (A & B) ∨ (A & C) | 8 ∨I |
| 1 | (10) | (A & B) ∨ (A & C) | 3,4,6,7,9 ∨E ◆ |

To repeat the idea: we are trying to derive '(A & B) ∨ (A & C)' and we have lines 2 and 3 to work with. Since (3) is a disjunction, ∨E is called for. This means that we need to show that '(A & B) ∨ (A & C)' can be inferred from the first disjunct of (3), and also from the second, in each case using any other lines which are helpful. So at line 4 we assume the first disjunct of (3) and at line 6 we show that '(A & B) ∨ (A & C)' follows from it, with the help of (2). At line 7 we assume the second disjunct and show at line 9 that '(A & B) ∨ (A & C)' follows from it, also with the help of (2). At line 10, therefore, we may assert that '(A & B) ∨ (A & C)' follows from the disjunction at (3), with the help of (2), labeling (10) with the five relevant numbers, as given by the schema for ∨E (page 112). We then adjust assumption numbers, using the rule stated above: on the left of (10) we write the numbers the disjunction depends upon, the numbers on the left of (6) except for 4, and the numbers on the left of (9) except for 7.

With this rule we have assembled all the rules for the first four of our connectives (we have still to discuss '↔'). We are now in a position to construct some rather more complex proofs:

*Example 5:* Show ~A & ~B $\vdash_{\mathrm{NK}}$ ~(A ∨ B).

Since the conclusion is negative, we will try to obtain it by using ~I, which means that we should assume 'A ∨ B' with a view to proving '⋏' by ~E, since we can then derive '~(A ∨ B)' by ~I. However, since 'A ∨ B' is a disjunction, we need to use ∨E to obtain '⋏' from it, which means we need to obtain '⋏' from each disjunct separately, with the assistance of the premise, '~A & ~B', in whatever way it may be helpful. Here is the proof:

| | | | |
|---|---|---|---|
| 1 | (1) | ~A & ~B | Premise |
| 2 | (2) | A ∨ B | Assumption |
| 3 | (3) | A | Assumption |
| 1 | (4) | ~A | 1 &E |
| 1,3 | (5) | ⋏ | 3,4 ~E |
| 6 | (6) | B | Assumption |
| 1 | (7) | ~B | 1 &E |
| 1,6 | (8) | ⋏ | 6,7 ~E |
| 1,2 | (9) | ⋏ | 2,3,5,6,8 ∨E |
| 1 | (10) | ~(A ∨ B) | 2,9 ~I ◆ |

Our next example looks trivial but actually involves something quite tricky, an application of ∨E within another application of ∨E.

*Example 6:* Show $A \vee (B \vee C) \vdash_{NK} (A \vee B) \vee C$.

Like Example 4, neither disjunct of the conclusion is by itself a semantic consequence of the premise (why not?), so the last line of the proof cannot be obtained straightforwardly by ∨I from one or the other disjunct. Instead, we have to use ∨I within an application of ∨E. So our subgoals are to obtain ‘$(A \vee B) \vee C$’ from each disjunct of the premise by itself. The extra complication is that the second disjunct of the premise *is another disjunction*, so ∨E will have to be used on it as well. Here is the proof of Example 6:

| | | | |
|---|---|---|---|
| 1 | (1) | $A \vee (B \vee C)$ | Premise |
| 2 | (2) | $A$ | Assumption |
| 2 | (3) | $A \vee B$ | 2 ∨I |
| 2 | (4) | $(A \vee B) \vee C$ | 3 ∨I |
| 5 | (5) | $B \vee C$ | Assumption |
| 6 | (6) | $B$ | Assumption |
| 6 | (7) | $A \vee B$ | 6 ∨I |
| 6 | (8) | $(A \vee B) \vee C$ | 7 ∨I |
| 9 | (9) | $C$ | Assumption |
| 9 | (10) | $(A \vee B) \vee C$ | 9 ∨I |
| 5 | (11) | $(A \vee B) \vee C$ | 5,6,8,9,10 ∨E |
| 1 | (12) | $(A \vee B) \vee C$ | 1,2,4,5,11 ∨E ◆ |

At line 4 we obtain the target formula from the first disjunct of (1) (which is assumed at line 2). At line 5 we assume the second disjunct of (1) and obtain the target formula from it at line 11, but since (5) is a disjunction, this second part of the main application of ∨E involves a subsidiary ∨E using the disjuncts of (5): the first disjunct is assumed at (6), the target formula derived at (8), the second is assumed at (9) and the target formula derived at (10). So line (11) is simultaneously the conclusion of the subsidiary ∨E and the completion of the second part of the main ∨E.

Finally, we prove another theorem:

*Example 7:* Show $\vdash_{NK} A \vee {\sim}A$.

This theorem is sometimes called the Law of Excluded Middle, and its theoremhood is a reflection of the truth-table for negation and the Principle of Bivalence (‘every sentence is either true or false’). Since it is a theorem, we have to begin its proof with an assumption, and it seems that our only option is to assume ‘${\sim}(A \vee {\sim}A)$’ with a view to deriving ‘$\curlywedge$’ and then using ~I and DN. But ‘${\sim}(A \vee {\sim}A)$’ is a negative formula, so heeding the third point of the list at the end of §4, it is likely to be one of a pair of contradictory formulae for an application of ~E. We should therefore attempt to infer ‘$A \vee {\sim}A$’. To do this we will have to make another assumption, one which we can also discharge in some way, and one which enables us to derive '$A \vee \sim A$'. The assumption '$A$' fits the description. Here is the proof:

| | | | |
|---|---|---|---|
| 1 | (1) | $\sim(A \vee \sim A)$ | Assumption |
| 2 | (2) | $A$ | Assumption |
| 2 | (3) | $A \vee \sim A$ | 2 ∨I |
| 1,2 | (4) | $\curlywedge$ | 1,3 ~E |
| 1 | (5) | $\sim A$ | 2,4 ~I |
| 1 | (6) | $A \vee \sim A$ | 5 ∨I |
| 1 | (7) | $\curlywedge$ | 1,6 ~E |
| | (8) | $\sim\sim(A \vee \sim A)$ | 1,7 ~I |
| | (9) | $A \vee \sim A$ | 8 DN ◆ |

Notice that it would be wrong to stop at line 6, since (6) depends on (1), but our goal is to derive '$A \vee \sim A$' depending on *no* premises or assumptions. Also, the use of DN at (9) cannot be avoided. Just as the rule of DN is characteristic of classical logic, so is the theoremhood of '$A \vee \sim A$': in the alternatives to classical logic which reject DN, such as intuitionistic logic, the Law of Excluded Middle is not a theorem.

## ❑ Exercises

Show the following ('$p \dashv\vdash_{NK} q$' abbreviates '$p \vdash_{NK} q$ and $q \vdash_{NK} p$'):

(1) $(A \vee B) \rightarrow C \vdash_{NK} A \rightarrow C$
(2) $A \rightarrow B, (B \vee C) \rightarrow D, D \rightarrow \sim A \vdash_{NK} \sim A$
(3) $A \vee B \vdash_{NK} B \vee A$
*(4) $A \vee \sim\sim B \vdash_{NK} A \vee B$
(5) $A \vee A \vdash_{NK} A$
(6) $A \rightarrow (B \vee C), \sim B \;\&\; \sim C \vdash_{NK} \sim A$
(7) $A \vee \curlywedge \vdash_{NK} A$
(8) $A \vee B, A \rightarrow B, B \rightarrow A \vdash_{NK} A \;\&\; B$
(9) $(A \;\&\; B) \vee (A \;\&\; C) \vdash_{NK} A \;\&\; (B \vee C)$
*(10) $A \vee B \vdash_{NK} (A \rightarrow B) \rightarrow B$
(11) $(A \rightarrow \curlywedge) \vee (B \rightarrow \curlywedge), B \vdash_{NK} \sim A$
(12) $\sim(A \rightarrow \sim A), \sim(A \vee B) \vdash_{NK} \curlywedge$
*(13) $A \vee B \vdash_{NK} \sim(\sim A \;\&\; \sim B)$
(14) $A \;\&\; B \dashv\vdash_{NK} \sim(\sim A \vee \sim B)$
(15) $\sim(\sim A \;\&\; \sim B) \vdash_{NK} A \vee B$
(16) $\sim A \vee \sim B \dashv\vdash_{NK} \sim(A \;\&\; B)$
(17) $A \rightarrow B \dashv\vdash_{NK} \sim A \vee B$
(18) $\sim(A \rightarrow B) \dashv\vdash_{NK} \sim(\sim A \vee B)$
(19) $(A \vee B) \vee C \vdash_{NK} A \vee (B \vee C)$
(20) $A \vee (B \;\&\; C) \dashv\vdash_{NK} (A \vee B) \;\&\; (A \vee C)$
(21) $\sim(A \vee (B \;\&\; C)) \vdash_{NK} \sim(A \vee B) \vee \sim(A \vee C)$
(22) $A \rightarrow (\sim B \vee \sim C), \sim C \rightarrow \sim D, B \vdash_{NK} \sim A \vee \sim D$

## 6 The biconditional

The remaining connective for which we have to give rules is ‘↔’. We could provide the biconditional with analogues of →I and →E. ↔E would be: given a biconditional and one of its sides, infer the other, pooling assumptions; and ↔I would be: if $q$ can be inferred from auxiliary premises and assumptions X and $p$ as assumption, and if $p$ can be inferred from auxiliary premises and assumptions Y and $q$ as assumption, then $\ulcorner p \leftrightarrow q \urcorner$ can be inferred from auxiliary premises and assumptions $X \cup Y$. However, when we provided ‘↔’ with a truth-table in §1 of Chapter 3, we derived the table via the table for a conjunction of conditionals. Although in the syntax of LSL we took the shortcut of treating ‘↔’ as if it were a *bona fide* binary connective in its own right, we shall otherwise work out its properties by relating it to the conditional form it abbreviates. So in NK, instead of the I and E rules described above, we shall employ a *rule of definition* which allows us to replace any LSL conjunction of conditionals with its notational abbreviation using ‘↔’ (which we think of as an abbreviation in LSL) and to expand any abbreviation into the formula it abbreviates, in both cases carrying over the same premises and assumptions. This amounts to the following rule:

> ***Rule of Definition for ‘↔’:*** If $\ulcorner (p \rightarrow q) \,\&\, (q \rightarrow p) \urcorner$ occurs as the entire formula at a line j, then at line k we may write $\ulcorner p \leftrightarrow q \urcorner$, labeling the line ‘j, Df’ and writing on its left the same numbers as are on the left of j. Conversely, if $\ulcorner p \leftrightarrow q \urcorner$ occurs as the entire formula at a line j, then at line k we may write $\ulcorner (p \rightarrow q) \,\&\, (q \rightarrow p) \urcorner$, labeling the line ‘j, Df’ and writing on its left the same numbers as are on the left of j.

Consequently, to infer a biconditional we would aim to deduce the corresponding conjunction of conditionals, normally using &I. This requires each conditional to be obtained by itself, so we would expect in the typical case to use →I twice: assume the antecedent of the first conditional, derive its consequent, apply →I, then repeat the process for the second conditional. Notice that the rule Df requires that the occurrence of ‘↔’ being expanded be the main connective of the formula on line j, and that the ‘&’ of a conjunction being abbreviated be the main connective. So expanding ‘$A \,\&\, (B \leftrightarrow C)$’ into ‘$A \,\&\, ((B \rightarrow C) \,\&\, (C \rightarrow B))$’ would *not* be a legitimate application of Df—one must use &E first. Here is a simple example of a proof employing Df:

*Example 1:* Show $\vdash_{NK} A \leftrightarrow A$.

| | | | |
|---|---|---|---|
| 1 | (1) | $A$ | Assumption |
| | (2) | $A \rightarrow A$ | 1,1 →I |
| | (3) | $(A \rightarrow A) \,\&\, (A \rightarrow A)$ | 2,2 &I |
| | (4) | $A \leftrightarrow A$ | 3 Df ◆ |

Since the problem is to derive the biconditional (4), we aim at proving the appropriate conjunction of conditionals (3).

Proofs involving biconditionals can often be long without requiring much ingenuity. The main problem is to keep track of where one is in the overall plan of execution of the proof. Here is an example:

*Example 2:* Show A ↔ ~B $\vdash_{NK}$ ~(A ↔ B).

Since the conclusion is a negative formula we are likely to obtain it by ~I, which requires that we assume 'A ↔ B' and infer '⋏', then use ~I to discharge the assumption (line 20 below). The premise and the assumption together give us four conditionals to work with (lines 3 and 4) but there is not much one can do with conditionals by themselves. So we must make a further assumption to get going, one which can also be discharged by ~I (line 11).

| | | | |
|---|---|---|---|
| 1 | (1) | A ↔ ~B | Premise |
| 2 | (2) | A ↔ B | Assumption |
| 2 | (3) | (A → B) & (B → A) | 2 Df |
| 1 | (4) | (A → ~B) & (~B → A) | 1 Df |
| 5 | (5) | A | Assumption |
| 2 | (6) | A → B | 3 &E |
| 1 | (7) | A → ~B | 4 &E |
| 2,5 | (8) | B | 5,6 →E |
| 1,5 | (9) | ~B | 5,7 →E |
| 1,2,5 | (10) | ⋏ | 8,9 ~E |
| 1,2 | (11) | ~A | 5,10 ~I |
| 2 | (12) | B → A | 3 &E |
| 13 | (13) | B | Assumption |
| 2,13 | (14) | A | 12,13 →E |
| 1,2,13 | (15) | ⋏ | 11,14 ~E |
| 1,2 | (16) | ~B | 13,15 ~I |
| 1 | (17) | ~B → A | 4 &E |
| 1,2 | (18) | A | 16,17 →E |
| 1,2 | (19) | ⋏ | 11,18 ~E |
| 1 | (20) | ~(A ↔ B) | 2,19 ~I ◆ |

Though lengthy, this proof is not devious. The overall strategy is to infer '⋏' (line 19) from 'A ↔ B' as assumption. For this to give '~(A ↔ B)' *depending only on line 1*, '⋏' must depend *only on lines 1 and 2*, as it does at (19), but not at (15) or (10). To obtain '⋏' depending just on lines 1 and 2, we use Df to expand (1) and (2). This gives us four conditionals, which inspection confirms cannot all be true. For the first conjuncts of (3) and (4) imply that 'A' is false, while the second conjuncts imply that 'A' is true. So if we assume 'A' we should be able to reach an explicit contradiction and thus '⋏' by ~E. The idea is to use the first conjuncts of (3) and (4) to reach '⋏' and so to reject 'A', yielding '~A' depending just on (1) and (2) (line 11) and then to use the second conjuncts to reach '⋏' again, still depending just on (1) and (2), so that we can reject (2) (lines 19 and 20). In order to get '⋏' the second time we have to make an inner application of ~I, as in lines 13–16.

This completes our core presentation of NK. We now turn to discussion of some simplifications and elaborations, after giving a summary of advice for doing proofs.

## ❑ Exercises

Show the following (‘$p \dashv\vdash_{NK} q$’ abbreviates ‘$p \vdash_{NK} q$ and $q \vdash_{NK} p$’):

(1) $A, A \leftrightarrow B \vdash_{NK} B$
(2) $A \leftrightarrow B \vdash_{NK} B \leftrightarrow A$
(3) $(A \,\&\, B) \leftrightarrow A \vdash_{NK} A \rightarrow B$
*(4) $(A \lor B) \leftrightarrow A \vdash_{NK} B \rightarrow A$
(5) $\sim A, A \leftrightarrow B \vdash_{NK} \sim B$
(6) $\sim A \leftrightarrow \sim B \dashv\vdash_{NK} A \leftrightarrow B$
(7) $\sim(A \leftrightarrow B) \dashv\vdash_{NK} \sim A \leftrightarrow B$
(8) $A \,\&\, B \vdash_{NK} A \leftrightarrow B$
(9) $A \leftrightarrow \sim A \dashv\vdash_{NK} A \,\&\, \sim A$
*(10) $(A \lor B) \lor C, B \leftrightarrow C \vdash_{NK} C \lor A$
(11) $A \rightarrow B, B \rightarrow C, C \rightarrow A \vdash_{NK} (A \leftrightarrow B) \,\&\, ((B \leftrightarrow C) \,\&\, (C \leftrightarrow A))$
(12) $A \leftrightarrow (B \lor C), B \rightarrow D, C \rightarrow E \vdash_{NK} D \lor (E \lor \sim A)$
(13) $(A \,\&\, B) \leftrightarrow (A \,\&\, C) \dashv\vdash_{NK} A \rightarrow (B \leftrightarrow C)$
(14) $\sim A \lor C, \sim B \lor \sim C \vdash_{NK} A \rightarrow \sim(A \leftrightarrow B)$
(15) $A \leftrightarrow (B \leftrightarrow C) \dashv\vdash_{NK} (A \leftrightarrow B) \leftrightarrow C$
(16) $A \leftrightarrow B \dashv\vdash_{NK} (A \,\&\, B) \lor (\sim A \,\&\, \sim B)$

## 7 Heuristics

Here are some rules of thumb (‘heuristics’) to remember when constructing proofs:

- If the conclusion of the sequent contains connectives, it is likely that the last line of the proof will be obtained by the introduction rule for the conclusion’s main connective (the common exception is ‘$\lor$’). This should indicate what assumptions, if any, need to be made, and what other formulae need to be derived. If these other formulae also contain connectives, iterate this rule of thumb on them. In this way, construct the proof backward from the conclusion as far as possible.
- When the above rule can no longer be applied, inspect the premises to see if they have any *obvious* consequences (e.g. if &E or →E can be applied to them).
- If you have applied the previous heuristics and still cannot obtain the formula $p$ you want, try assuming $\ulcorner \sim p \urcorner$ and aim for $\ulcorner \sim\sim p \urcorner$ by ~E, ~I; then use DN.

- In constructing a proof, any assumptions you make must eventually be discharged, so you should only make assumptions in connection with the three rules which discharge assumptions. In other words, if you make an assumption *p* in a proof, you *must* be able to give one of the following reasons: (a) *p* is the antecedent of a conditional you are trying to derive; (b) you are trying to derive $\ulcorner {\sim}p \urcorner$, so you assume *p* with a view to using ~I; (c) you are using ∨E and *p* is one of the disjuncts of the disjunction to which you will be applying ∨E. *If you make an assumption and cannot justify it by one of* (*a*), (*b*) *or* (*c*), *you are almost certainly pursuing the wrong strategy.*

## 8 Sequent and Theorem Introduction

Logic, like other branches of knowledge, should be cumulative—we should be able to use what we already know in making new discoveries. At the moment, there is no way to do this. For instance, recall Example 4.1:

*Example 1:* Show $A \rightarrow B, {\sim}B \vdash_{NK} {\sim}A$.

| | | | |
|---|---|---|---|
| 1 | (1) | $A \rightarrow B$ | Premise |
| 2 | (2) | ${\sim}B$ | Premise |
| 3 | (3) | $A$ | Assumption |
| 1,3 | (4) | $B$ | 1,3 →E |
| 1,2,3 | (5) | $\curlywedge$ | 2,4 ~E |
| 1,2 | (6) | ${\sim}A$ | 3,5 ~I ◆ |

Now suppose we are asked to prove the following:

*Example 2:* Show $C \rightarrow D, {\sim}D \vdash_{NK} {\sim}C$.

Though Example 2 is not *literally* the same as Example 1, there is a clear sense in which it is ‘essentially’ the same. A proof of Example 2 can be obtained simply by going through Example 1’s proof, substituting ‘C’ for ‘A’ and ‘D’ for ‘B’. We will forego writing this proof out. Next, consider the following problem:

*Example 3:* Show $R \rightarrow (V \rightarrow (S \vee T)), {\sim}(S \vee T) \vdash_{NK} R \rightarrow {\sim}V$.

| | | | |
|---|---|---|---|
| 1 | (1) | $R \rightarrow (V \rightarrow (S \vee T))$ | Premise |
| 2 | (2) | ${\sim}(S \vee T)$ | Premise |
| 3 | (3) | $R$ | Assumption |
| 1,3 | (4) | $V \rightarrow (S \vee T)$ | 1,3 →E |
| 5 | (5) | $V$ | Assumption |
| 1,3,5 | (6) | $S \vee T$ | 4,5 →E |
| 1,2,3,5 | (7) | $\curlywedge$ | 2,6 ~E |
| 1,2,3 | (8) | ${\sim}V$ | 5,7 ~I |
| 1,2 | (9) | $R \rightarrow {\sim}V$ | 3,8 →I ◆ |

This proof involves much reinventing of the wheel, since lines 2 and 4–8 recapitulate the proof of Example 1. These lines can be obtained from Example 1 by substituting ‘V’ for ‘A’ and ‘S ∨ T’ for ‘A’. So it is inefficient to write them out: it ought to be possible to simplify the proof of a new sequent when part or all of its proof can be obtained by making substitutions in a proof already done.

The technique we need here is known as *Sequent Introduction*. First we define the idea of one sequent’s being ‘essentially the same’ as another, or as we now put it, of its being a *substitution-instance* of another. The idea is that the sequents

(a) $\mathrm{C} \rightarrow \mathrm{D}, {\sim}\mathrm{D} \vdash_{\mathrm{NK}} {\sim}\mathrm{C}$

and

(b) $(\mathrm{V} \rightarrow (\mathrm{S} \vee \mathrm{T})), {\sim}(\mathrm{S} \vee \mathrm{T}) \vdash_{\mathrm{NK}} {\sim}\mathrm{V}$

are essentially the same as the sequent

(c) $\mathrm{A} \rightarrow \mathrm{B}, {\sim}\mathrm{B} \vdash_{\mathrm{NK}} {\sim}\mathrm{A}$

because each of (a) and (b) can be obtained from (c) by substituting specific formulae for the sentence-letters in (c). (a) is obtained by substituting ‘C’ for ‘A’ and ‘D’ for ‘B’ in (c) while (b) is obtained by substituting ‘V’ for ‘A’ and ‘S ∨ T’ for ‘B’ in (c). For this reason, (a) and (b) are called *substitution-instances* of (c). This general criterion is given in the following rather wordy definition:

> Let $\pi_1,\ldots,\pi_t$ be the sentence-letters other than ‘⋏’ in the formulae $p_1,\ldots,p_n$ and $q$. Then the sequent $r_1,\ldots,r_n \vdash_{\mathrm{NK}} s$ is said to be a *substitution-instance* of the sequent $p_1,\ldots,p_n \vdash_{\mathrm{NK}} q$ if and only if there are (not necessarily distinct) formulae $u_1,\ldots,u_t$ such that if $u_j$ replaces $\pi_j$ throughout $p_1,\ldots,p_n$ and $q$, $1 \leqslant j \leqslant t$, then the sequent $r_1,\ldots,r_n \vdash_{\mathrm{NK}} s$ results.

When $r_1,\ldots,r_n \vdash_{\mathrm{NK}} s$ is a substitution-instance of $p_1,\ldots,p_n \vdash_{\mathrm{NK}} q$, we also say that each $r_i$ is a substitution-instance of the corresponding $p_i$ and $s$ is a substitution-instance of $q$. Thus (a) is a substitution-instance of (c), since we have $n = t = 2$, $\pi_1$ = ‘A’, $\pi_2$ = ‘B’, $p_1$ = ‘A → B’, $p_2$ = ‘~B’, $q$ = ‘~A’, $u_1$ = ‘C’ and $u_2$ = ‘D’. Putting $u_1$ and $u_2$ for $\pi_1$ and $\pi_2$ respectively in sequent (c), $\mathrm{A} \rightarrow \mathrm{B}, {\sim}\mathrm{B} \vdash_{\mathrm{NK}} {\sim}\mathrm{A}$, yields sequent (a), $\mathrm{C} \rightarrow \mathrm{D}, {\sim}\mathrm{D} \vdash_{\mathrm{NK}} {\sim}\mathrm{C}$, so the latter is a substitution-instance of the former, as desired. Similarly, (b) is a substitution-instance of (c), since we have $n = t = 2$, $\pi_1$ = ‘A’, $\pi_2$ = ‘B’, $p_1$ = ‘A → B’, $p_2$ = ‘~B’, $q$ = ‘~A’, $u_1$ = ‘V’ and $u_2$ = ‘S ∨ T’. Putting $u_1$ and $u_2$ for $\pi_1$ and $\pi_2$ respectively in (c) yields (b), which is what it means for (b) to be a substitution-instance of (c). Another example: the sequent

(d) ${\sim}({\sim}(\mathrm{R} \,\&\, \mathrm{S}) \,\&\, {\sim}(\mathrm{U} \leftrightarrow {\sim}\mathrm{V})) \vdash_{\mathrm{NK}} (\mathrm{R} \,\&\, \mathrm{S}) \vee (\mathrm{U} \leftrightarrow {\sim}\mathrm{V})$

is a substitution-instance of the sequent

(e) $\sim(\sim A \mathbin{\&} \sim B) \vdash_{NK} A \vee B$,

since if we replace 'A' in (e) by '(R & S)' and 'B' by '(U ↔ ~V)', we obtain (d).

Notice that the definition of 'substitution-instance' allows some or all of the $u_j$ to be the same; in other words, we can substitute the same formulae for different sentence-letters, though we cannot substitute different formulae for the same sentence-letter. Note also that we cannot replace '$\curlywedge$'. For example, if we counted $A \vee B \vdash_{NK} A$ a substitution-instance of $A \vee \curlywedge \vdash_{NK} A$, then putting 'B' for '$\curlywedge$' throughout a proof of $A \vee \curlywedge \vdash_{NK} A$ (Exercise 5.7) should yield a proof of $A \vee B \vdash_{NK} A$. But it does not. Indeed, there are no proofs of $A \vee B \vdash_{NK} A$, since $A \vee B \nvDash A$.

The method of sequent introduction allows us to move from any lines in a proof to a new line if the corresponding sequent is a substitution-instance of one we have already proved. The new line depends on all premises and assumptions which the lines we are using depend on. More precisely:

> ***Rule of Sequent Introduction:*** Suppose the sequent $r_1,\ldots,r_n \vdash_{NK} s$ is a substitution-instance of the sequent $p_1,\ldots,p_n \vdash_{NK} q$, that we have already proved the sequent $p_1,\ldots,p_n \vdash_{NK} q$, and that the formulae $r_1,\ldots,r_n$ occur at lines $j_1,\ldots,j_n$ in a proof. Then we may infer $s$ at line k, labeling the line '$j_1,\ldots,j_n$ SI ⟨Identifier⟩' and writing on the left all the numbers which occur on the left of lines $j_1,\ldots,j_n$. As a special case, when $n = 0$ and $\vdash_{NK} s$ is a substitution-instance of some theorem $\vdash_{NK} q$ which we have already proved, we may introduce a new line k into a proof with the formula $s$ at it and no numbers on the left, labeling the line 'TI ⟨Identifier⟩'.

In this statement of the rule, 'TI' stands for 'Theorem Introduction', and the placeholder '⟨Identifier⟩' stands for a reference to the previously proved sequent or theorem which is being used.

We can now solve Examples 2 and 3 quite rapidly. As an identifier for the sequent $A \rightarrow B, \sim B \vdash_{NK} \sim A$, we use its traditional Latin tag *Modus Tollens*, or MT for short. Hence:

*Example 2:* Show $C \rightarrow D, \sim D \vdash_{NK} \sim C$.

| | | | | |
|---|---|---|---|---|
| 1 | (1) | $C \rightarrow D$ | Premise | |
| 2 | (2) | $\sim D$ | Premise | |
| 1,2 | (3) | $\sim C$ | 1,2 SI (MT) | ◆ |

We employ the identifier 'MT' to indicate which previously proved sequent we are using in deducing line 3; the lines we cite on the right are the lines where the premises of the sequent which is the substitution-instance of MT occur. This example is a rather special case, in that the lines which are the substitution-instances of the premises of MT, lines 1 and 2, are themselves the premises of the sequent we are trying to prove. More generally, we can use SI on any previous lines in a proof to produce a new line, not just on the premises; all that is required is that the lines in question be substitution-instances of the premises and conclusion of the sequent we use in applying SI. Thus:

*Example 3:* Show $R \rightarrow (V \rightarrow (S \vee T)), {\sim}(S \vee T) \vdash_{\mathrm{NK}} R \rightarrow {\sim}V$.

| | | | |
|---|---|---|---|
| 1 | (1) | $R \rightarrow (V \rightarrow (S \vee T))$ | Premise |
| 2 | (2) | ${\sim}(S \vee T)$ | Premise |
| 3 | (3) | $R$ | Assumption |
| 1,3 | (4) | $V \rightarrow (S \vee T)$ | 1,3 $\rightarrow$E |
| 1,2,3 | (5) | ${\sim}V$ | 4,2 SI (MT) |
| 1,2 | (6) | $R \rightarrow {\sim}V$ | 3,5 $\rightarrow$I ◆ |

Though SI and TI allow the use of *any* previously proved sequent or theorem, the sequents and theorem listed below are the most useful (in this list, '$p \dashv\vdash_{\mathrm{NK}} q$' abbreviates '$p \vdash_{\mathrm{NK}} q$ and $q \vdash_{\mathrm{NK}} p$'):

| | | |
|---|---|---|
| (a) | $A \vee B, {\sim}A \vdash_{\mathrm{NK}} B$; or: $A \vee B, {\sim}B \vdash_{\mathrm{NK}} A$ | (DS) |
| (b) | $A \rightarrow B, {\sim}B \vdash_{\mathrm{NK}} {\sim}A$ | (MT) |
| (c) | $A \vdash_{\mathrm{NK}} B \rightarrow A$ | (PMI) |
| (d) | ${\sim}A \vdash_{\mathrm{NK}} A \rightarrow B$ | (PMI) |
| (e) | $A \vdash_{\mathrm{NK}} {\sim}{\sim}A$ | ($DN^{+}$) |
| (f) | ${\sim}(A \& B) \dashv\vdash_{\mathrm{NK}} {\sim}A \vee {\sim}B$ | (DeM) |
| (g) | ${\sim}(A \vee B) \dashv\vdash_{\mathrm{NK}} {\sim}A \& {\sim}B$ | (DeM) |
| (h) | ${\sim}({\sim}A \vee {\sim}B) \dashv\vdash_{\mathrm{NK}} A \& B$ | (DeM) |
| (i) | ${\sim}({\sim}A \& {\sim}B) \dashv\vdash_{\mathrm{NK}} A \vee B$ | (DeM) |
| (j) | $A \rightarrow B \dashv\vdash_{\mathrm{NK}} {\sim}A \vee B$ | (Imp) |
| (k) | ${\sim}(A \rightarrow B) \dashv\vdash_{\mathrm{NK}} A \& {\sim}B$ | (Neg-Imp) |
| (l) | $A * B \vdash_{\mathrm{NK}} B * A$ | (Com) |
| (m) | $A \& (B \vee C) \dashv\vdash_{\mathrm{NK}} (A \& B) \vee (A \& C)$ | (Dist) |
| (n) | $A \vee (B \& C) \dashv\vdash_{\mathrm{NK}} (A \vee B) \& (A \vee C)$ | (Dist) |
| (p) | $\vdash_{\mathrm{NK}} A \vee {\sim}A$ | (LEM) |
| (q) | $A * B \dashv\vdash_{\mathrm{NK}} {\sim}{\sim}A * {\sim}{\sim}B$; or: ${\sim}{\sim}A * B$; or: $A * {\sim}{\sim}B$ | (SDN) |
| (r) | ${\sim}(A * B) \dashv\vdash_{\mathrm{NK}} {\sim}({\sim}{\sim}A * {\sim}{\sim}B)$; or: ${\sim}({\sim}{\sim}A * B)$; or: ${\sim}(A * {\sim}{\sim}B)$ | (SDN) |

As identifiers we can write out the sequents, or use the letters ordering this list, or else we can use the traditional names the sequents are known by, as indicated on the right. (a) is called Disjunctive Syllogism or Modus Tollendo Ponens; (b), as already remarked, is called Modus Tollens (traditionally, our rule →E is known as Modus Ponens); (c) and (d) are known as the Paradoxes of Material Implication (though they are not really paradoxical, by the account in §8 of Chapter Three); (e) is called Double Negation Addition; (f) through (i) are called De Morgan's Laws (after the mathematician who first investigated them); (j) is Implication and (k) Negation Implication; (l) is the Law of Commutation for any binary connective * *other than* '→'; (m) and (n) are the Laws of Distribution and

(p) is the Law of Excluded Middle. (q) and (r), Subformula Double Negation, require special comment. In (q), ‘*’ stands for any binary connective, and so SDN allows adding or eliminating a ‘~~’ prefix to one or other of the *main subformulae* of a formula—this is (q)—or the *main subformulae of the main subformula* of a negation—this is (r). By contrast, DN allows prefixing ‘~~’ only to the whole formula. For example, applying DN to ‘(A & B)’ would only produce ‘~~(A & B)’, but with (q) we can, as it were, apply DN to the subformulae ‘A’ and ‘B’ without first using &E to get each by itself. So (q) could produce any of ‘(A & ~~B)’, ‘(~~A & B)’ and ‘(~~A & ~~B)’; and in the other direction, from any of these three SDN could produce ‘(A & B)’. Similarly, from ‘~(A ∨ ~~B)’, (r) could produce ‘~(A ∨ B)’ and vice versa.

Here is another example where we use SI, this time appealing to DS, to shorten a proof:

*Example 4:* Show $R \rightarrow ((S \rightarrow T) \vee V), {\sim}(S \rightarrow T), R \rightarrow {\sim}V \vdash_{NK} {\sim}R$.

| | | | |
|---|---|---|---|
| 1 | (1) | R → ((S → T) ∨ V) | Premise |
| 2 | (2) | ~(S → T) | Premise |
| 3 | (3) | R → ~V | Premise |
| 4 | (4) | R | Assumption |
| 1,4 | (5) | (S → T) ∨ V | 1,4 →E |
| 1,2,4 | (6) | V | 5,2 SI (DS) |
| 3,4 | (7) | ~V | 3,4 →E |
| 1,2,3,4 | (8) | ⋏ | 7,6 ~E |
| 1,2,3 | (9) | ~R | 4,8 ~I ◆ |

The sequent $(S \rightarrow T) \vee V, {\sim}(S \rightarrow T) \vdash_{NK} V$ is a substitution-instance of the sequent DS ((a) in the list on page 123) and we have already proved DS. Also, the premises of the sequent $(S \rightarrow T) \vee V, {\sim}(S \rightarrow T) \vdash_{NK} V$ occur at lines 5 and 2 respectively in our proof. So at line 6 we cite these lines and say which sequent it is whose premises they instantiate. On the left at (6) we pool together all the premises and assumptions on which lines 5 and 2 depend.

It is never compulsory to use Sequent Introduction. By ‘NK’ we mean a particular collection of rules, so adding a new rule to NK by definition *extends* NK and gives us a new system. However, when we add a new rule to a system, it is an open question whether or not the new rule allows us to prove any *new sequents*, sequents which could not be proved without the new rule. If we extend a system by adding a rule which does *not* allow us to prove new sequents, then the extension is said to be a *conservative* extension. In adding SI to NK, we are only making a conservative extension of NK, since there is nothing we can prove using SI that we could not already prove—albeit at greater length—without. For we can eliminate any step of SI in a proof by inserting into the proof the relevant substitution-instance of the proof of the sequent used by the application of SI in question.

To illustrate Theorem Introduction, TI, which is the special case of SI with no premises, we give a proof which uses LEM. LEM is typically used in combination with ∨E.

*Example 5:* Show $\vdash_{NK} (A \rightarrow B) \vee (B \rightarrow A)$.

Intuitively, ‘A’ is either true or false; if ‘A’ is true, then ‘$(B \rightarrow A)$’ holds and so ‘$(A \rightarrow B) \vee (B \rightarrow A)$’ holds, while if ‘A’ is false, ‘$(A \rightarrow B)$’ holds and so once again ‘$(A \rightarrow B) \vee (B \rightarrow A)$’ holds. Either way, then, ‘$(A \rightarrow B) \vee (B \rightarrow A)$’ holds. This semantic argument uses the Principle of Bivalence, that ‘A’ is either true or false, and the table for material implication, that a true consequent or a false antecedent is sufficient for the truth of a conditional, to infer that $\vDash (A \rightarrow B) \vee (B \rightarrow A)$. Its proof-theoretic counterpart is:

| | | | |
|---|---|---|---|
| | (1) | $A \vee {\sim}A$ | TI (LEM) |
| 2 | (2) | $A$ | Assumption |
| 2 | (3) | $B \rightarrow A$ | 2 SI (PMI) |
| 2 | (4) | $(A \rightarrow B) \vee (B \rightarrow A)$ | 3 ∨I |
| 5 | (5) | ${\sim}A$ | Assumption |
| 5 | (6) | $A \rightarrow B$ | 5 SI (PMI) |
| 5 | (7) | $(A \rightarrow B) \vee (B \rightarrow A)$ | 6 ∨I |
| | (8) | $(A \rightarrow B) \vee (B \rightarrow A)$ | 1,2,4,5,7 ∨E ◆ |

One complaint about natural deduction that is sometimes made is that it often does not seem very *natural*, for example in proofs which require non-obvious assumptions and uses of DN. Those who make this complaint are contrasting formal proofs with ordinary reasoning. However, in ordinary reasoning, of which we have an exemplar in the argument in §1 of this chapter about who helped Smith open the safe, we use Sequent Introduction all the time. So some proofs earlier in this chapter, where SI was not available, seem unnatural by comparison. But when NK is equipped with SI we can produce natural proofs which exactly mirror the way one would reason in ordinary language. For example, here is such a proof of the argument about Smith’s accomplice.

> If the safe was opened, it must have been opened by Smith, with the assistance of Brown or Robinson. None of these three could have been involved unless he was absent from the meeting. But we know that either Smith or Brown was present at the meeting. So since the safe was opened, it must have been Robinson who helped open it.

We symbolize this argument using the following dictionary:

O: The safe was opened
S: Smith opened the safe
B: Brown assisted
R: Robinson assisted
X: Smith was absent from the meeting
Y: Brown was absent from the meeting
Z: Robinson was absent from the meeting

We obtain the following sequent:

$$O \rightarrow (S \mathbin{\&} (B \lor R)),\ ({\sim}S \lor X) \mathbin{\&} (({\sim}B \lor Y) \mathbin{\&} ({\sim}R \lor Z)),\ {\sim}X \lor {\sim}Y,\ O \vdash_{NK} R.$$

The following proof articulates exactly the principles of inference that the informal proof in §1 implicitly employs.

| | | | |
|---|---|---|---|
| 1 | (1) | O → (S & (B ∨ R)) | Premise |
| 2 | (2) | (~S ∨ X) & ((~B ∨ Y) & (~R ∨ Z)) | Premise |
| 3 | (3) | ~X ∨ ~Y | Premise |
| 4 | (4) | O | Premise |
| 2 | (5) | ~S ∨ X | 2 &E |
| 2 | (6) | (~B ∨ Y) & (~R ∨ Z) | 2 &E |
| 2 | (7) | ~B ∨ Y | 6 &E |
| 2 | (8) | ~R ∨ Z | 6 &E |
| 1,4 | (9) | S & (B ∨ R) | 1,4 →E |
| 1,4 | (10) | S | 9 &E |
| 1,4 | (11) | B ∨ R | 9 &E |
| 1,4 | (12) | ~~S | 10 SI (DN$^{+}$) |
| 1,2,4 | (13) | X | 5,12 SI (DS) |
| 1,2,4 | (14) | ~~X | 13 SI (DN$^{+}$) |
| 1,2,3,4 | (15) | ~Y | 3,14 SI (DS) |
| 1,2,3,4 | (16) | ~B | 7,15 SI(DS) |
| 1,2,3,4 | (17) | R | 11,16 SI (DS) ◆ |

Note carefully the need for SI at lines 12 and 14. In order to use the sequent DS, we need a disjunction and the *negation* of one of its disjuncts. (10) is not *the negation of* the first disjunct of (5), but (12) is.

As the example illustrates, with the addition of SI, we can use NK to mimic actual human reasoning, for in the latter an informal analogue of SI is used repeatedly. The system NK, then, provides us with not only a tool for executing proofs but also an idealized model of the human psychological faculty of deductive inference. If a machine with the same reasoning potential as humans is ever built, we can expect to find something like NK at the core of its program. On the other hand, to say that NK is an idealized model of the deductive reasoning faculty is not to say that people can reason deductively because rules like those of NK are encoded in their brains. The nature of the actual psychological mechanisms of deduction is an ongoing field of research in cognitive psychology; for further information, the reader should consult Johnson-Laird and Byrne. And obviously it is not being claimed that people *do* reason deductively in accordance with the rules of NK; we said, after all, that NK is an *idealized* model. There is a general distinction in cognitive psychology between *competence* and *performance*. A person's deductive competence is determined by the nature of the psychological mechanisms with which human beings are equipped. But one's individual performance is determined by many other factors, such as other aspects of one's psychological make-up, or just the kind of opportunities for practice which one has previously enjoyed. Though performance cannot exceed competence, it can fall well short of it for these kinds of reasons.

## ❑ Exercises

I Some sequents exhibited below are substitution-instances of sequents in the list on page 123. For each such sequent below, identify the sequent in the list of which it is a substitution-instance. Justify your answer in every case by stating, for each sentence-letter $\pi_i$ in the sequent in which substitution is made, which formula $u_j$ has been substituted for $\pi_i$ (refer to the definition of 'substitution-instance' on page 121).

(1) $\sim\sim(R \,\&\, S) \vee \sim T, T \vdash_{NK} \sim(R \,\&\, S)$
(2) $(A \rightarrow B) \rightarrow C, \sim C \vdash_{NK} \sim(A \rightarrow B)$
(3) $\sim(\sim(R \vee S) \vee \sim(\sim R \vee \sim S)) \vdash_{NK} \sim\sim(R \vee S) \,\&\, \sim\sim(\sim R \vee \sim S)$
(4) $((P \rightarrow Q) \vee R) \,\&\, ((P \rightarrow Q) \vee S) \vdash_{NK} (P \rightarrow Q) \vee (R \,\&\, S)$
*(5) $\sim(M \vee N) \vee (W \,\&\, U) \vdash_{NK} (M \vee N) \rightarrow (W \,\&\, U)$

II Below there are two lists of sequents, and each sequent in the first list is a substitution-instance of a sequent in the second. Say which sequents are substitution-instances of which, justifying your answer in the same way as in I.

List 1:

(i) $\sim(R \,\&\, S) \vee \sim\sim(\sim T \,\&\, S), \sim W \vee \sim\sim T, \sim(R \,\&\, S) \rightarrow \sim\sim W \vdash_{NK} (\sim T \,\&\, S) \rightarrow \sim T$
(ii) $\sim\sim(R \,\&\, S) \vee \sim(\sim T \,\&\, S), \sim\sim W \vee \sim T, \sim\sim(R \,\&\, S) \rightarrow \sim\sim\sim W \vdash_{NK} \sim(\sim T \,\&\, S) \rightarrow \sim T$
*(iii) $\sim\sim(R \,\&\, S) \vee \sim(\sim T \,\&\, S), \sim\sim W \vee \sim\sim T, \sim\sim(R \,\&\, S) \rightarrow \sim\sim\sim W \vdash_{NK} \sim(\sim T \,\&\, S) \rightarrow \sim\sim T$

List 2:

(a) $A \vee \sim B, C \vee \sim D, A \rightarrow \sim C \vdash_{NK} \sim B \rightarrow \sim D$
(b) $\sim\sim A \vee B, \sim\sim C \vee D, \sim\sim A \rightarrow \sim C \vdash_{NK} B \rightarrow D$
(c) $\sim A \vee \sim\sim B, C \vee \sim\sim D, \sim A \rightarrow \sim C \vdash_{NK} B \rightarrow \sim D$

III Show the following. Wherever you apply Sequent Introduction, be sure to indicate which previously proved sequent you are using.

(1) $\sim A \vdash_{NK} \sim B \rightarrow \sim(A \vee B)$
(2) $A \rightarrow (B \vee \sim C), \sim A \rightarrow (B \vee \sim C), \sim B \vdash_{NK} \sim C$
(3) $\vdash_{NK} A \vee (A \rightarrow B)$
*(4) $\sim B \rightarrow A \vdash_{NK} (B \rightarrow A) \rightarrow A$
(5) $\vdash_{NK} (A \rightarrow B) \rightarrow [(A \rightarrow \sim B) \rightarrow \sim A]$
(6) $\sim[A \rightarrow (B \vee C)] \vdash_{NK} (B \vee C) \rightarrow A$
(7) $A \rightarrow B, (\sim B \rightarrow \sim A) \rightarrow (C \rightarrow D), \sim D \vdash_{NK} \sim C$
*(8) $(A \vee B) \rightarrow (A \vee C) \vdash_{NK} A \vee (B \rightarrow C)$
(9) $(A \,\&\, B) \leftrightarrow C, \sim(C \vee \sim A) \vdash_{NK} \sim B$
(10) $A \rightarrow (B \vee C) \dashv\vdash_{NK} (A \rightarrow B) \vee (A \rightarrow C)$

(11) $\sim(A \mathbin{\&} \sim B) \vee \sim(\sim D \mathbin{\&} \sim E), \sim(E \vee B), C \rightarrow (\sim E \rightarrow (\sim D \mathbin{\&} A)) \vdash_{NK} \sim C$
(12) $(A \vee B) \rightarrow (C \mathbin{\&} D), (\sim E \vee C) \rightarrow [(F \vee G) \rightarrow H],$
$(\sim I \rightarrow J) \rightarrow [G \mathbin{\&} (H \rightarrow \sim K)] \vdash_{NK} K \rightarrow (\sim A \vee \sim I)$
*(13) $(A \leftrightarrow B) \leftrightarrow (C \leftrightarrow D) \vdash_{NK} (A \leftrightarrow C) \leftrightarrow (B \leftrightarrow D)$
(14) $(A \vee B) \mathbin{\&} (C \vee D) \vdash_{NK} (B \vee C) \vee (A \mathbin{\&} D)$
(15) $\sim(A \leftrightarrow B), \sim(B \leftrightarrow C), \sim(C \leftrightarrow A) \vdash_{NK} \curlywedge$

IV Symbolize the following arguments (state your dictionary explicitly). Then give proofs of the resulting argument-forms.

(1) If God is omnipotent then He can do anything. So He can create a stone which is too heavy to be lifted. But that means *He* can't lift it, so there's something He can't do. Therefore, He isn't omnipotent.

(2) If there is an empirical way of distinguishing between absolute rest and absolute motion, Newton was right to think that space is absolute, not relative. Also, if there is absolute space, there is a real difference between absolute rest and absolute motion—whether or not they are empirically distinguishable. So if, as some argue, there cannot really be a difference between absolute rest and absolute motion unless they are empirically distinguishable, an empirical way of distinguishing between absolute rest and absolute motion is necessary and sufficient for the existence of absolute space.

(3) If God is willing to prevent evil but is unable to do so, He is impotent. If God is able to prevent evil but unwilling to do so, He is malevolent. If He is neither able nor willing, then he is both impotent and malevolent. Evil exists if and only if God is unwilling or unable to prevent it. God exists only if He is neither impotent nor malevolent. Therefore, if God exists evil does not.

## 9 Alternative formats for proofs

The format in which we have been setting out our proofs, taken from Lemmon, sits midway between two other formats, known respectively as *tree* format (this was Gentzen's original format) and *sequent-to-sequent* format. Lemmon format can be regarded either as a linearization of tree format or as a notational variant of sequent-to-sequent format. Since both other formats are revealing, we present them briefly here.

By contrast with parse trees and semantic tableaux, proof trees are not inverted: leaves are at the top and roots at the bottom. But like other trees, construction proceeds downward. A proof begins with a listing of premises and assumptions *across* the page, and an application of an inference rule extends a path or paths by adding a node below the current level and labeling it with the formula which that rule-application produces. In Lemmon format, the numbers of the premises and assumptions on which a formula occurrence $\phi$ depends are explicitly stated on its left. In tree format, what a formula occurrence $\phi$ depends on can be determined merely by tracing up through the proof from $\phi$

and following all paths; the top nodes (with undischarged formulae) at which one arrives are the relevant premises and assumptions. For instance, the proof of Example 2.1 on page 89 can be arranged as a tree as follows:

*Example 1:* Show A & B, C & D, (A & D) → H $\vdash_{NK}$ H.

$$\dfrac{\dfrac{\dfrac{A \mathbin{\&} B}{A}\,{\scriptstyle \&E} \qquad \dfrac{C \mathbin{\&} D}{D}\,{\scriptstyle \&E}}{A \mathbin{\&} D}\,{\scriptstyle \&I} \qquad (A \mathbin{\&} D) \to H}{H}\,{\scriptstyle \to E}$$

We see that ‘A & D’ depends on ‘A & B’ and ‘C & D’, since these are the formulae at the leaves of the branches that lead upwards from the node labeled ‘A & D’. More generally, in tree format the *root* formula (the conclusion) depends on whatever formulae there are on the leaves of the tree, except those which are discharged (none are discharged in this example). A *path* in a tree *T* is a sequence of formulae occurrences $\phi_1,\ldots,\phi_n$ such that $\phi_1$ is a leaf of *T*, $\phi_n$ is the root formula of *T*, and $\phi_{i+1}$ is the formula occurring in *T* below the line beneath $\phi_i$. Thus the paths in Example 1 are ⟨A & B, A, A & D, H⟩, ⟨C & D, D, A & D, H⟩, and ⟨(A & D) → H, H⟩.

To discharge an assumption we draw a line over it. We indicate which step in the proof causes an assumption to be discharged by numbering the assumptions as we introduce them and using an assumption’s number to label the rule-application which discharges it. Another feature of tree format is that if an assumption or premise is to be used twice, then we start *two* paths of the proof tree with that assumption or premise at the top. Both these features are illustrated by the following:

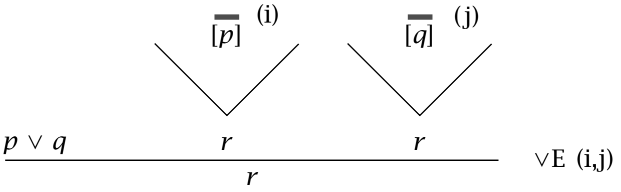
*Example 2:* Show A → (B & C) $\vdash_{NK}$ (A → B) & (A → C)

$$\dfrac{\dfrac{\dfrac{\dfrac{A \to (B \mathbin{\&} C) \qquad \overline{A}^{\,(1)}}{B \mathbin{\&} C}\,{\scriptstyle \to E}}{B}\,{\scriptstyle \&E}}{A \to B}\,{\scriptstyle \to I\,(1)} \qquad \dfrac{\dfrac{\dfrac{A \to (B \mathbin{\&} C) \qquad \overline{A}^{\,(2)}}{B \mathbin{\&} C}\,{\scriptstyle \to E}}{C}\,{\scriptstyle \&E}}{A \to C}\,{\scriptstyle \to I\,(2)}}{(A \to B) \mathbin{\&} (A \to C)}\,{\scriptstyle \&I}$$

In Lemmon format we would assume ‘A’ at line 2 and apply →I to it twice. In the corresponding proof in tree format, we put ‘A’ at the top of two paths, and since each of these formula occurrences will be used in an application of →E, we have to write the conditional premise ‘A → (B & C)’ at the top of two paths as well, one for each use of →E (hence the difference between a formula and its occurrences). The assumptions are numbered (1) and (2) respectively, and each is discharged by an application of →I. When we discharge an assumption, we indicate which assumption is being discharged by writing its number next to the rule-application that does the discharging. At that point, we draw a line over the assumption, indicating that it has been discharged. Undeniably, something of the dynamics of this process is lost in the final display.

One feature of tree format which Examples 1 and 2 illustrate is the way it factors out certain arbitrary features of a proof in Lemmon format. In Example 1 we have to apply &E to premises 1 and 2; it does not matter which application we make first, but in Lemmon format we must put one before the other. Similarly, in a Lemmon-format version of Example 2 we must decide which conjunct of the conclusion to derive first. But in tree format we can represent parts of the proof whose relative order is of no matter as proceeding in parallel.

The rule of ∨E is perspicuously represented as a schema in tree notation:



Here we represent a deduction of *r* from *p*, perhaps with the aid of other premises and assumptions (leaf formulae of paths which merge with the displayed *p*-paths before the first occurrence of *r*), and a deduction of *r* from *q*, again perhaps with aid, followed by an application of ∨E to discharge the assumptions *p* and *q*. The brackets indicate that the bracketed formula may be the leaf formula of more than one path and that all its occurrences are discharged. Here is an example which involves the negation rules as well as ∨E:

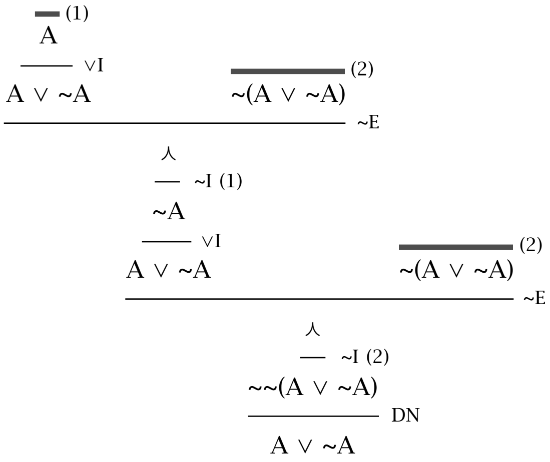
*Example 3:* Show A → (B ∨ C), ~B & ~C $\vdash_{NK}$ ~A.

$$\dfrac{\dfrac{\dfrac{A \to (B \vee C) \qquad \overline{A}^{\,(1)}}{B \vee C}\,{\to}\mathrm{E} \qquad \dfrac{\overline{B}^{\,(2)} \qquad \dfrac{{\sim}B \,\&\, {\sim}C}{{\sim}B}\,\&\mathrm{E}}{\curlywedge}\,{\sim}\mathrm{E} \qquad \dfrac{\overline{C}^{\,(3)} \qquad \dfrac{{\sim}B \,\&\, {\sim}C}{{\sim}C}\,\&\mathrm{E}}{\curlywedge}\,{\sim}\mathrm{E}}{\curlywedge}\,{\vee}\mathrm{E}\ (2,3)}{{\sim}A}\,{\sim}\mathrm{I}\ (1)$$

The undischarged leaf formulae are all premises of the sequent.

A formula can occur as leaf formula on two or more paths and have all its occurrences discharged by one rule-application when all the occurrences of the assumption lie on paths which merge at or before the discharging rule-application. In such a case, the occurrences of the assumption as leaf formulae may be given the same number. An example is the proof of the Law of Excluded Middle (compare the proof on page 116):

*Example 4:* Show $\vdash_{NK} A \vee {\sim}A$.



In this tree, all leaf formulae are discharged, and so the root formula depends on nothing. We use '~(A ∨ ~A)' in two applications of ~E, but discharge it only once, with ~I as indicated. The one application of ~I discharges different occurrences of the assumption-formula because the paths which have their leaves labeled by those occurrences merge at the application of ~I in question.

The other format for proofs to which Lemmon format is related is sequent-to-sequent format. We presented rules of inference in previous sections as devices for deriving formulae from formulae while adjusting premise and assumption numbers on the left as a bookkeeping device. But it is quite revealing to realize that we can also regard our rules as devices for deriving *sequents* from *sequents*. This effect of the rules can be brought out by writing the relevant premise and assumption formulae on the left explicitly and putting a turnstile in the position where the line number goes in Lemmon format. Here is the proof of Example 2 above in Lemmon format (renumbered as Example 5), followed by the same proof in sequent-to-sequent format:

*Example 5:* Show $A \rightarrow (B \,\&\, C) \vdash_{NK} (A \rightarrow B) \,\&\, (A \rightarrow C)$.

| | | | |
|---|---|---|---|
| 1 | (1) | $A \rightarrow (B \,\&\, C)$ | Premise |
| 2 | (2) | $A$ | Assumption |
| 1,2 | (3) | $B \,\&\, C$ | 1,2 →E |
| 1,2 | (4) | $B$ | 3 &E |
| 1,2 | (5) | $C$ | 3 &E |
| 1 | (6) | $A \rightarrow B$ | 2,4 →I |
| 1 | (7) | $A \rightarrow C$ | 2,5 →I |
| 1 | (8) | $(A \rightarrow B) \,\&\, (A \rightarrow C)$ | 6,7 &I ◆ |

And now in sequent-to-sequent format:

| | | |
|---|---|---|
| (1) | $A \rightarrow (B \& C) \vdash A \rightarrow (B \& C)$ | Premise |
| (2) | $A \vdash A$ | Assumption |
| (3) | $A \rightarrow (B \& C), A \vdash B \& C$ | 1,2 →E |
| (4) | $A \rightarrow (B \& C), A \vdash B$ | 3 &E |
| (5) | $A \rightarrow (B \& C), A \vdash C$ | 3 &E |
| (6) | $A \rightarrow (B \& C) \vdash A \rightarrow B$ | 4 →I |
| (7) | $A \rightarrow (B \& C) \vdash A \rightarrow C$ | 5 →I |
| (8) | $A \rightarrow (B \& C) \vdash (A \rightarrow B) \& (A \rightarrow C)$ | 6,7 &I ◆ |

We see that each line in the Lemmon-format proof can be transcribed into the corresponding line in the sequent-to-sequent proof simply by replacing the numbers on the left in the Lemmon-format proof by the formulae for which they stand, and the line number by the turnstile. Notice that such a procedure always produces sequents of the form $p \vdash_{NK} p$ for premises and assumptions. If we think of Lemmon-format proofs as notational variants of sequent-to-sequent proofs, then the ultimate justification for the Rule of Assumptions is simply that in making an assumption, we are claiming no more than that a certain formula follows from itself.

Similarly, the justification for dropping an assumption-number on the left when applying →I is evident from inspection of lines 6 and 7 in the sequent-to-sequent proof above: →I moves a formula on which another depends across the turnstile and makes it the antecedent of a conditional, with the dependent formula as consequent. However, →I occasions a small divergence from simple transcription into sequent-to-sequent format, in that we need only cite a single line number on the right, the number of the sequent on which we are performing the move operation. In the same way, when applying ~I or ∨E in a sequent-to-sequent proof, one would only cite, respectively, one and three earlier lines, omitting the numbers of the assumption sequents.

Here are the sequent-to-sequent formulations of the rules of NK. We use uppercase Greek gamma, '$\Gamma$' and uppercase Greek sigma, '$\Sigma$', to stand for sets of LSL formulae, and '$\Gamma,\Sigma$' to stand for the union of the two sets $\Gamma$ and $\Sigma$ (this is an alternative to the usual notation '$\Gamma \cup \Sigma$'). Recall that the union of two sets X and Y is the set whose members are all the members of X together with all the members of Y. So '$\Gamma,\Sigma$' stands for the collection of all the formulae in $\Gamma$ together with all those in $\Sigma$. For the set of formulae which results from removing $p$ from $\Gamma$, if $p$ is in $\Gamma$, we write '$\Gamma/p$'; if $p$ is not in $\Gamma$, then $\Gamma/p = \Gamma$.

> ***Rule of Assumptions:*** At any point in a proof, for any LSL formula $p$, we may introduce the sequent $p \vdash p$.
> ***Rule of*** **&*I:*** From sequents $\Gamma \vdash p$ and $\Sigma \vdash q$ we may infer the sequent $\Gamma,\Sigma \vdash p \& q$.
> ***Rule of*** **&*E:*** From the sequent $\Gamma \vdash p \& q$ we may infer the sequent $\Gamma \vdash p$ or the sequent $\Gamma \vdash q$.
> ***Rule of →I:*** From the sequent $\Gamma \vdash q$ we may infer the sequent $\Gamma/p \vdash p \rightarrow q$.

***Rule of →E:*** From the sequents $\Gamma \vdash p \rightarrow q$ and $\Sigma \vdash p$ we may infer the sequent $\Gamma, \Sigma \vdash q$.
***Rule of ~I:*** From the sequent $\Gamma \vdash \curlywedge$ we may infer the sequent $\Gamma / p \vdash {\sim}p$.
***Rule of ~E:*** From the sequents $\Gamma \vdash q$, $\Sigma \vdash {\sim}q$ we may infer the sequent $\Gamma, \Sigma \vdash \curlywedge$.
***Rule of DN:*** From the sequent $\Gamma \vdash {\sim}{\sim}p$ we may infer the sequent $\Gamma \vdash p$.
***Rule of ∨I:*** From the sequent $\Gamma \vdash p$ we may infer either the sequent $\Gamma \vdash p \vee q$ or the sequent $\Gamma \vdash q \vee p$.
***Rule of ∨E:*** From the sequents $\Gamma \vdash p \vee q$, $\Sigma \vdash r$ and $\Delta \vdash r$ we may infer the sequent $\Gamma, \Sigma / p, \Delta / q \vdash r$.
***Rule of Df:*** The sequent $\Gamma \vdash p \leftrightarrow q$ may be expanded into the sequent $\Gamma \vdash (p \rightarrow q) \,\&\, (q \rightarrow p)$; and the sequent $\Gamma \vdash (p \rightarrow q) \,\&\, (q \rightarrow p)$ may be contracted into the sequent $\Gamma \vdash p \leftrightarrow q$.

### ❑ Exercises

*(1) Write out a schema for each rule of inference for use in tree format (see the example of ∨E in this section).
(2) Arrange the proofs of Examples 5.1–5.7 in tree format. [*(5.6)]
(3) Arrange the proofs of Examples 5.1–5.7 in sequent-to-sequent format.

## 10 Systems equivalent to NK

We remarked at the point of explaining the rule DN that it would be *redundant* to extend NK by adding a rule for prefixing '~~' to a wff also. Redundancy means that we would not be able to prove any *new* sequents, sequents not already provable without this rule; in the terminology of §8, the addition of such a rule would only yield a conservative extension of NK. However, not every change in NK which we might contemplate involves the *addition* of a rule: we might also consider *replacing* one of our rules by some other rule. To discuss the effects of both addition and replacement we need a more general concept than that of conservative extension, since replacement is a different kind of modification than extension.

> Two systems of natural deduction are *equivalent* if and only if every sequent provable in one is provable in the other, and vice versa.

We can show that a system *S* is equivalent to NK if we can show that the following two conditions hold:

- For every rule *R* of *S* which is not a rule of NK, there is some combination of rule-applications in NK which has the same effect as *R*.

- For every rule of $R$ of NK which is not a rule of $S$, there is some combination of rule-applications in $S$ which has the same effect as $R$.

We assume that every system $S$ permits use of SI (SI is a conservative extension of any system). A combination of rule-applications in a system $S$ has the same effect as a rule $R$ in $S'$ if and only if, whenever $R$ is used in an $S'$-proof to infer a formula $q$ depending on assumptions $\Sigma$, that combination of rule-applications could be used in $S$ instead, resulting in the inference of the same $q$ depending on the same assumptions $\Sigma$. The simplest cases involve rules which do not discharge assumptions. Suppose $S$ is a system with no rule of &I. Then the most direct way of showing that there is a combination of rule-applications in $S$ which has the same effect as &I in NK is to show that $A, B \vdash_S A \,\&\, B$. For then, wherever an NK-proof uses &I, we can employ SI in $S$ using this sequent. The problem will be to show, for the particular $S$ in question, that $A, B \vdash_S A \,\&\, B$.

If the rule that is not present in $S$ involves discharge of assumptions, it may be less straightforward to explain how to get its effect in $S$. Suppose, for example, that $S$ does not have the rule ∨E. What combination of rules would get the same effect as ∨E? A use of ∨E to infer $r$ from $\ulcorner p \vee q \urcorner$ requires a derivation of $r$ from $p$ and then again from $q$. So if $S$ contains the rule of →I, then assuming that the derivation of $r$ from $p$ does not itself use ∨E (or any other rule missing from $S$), we can derive $r$ from $p$ in $S$ and then add an extra application of →I to obtain $\ulcorner p \rightarrow r \urcorner$. Similarly, we can get $\ulcorner q \rightarrow r \urcorner$ in $S$. So if we also have

$$A \vee B, A \rightarrow C, B \rightarrow C \vdash_S C \qquad (*)$$

we can then use SI in $S$ to obtain $r$. Schematically, the $S$-proof will look like this:

| | | | |
|---|---|---|---|
| $a_1,\ldots,a_n$ | (g) | $p \vee q$ | |
| | ⋮ | | |
| h | (h) | $p$ | Assumption |
| | ⋮ | | |
| $b_1,\ldots,b_u$ | (i) | $r$ | |
| $b_1,\ldots,b_u$/h | (i′) | $p \rightarrow r$ | h,i →I |
| | ⋮ | | |
| j | (j) | $q$ | Assumption |
| | ⋮ | | |
| $c_1,\ldots,c_w$ | (k) | $r$ | |
| $c_1,\ldots,c_w$/j | (k′) | $q \rightarrow r$ | j,k →I |
| | ⋮ | | |
| X | (m) | $r$ | g,i′,k′ SI (*) |

where X is the set $\{a_1,\ldots,a_n\} \cup \{b_1,\ldots,b_u\}/h \cup \{c_1,\ldots,c_w\}/j$. In sum, in a system $S$ with →I and the sequent (*), two applications of →I and then SI using (*) yield the same result as an application of ∨E in NK (compare the schema for ∨E on page 112 with the schema just displayed). The problem will be to show, for the particular system $S$ in question, that (*) is provable (see Exercise 10.2).

Returning now to our two conditions for a system $S$ to be equivalent to NK, we see that the first condition guarantees that any sequent which has a proof in *S* has a proof in NK, for if a line in an *S*-proof is derived by a rule *R* of *S*, then we can infer the same line in NK using the appropriate combination of NK-rules. Conversely, the second condition guarantees that any line in an NK-proof inferred by a rule *R* of NK can be inferred in *S*, using the appropriate combination of *S*-rules. So the two conditions are exactly what is required for equivalence of the systems to hold.

*Example 1:* Let *S* be the system consisting in the rules of NK plus the rule DNI, from *p* to infer ⌜~~*p*⌝ depending on the same premises and assumptions. Show that *S* and NK are equivalent (hence DNI is redundant).

We have to show that the two conditions for equivalence of *S* and NK hold. Since every rule of NK is a rule of *S*, the second condition for *S* and NK to be equivalent is met trivially. To show that the first condition holds, we have to show that the effect of DNI can be obtained in NK by some combination of rules of NK. The simplest demonstration of this is to prove A $\vdash_{\mathrm{NK}}$ ~~A, that is, DN$^+$, since then wherever an *S*-proof uses DNI, an NK-proof can use SI (DN$^+$). The proof of DN$^+$ is straightforward:

| | | | |
|---|---|---|---|
| 1 | (1) | A | Premise |
| 2 | (2) | ~A | Assumption |
| 1,2 | (3) | ⋏ | 1,2 ~E |
| 1 | (4) | ~~A | 2,3 ~I ◆ |

Another rule that sometimes features in Gentzen-style systems is the rule known as *ex falso quodlibet*, or *Absurdity*, that anything follows from an absurdity. This is the rule that if '⋏' has been inferred at line j, then at a later line k, any formula *p* may be inferred on the same assumptions as j, and the line labeled 'j EFQ'. Schematically,

| | | | |
|---|---|---|---|
| $a_1,\ldots,a_n$ | (j) | ⋏ | |
| | ⋮ | | |
| $a_1,\ldots,a_n$ | (k) | *p* | j EFQ |

The system with EFQ *in place of* DN is known as NJ. NJ is a collection of rules for the alternative to classical logic which we have already mentioned, intuitionistic logic. NJ is not equivalent to NK, for there are many sequents provable in NK which are not provable in NJ. In particular, the sequent corresponding to DN is not provable in NJ. However, if we define *S* as the system resulting from *adding* EFQ to NK, *S* is equivalent to NK—no new sequents can be proved.

*Example 2:* Let *S* be the system which results from adding EFQ to NK. Show that *S* is equivalent to NK.

As before, the second condition holds trivially. The simplest way of showing that the first condition holds is again by appeal to SI, this time using the sequent ⋏ $\vdash_{\mathrm{NK}}$ A, since then a use of EFQ in an *S*-proof can be mimicked by SI in a corresponding NK-proof. To show ⋏ $\vdash_{\mathrm{NK}}$ A:

| | | | |
|---|---|---|---|
| 1 | (1) | ⅄ | Premise |
| 2 | (2) | ~A | Assumption |
| 1 | (3) | ~~A | 2,1 ~I |
| 1 | (4) | A | 3 DN ◆ |

We also remarked earlier that the Law of Excluded Middle, $\vdash_{NK}$ A ∨ ~A, like the rule DN, is characteristic of classical logic; indeed, intuitionistic logic is better known for rejecting the Law than it is for rejecting DN. The Law can also be formulated as a rule, namely, that at any line in a proof one may write down any substitution-instance of ⌜*p* ∨ ~*p*⌝, labeling the line 'LEM' and writing *no* numbers on its left (this is a special case of TI in NK). In view of the common fate of DN and the Law in intuitionistic logic, then, one might speculate that a system *S* equivalent to NK can be obtained by replacing DN in NK with the rule LEM. But this is not so. We have already seen how to prove the sequent corresponding to LEM in NK, but if *S* has only LEM in place of DN, there is no proof in *S* of the sequent corresponding to DN. Note in particular that it would be a mistake to offer the following proof:

| | | | |
|---|---|---|---|
| 1 | (1) | ~~A | Premise |
| | (2) | A ∨ ~A | LEM |
| 1 | (3) | A | 1,2 SI (DS) ◆ |

The problem here is that our proof of the sequent DS (Example 5.3 on page 112) was an NK-proof, not an *S*-proof, and used the rule DN, which is not available in *S*. However, if instead we define *S* to be the result of replacing DN in NK with the *two* rules EFQ and LEM, then we do obtain a system equivalent to NK.

*Example 3:* Let *S* be the system which results from replacing DN in NK with the rules EFQ and LEM. Then *S* is equivalent to NK.

The first condition for equivalence is met since we have already given NK-proofs of the sequents corresponding to EFQ and LEM (both used DN), namely, Examples 5.7 and 10.2. To show that the effect of DN can be obtained in *S* using SI we give the following *S*-proof of the sequent corresponding to DN, ~~A $\vdash_S$ A:

| | | | |
|---|---|---|---|
| 1 | (1) | ~~A | Premise |
| | (2) | A ∨ ~A | LEM |
| 3 | (3) | A | Assumption |
| 4 | (4) | ~A | Assumption |
| 1,4 | (5) | ⅄ | 1,4 ~E |
| 1,4 | (6) | A | 5 EFQ |
| 1 | (7) | A | 2,3,3,4,6 ∨E ◆ |

This proof again illustrates the standard way of using LEM, that is, in combination with ∨E.

In this section, then, we have seen that there is more than one way of formulating a natural deduction system for classical logic (more are described in the exercises). But the system NK has a certain naturalness as a formulation of the logic of our five connectives: each connective has an introduction and an elimination rule, and there is one extra rule, DN, which captures an aspect of the meaning of negation which is missed by ~E and ~I. Furthermore, DN is the simplest way of capturing this extra aspect, for other rules or combinations of rules with the same effect seem to be in some sense *justified* by the fact that double negations collapse, rather than this fact being a consequence of something else to which those other rules make more direct appeal. For example, we proved LEM by showing that we could *reject* the *denial* of LEM, but it is only against the background of DN that rejecting the denial of LEM amounts to endorsing LEM. Similarly, EFQ's various possible justifications all depend on DN—LEM by itself is insufficient. Nor is it plausible that EFQ needs no justification (in the way that &E might be said to need no justification) until it is made plausible that our understanding of '⋏' does not involve our understanding of negation. On the other hand, it is hard to believe that our proof of $\sim\sim A \vdash_S A$ using LEM and EFQ articulates a more fundamental justification for identifying the meaning of the rejection of the denial of *p* with the meaning of the assertion of *p*.

There is also a question about our formulation of the disjunction rules. Some readers may be familiar with systems of logic in which reasoning from a disjunction is handled by the rule known variously as Modus Tollendo Ponens (MTP) or Disjunctive Syllogism, described in Exercise 2 below, in place of our own more complicated rule of ∨E. Why prefer the more complicated rule, granted the result of Exercise 2 that a system with MTP in place of ∨E, but otherwise like NK, is equivalent to NK? The reason is that MTP essentially involves negation: it says that from a disjunction and the *negation* of one of its disjuncts, we may infer the other disjunct. But the logic of disjunction does not involve negation *essentially*. There is no reason at all why we should have to use negation rules to prove, say, $A \vee B \vdash B \vee A$, yet in a system *S* with MTP in place of ∨E, the negation rules would have to be used in proving this sequent (the reader may find it entertaining to construct such a proof). In fact, in NK any semantically correct sequent which contains no occurrences of negation, the conditional, or '⋏', can be proved without use of negation rules, whereas in the system *S* with MTP, many such sequents are unprovable without use of negation rules. This suggests that someone who regards a system with MTP in place of ∨E as providing the fundamental account of the logic of the sentential connectives is committed to the prediction that users of a language without negation would be unable to reason deductively from such premises as '$A \vee B$' to such conclusions as '$B \vee A$', or from '$(A \& B) \vee (A \& C)$' to 'A'. But it seems highly implausible that this is true. Rather, the pattern of reasoning from a disjunction encapsulated in ∨E is the basic one, and so we prefer that rule to superficially simpler ones because of its fundamental nature.

## ❑ Exercises

(1) The rule of *Modus Tollendo Tollens* is the following:

> ***Modus Tollendo Tollens:*** For any formulae $p$ and $q$, if $\ulcorner p \rightarrow q \urcorner$ occurs at a line j and $\ulcorner {\sim} q \urcorner$ occurs at a line k then $\ulcorner {\sim} p \urcorner$ may be inferred at line m, labeling the line 'j,k MTT' and writing on the left all numbers which appear on the left of line j and all which appear on the left of line k. We may have j > k or k > j. Schematically:

| | | | |
|---|---|---|---|
| $a_1,\ldots,a_n$ | (j) | $p \rightarrow q$ | |
| | ⋮ | | |
| $b_1,\ldots,b_u$ | (k) | ${\sim} q$ | |
| | ⋮ | | |
| $a_1,\ldots,a_n,b_1,\ldots,b_u$ | (m) | ${\sim} p$ | j,k MTT |

Show that the system *S* defined as the result of replacing →E in NK with MTT is equivalent to NK.

(2) The rule of *Modus Tollendo Ponens* is the following:

> ***Modus Tollendo Ponens:*** for any formulae $p$ and $q$, if $\ulcorner p \vee q \urcorner$ occurs at a line j and $\ulcorner {\sim} q \urcorner$ occurs at a line k then $p$ may be inferred at line m, labeling the line 'j,k MTP' and writing on the left all numbers which appear on the left of line j and all which appear on the left of k. Alternatively, if $\ulcorner {\sim} p \urcorner$ occurs at k, $q$ may be inferred at m with all the numbers from j and k on the left. Schematically:

| | | | |
|---|---|---|---|
| $a_1,\ldots,a_n$ | (j) | $p \vee q$ | |
| | ⋮ | | |
| $b_1,\ldots,b_u$ | (k) | ${\sim} q$ (or ${\sim} p$) | |
| | ⋮ | | |
| $a_1,\ldots,a_n,b_1,\ldots,b_u$ | (m) | $p$ (or $q$) | j,k MTP |

Show that the system *S* defined as the result of replacing ∨E in NK with MTP is equivalent to NK. (Refer to the discussion of systems without ∨E on page 134 for guidance.)

*(3) The rule of *Classical Reductio* is the following:

> ***Classical Reductio:*** If '$\curlywedge$' has been inferred at line k in a proof and $\{a_1,\ldots,a_n\}$ are the premise and assumption numbers $\curlywedge$ depends upon, then if $\ulcorner {\sim} p \urcorner$ is the formula at line j, $p$ may be inferred at line m, labeling the line 'j,k CR' and writing on its left the assumption numbers in $\{a_1,\ldots,a_n\}/\text{j}$. Schematically:

| | | | |
|---|---|---|---|
| j | (j) | $\sim p$ | Assumption |
| | ⋮ | | |
| $a_1,\ldots,a_n$ | (k) | $\curlywedge$ | |
| | ⋮ | | |
| $\{a_1,\ldots,a_n\}/j$ | (m) | $p$ | j,k CR |

As usual, j need not be in $\{a_1,\ldots,a_n\}$. (The difference between CR and ~I, therefore, is that CR can be used to reduce the number of negation symbols prefixing an assumption.) Show that the system *S* defined as the result of replacing ~I and DN in NK with the rule of Classical Reductio is equivalent to NK.

(4) The rule of *Nonconstructive Dilemma* is the following:

> ***Nonconstructive Dilemma:*** If *p* is assumed at line h, *q* is derived at line i, $\ulcorner\sim p\urcorner$ is assumed at line j and *q* is derived at line k, then at line m we may infer *q*, labeling the line 'h,i,j,k NCD' and writing on its left every number on the left of line i except h and every number on the left of k except j. Schematically:
>
> | | | | |
> |---|---|---|---|
> | h | (h) | $p$ | Assumption |
> | | ⋮ | | |
> | $a_1,\ldots,a_n$ | (i) | $q$ | |
> | | ⋮ | | |
> | j | (j) | $\sim p$ | Assumption |
> | | ⋮ | | |
> | $b_1,\ldots,b_u$ | (k) | $q$ | |
> | | ⋮ | | |
> | X | (m) | $q$ | h,i,j,k NCD |
>
> where X is the set $\{a_1,\ldots,a_n\}/h \cup \{b_1,\ldots,b_u\}/j$.

Show that the system *S* defined as the result of replacing DN in NK with the rules of Nonconstructive Dilemma and EFQ is equivalent to NK.

(5) The rule of *Negation-Definition* is the following:

> ***Negation-Definition:*** If $\ulcorner(p \rightarrow \curlywedge)\urcorner$ occurs as the entire formula at a line j, then at line k we may write $\ulcorner\sim p\urcorner$, labeling the line 'j, Df~' and writing on its left the same numbers as on the left of j. Conversely, if $\ulcorner\sim p\urcorner$ occurs as the entire formula at a line j, then at line k we may write $\ulcorner(p \rightarrow \curlywedge)\urcorner$, labeling the line 'j, Df~' and writing on its left the same numbers as on the left of j.

Show that the system *S* in which Df~ replaces ~E and ~I (but not DN) is equivalent to NK.

*(6) The rule of *Generalized Negation-Definition* is the following:

> ***Generalized Negation-Definition:*** If $\ulcorner(p \rightarrow \curlywedge)\urcorner$ occurs as a subformula of a formula $r$ at a line j, then at line k we may write the formula $s$, which is the same as $r$ except that it has $\ulcorner{\sim}p\urcorner$ in place of that occurrence of $\ulcorner(p \rightarrow \curlywedge)\urcorner$, labeling the line 'j, Df$\sim_{Gen}$' and writing on its left the same numbers as on the left of j. Conversely, if $\ulcorner{\sim}p\urcorner$ occurs as a subformula of a formula $r$ at a line j then at line k we may write the formula $s$, which is the same as $r$ except that $\ulcorner(p \rightarrow \curlywedge)\urcorner$ replaces that occurrence of $\ulcorner{\sim}p\urcorner$, labeling the line 'j, Df$\sim_{Gen}$' and writing on its left the same numbers as on the left of j.

Show that the system *S* in which Df$\sim_{Gen}$ replaces ~E and ~I (but not DN) is equivalent to NK. (One part of this problem is a trivial consequence of the result of Problem 5. The other part is hard.)

## 11 Semantic and deductive consequence compared

We now have before us two very different ways of explaining what it is for the conclusion of an argument to follow from its premises. According to our first account, the conclusion of an argument follows from its premises when no interpretation of the argument's form makes the premises true and the conclusion false. In this situation, we write $p_1,\ldots,p_n \vDash q$. According to our second account, the conclusion follows from the premises when the argument's form permits the derivation of the conclusion from the premises using the rules of inference of NK. In this situation, we write $p_1,\ldots,p_n \vdash_{\mathrm{NK}} q$. The second account says nothing about truth or interpretations, while the first says nothing about derivation or rules of inference. So it is not at all obvious how our two accounts are related.

What *is* obvious, however, is how we would *like* them to be related: we would like the two accounts to agree. In other words, (i) whenever the conclusion of a sequent can be derived from the sequent's premises in NK, we would like that conclusion also to be a semantic consequence of those premises. Conversely, (ii), whenever the conclusion of a sequent is a semantic consequence of the premises, we would also like to be able to derive it from those premises in NK. The desirability of both conditions can be brought out by an analogy. Suppose we have a lie detector which is designed to indicate when a subject is lying by flashing a green light to indicate truth-telling. Such a lie detector would be inadequate if *only* the following is true:

(1) If the green light flashes, then the subject is telling the truth

since (1) is consistent with the green light *failing* to flash when the subject is telling the truth, and so misleadingly implying a lie. But equally, the lie detector is inadequate if only the following is true:

(2) If the subject is telling the truth, then the green light flashes

since (2) is consistent with the green light flashing when the subject is lying—indeed, (2) is true if the green light constantly flashes, even when no one is attached to the machine, and (1) is true if the green light never flashes, no matter what the subject is saying. Evidently, what we require of a lie detector is that (1) and (2) *both* be true.

Analogously, it would be unsatisfactory if only the following is true:

(3) If $p_1,\ldots,p_n \vdash_{\mathrm{NK}} q$ then $p_1,\ldots,p_n \vDash q$

since this is consistent with there being many valid arguments which we *cannot* prove in NK. But equally, it would be unsatisfactory if only the following is true:

(4) If $p_1,\ldots,p_n \vDash q$ then $p_1,\ldots,p_n \vdash_{\mathrm{NK}} q$

since this is consistent with there being many invalid arguments which are provable. Ideally, what we require of our system of inference NK is that both (3) and (4) be true.

One way of looking at (3) and (4) is as expressing claims about the strength of the rules of NK. According to (4), the rules are *strong enough*, in that they allow us to prove every valid argument-form, while according to (3), the rules are *not too strong:* reading (3) contrapositively, it says that the rules will not permit a proof to be given of an invalid argument-form. The two properties of *sufficient* yet *not excessive* strength in a collection of rules are known respectively as the *completeness* and *soundness* of the rules. We have the following definitions:

> A system $S$ of rules of inference is said to be *sound* if and only if every $S$-provable argument-form is valid, that is, if and only if whenever $p_1,\ldots,p_n \vdash_S q$, then $p_1,\ldots,p_n \vDash q$.
>
> A system $S$ of rules of inference is said to be *complete* if and only if every valid argument-form is $S$-provable, that is, if and only if whenever $p_1,\ldots,p_n \vDash q$, then $p_1,\ldots,p_n \vdash_S q$.

Our analogy with the lie detector suggests that the definition of semantic consequence is the basic elucidation of the intuitive idea of a conclusion following from premises, and it is then up to us to find some collection $S$ of rules of inference (some machine) that is in agreement with the semantic criterion. But it should be emphasized that this is not the only possible perspective. There is an alternative view on which the rules of inference embody the basic way of capturing the meanings of the connectives, and it is then up to us to find a notion of semantic consequence that is in agreement with the rules (this was Gentzen's own perspective). If we adopt this perspective, the lie-detector analogy is misleading, for from this point of view it is the derivability of a conclusion from certain premises that *makes* the argument correct, but we would not want to say that it is the green light flashing that *makes* it the case that the subject is telling the truth.

But no matter which perspective we adopt, we require answers to the questions *is* NK sound, and *is* NK complete? The answer to both questions is yes, and there exist rigorous proofs of this, though ones beyond the scope of this book. However, we can see informally that it is quite plausible that NK is sound, since each rule by itself is perfectly acceptable. For example, if $\ulcorner p \,\&\, q \urcorner$ is a semantic consequence of some premises $\Gamma$, it is clear from the truth-table for '&' that each of the formulae we can infer from $\ulcorner p \,\&\, q \urcorner$ by &E is also a semantic consequence of $\Gamma$. Rules such as $\vee$E are more complex, but in the light of the truth-tables for the connectives, reflection indicates that they are semantically unobjectionable. Consequently, they could not be used to prove an invalid sequent. In this respect, the system NK contrasts with the system $S$ defined as the result of adding to NK the rule of (for example) Disjunction Reduction, which is the fallacious rule that given a disjunction $\ulcorner p \vee q \urcorner$ one may infer either disjunct, depending on the same premises and assumptions as $\ulcorner p \vee q \urcorner$. Obviously, there are invalid sequents which have $S$-proofs. The simplest example is that we can show $A \vee B \vdash_S A$ though of course $A \vee B \nvDash A$. Referring to the definition of 'sound', we conclude that $S$ as defined is an *unsound* system. And it is because none of NK's rules have the fallacious character of Disjunction Reduction that the soundness of NK seems evident.

Completeness is another matter, however. To say that NK is complete is to say that every valid argument-form can be proved in it. But while we can tell by inspecting the list of NK's rules that none are objectionable, we cannot tell by inspecting the list that we have *every rule we need* in order to provide proofs for all valid arguments. In other words, NK might be incomplete because there are certain valid arguments whose proof requires a rule we have forgotten about, and we cannot tell by inspecting the list of rules that this is not the case: the presence of a bad rule is perceptible, but not the absence of a good one.

However, NK is indeed complete (see Hodges), and we can use this fact to give examples of incomplete systems, exploiting the results of the previous section. First, the system $S$ which consists in NK without DN is incomplete, since although $\sim\sim A \vDash A$, we cannot show $\sim\sim A \vdash_S A$, since $S$ does not have DN or an equivalent rule such as Classical Reductio or a combination of rules such as EFQ/LEM. Similarly, replacing DN in NK with just EFQ or just LEM yields an incomplete system $S$, since we will still be unable to show $\sim\sim A \vdash_S A$. In general, then, if $S$ is a system which NK extends *non*conservatively, $S$ is incomplete.

Here are two final comments about completeness. First, we have previously mentioned a kind of logic called intuitionistic logic, for which a system of rules is obtained by replacing DN with EFQ in NK; this collection of rules, called 'NJ', does not suffice to prove the sequents corresponding to LEM and DN. This means that NJ is incomplete, and since we have said that completeness is a *desideratum* in a system of sentential logic, the reader may wonder what interest there can be in intuitionistic logic. The answer is that its interest lies in a perspective from which NJ is *not* incomplete. For intuitionists reject the classical definition of '$\vDash$' which we are implicitly invoking when we say NJ is incomplete, and which is based on the Principle of Bivalence. They have their own, completely different, way of defining '$\vDash$', and on *their* definition, NJ is complete—in particular, the sequents corresponding to LEM and DN are invalid (see Chapter 10). Just as we have been subscripting the single turnstile to indicate what system of rules is in question, so, according to intuitionists, we should subscript the *double* turnstile to indicate what definition of semantic consequence is in question. Suppose we use '$\vDash_I$' for the intuitionistic definition (whose details we do not need to know for the purposes of this discussion) and '$\vDash_C$' for the classical definition. Then although NJ is *classically incomplete*, since $\sim\sim A \vDash_C A$ while $\sim\sim A \nvdash_{NJ} A$, it is *intuitionistically complete* (hence $\sim\sim A \nvDash_I A$). Similarly, though NK is classically sound, it is intuitionistically unsound, since $\sim\sim A \vdash_{NK} A$ but $\sim\sim A \nvDash_I A$. So the notion of semantic consequence also has a kind of relativity in it, and the main area of dispute between classical and intuitionistic logic concerns the relative merits of the two definitions of semantic consequence (see Dummett). However, in most of this book we are only concerned with the classical account, so we will not bother to continue subscripting the double-turnstile.

The second point about completeness is that although it is a desirable property, it is less important than soundness: an unsound system is epistemically dangerous, since it could cause us to accept falsehoods on the basis of truths or even to accept implicit contradictions. When it is *possible* to formulate a sound and complete system of rules for a notion of semantic consequence, an incomplete system would be unsatisfactory. But if we had to choose *between* soundness and completeness, it would be better to choose soundness. And this choice is not hypothetical. Sentential logic is the simplest form of classical logic. The next simplest, to which the rest of this book is devoted, is classical first-order logic, for which we can also give sound and complete rules. But beyond first-order logic there lie classical 'higher-order' logics, for which it is provable that no set of rules can be both sound and complete (see Van Benthem and Doets). For the higher-order logics we have to content ourselves with sound systems of natural deduction only.

## ❑ Exercises

(1) Give your own example of a system *S* which is complete but unsound. Demonstrate its unsoundness by exhibiting a proof of an argument-form which you show to be invalid. (Compare our discussion of the fallacious rule of Disjunction Reduction.)

(2) Say that a set of sentences $\Sigma$ is *NK-consistent* if and only if $\Sigma \nvdash_{NK} \curlywedge$. Which of the following sets of sentences are consistent, which inconsistent? For each set that you judge inconsistent, give a proof which demonstrates its inconsistency.

(a) $\{A \vee B, \sim A \rightarrow \sim B, \sim A\}$ (b) $\{A \rightarrow (B \rightarrow C), \sim A, \sim B, \sim C\}$
(c) $\{A \vee B, \sim A \vee \sim B, A \leftrightarrow B\}$

(3) Say that a set of sentences $\Sigma$ is *satisfiable* if and only if there is an interpretation $\mathcal{I}$ on which every member of $\Sigma$ is true. Which of the following are satisfiable, which unsatisfiable? Explain your reasoning, and for each set that you judge to be satisfiable, give an interpretation which establishes its satisfiability.

(a) $\{\sim A \vee \sim B, B \rightarrow A\}$
(b) $\{(A \rightarrow A) \rightarrow B, \sim B\}$
(c) $\{\sim(A \rightarrow \sim A), \sim(B \rightarrow \sim B), \sim(A \& B)\}$

*(4) Let $\Sigma$ be a set of LSL sentences. Argue informally that the following is correct:

(a) $\Sigma \vdash_{NK} A$ if and only if $\Sigma,\sim A$ is inconsistent.

Here '$\Sigma,\sim A$' stands for the set of sentences whose members are '$\sim A$' and all the members of $\Sigma$. (Hint: for the left-to-right direction, show that if $\Sigma \vdash_{NK} A$ then $\Sigma,\sim A \vdash_{NK} \curlywedge$. To establish this conditional, assume that you have an NK-proof of A from premises in $\Sigma$. Describe how you would construct an NK-proof of '$\curlywedge$' from '$\sim A$' and premises in $\Sigma$. Alternatively, using natural-deduction rules in sequent-to-sequent versions, there is a three-line proof whose premise is '$\Sigma \vdash_{NK} A$' and whose conclusion is '$\Sigma,\sim A \vdash_{NK} \curlywedge$'. This is the left-to-right direction of (a). Now establish the right-to-left direction.)

(5) Let $\Sigma$ be a set of LSL sentences. Argue informally that the following is correct:

(a) $\Sigma \vDash A$ if and only if $\Sigma,\sim A$ is unsatisfiable.

(6) The completeness of NK is usually stated this way:

(Comp) If $\Sigma \vDash A$ then $\Sigma \vdash_{NK} A$.

Using Exercises 4 and 5 restate Comp in terms of consistency and satisfiability, simplifying the restatement as much as possible. (It is in this restated form that completeness results are normally proved, using a method discovered by Leon Henkin.)

## 12 Summary

- Every connective is governed by an introduction rule and an elimination rule, and there is an extra rule, known as DN, to remove double negations.

- A proof can be regarded either as a progression from formulae to formulae or as a progression from sequents to sequents.
- Sequents which have already been proved can be used again to shorten proofs of new sequents, using an abbreviatory technique called Sequent Introduction.
- There are other collections of rules which define systems that are equivalent to NK, in the sense that they have the same provable sequents as NK. There are also collections which define systems that are weaker than NK, in the sense that their provable sequents are a proper subset of NK's.
- NK is sound and complete with respect to the classical definition of '$\vDash$'.