Degenerate Fermi Gases with p-wave interactions

Victor Gurarie

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Most of these notes follow the reference V. Gurarie and L. Radzihovsky, Ann Phys 322, 2 (2007), arXiv:cond-mat/0611022.

1 Lecture 1: Narrow s-wave resonances

1.1 2-channel model

The two channel model is defined by its Hamiltonian

$$\hat{H} = \sum_{\mathbf{k},\sigma=\uparrow,\downarrow} \frac{k^2}{2m}\, \hat{a}^\dagger_{\mathbf{k},\sigma}\hat{a}_{\mathbf{k},\sigma} + \sum_{\mathbf{p}} \left(\epsilon_0 + \frac{p^2}{4m}\right) \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}} + \frac{g}{\sqrt{V}} \sum_{\mathbf{k},\mathbf{p}} \left(\hat{b}_{\mathbf{p}}\ \hat{a}^\dagger_{\mathbf{k}+\frac{\mathbf{p}}{2},\uparrow}\hat{a}^\dagger_{-\mathbf{k}+\frac{\mathbf{p}}{2},\downarrow} + \hat{b}^\dagger_{\mathbf{p}}\ \hat{a}_{-\mathbf{k}+\frac{\mathbf{p}}{2},\downarrow}\hat{a}_{\mathbf{k}+\frac{\mathbf{p}}{2},\uparrow}\right), \tag{1.1}$$

where V is the volume. It describes a physically appealing picture: fermions with spin which can glue together to form a boson with energy ϵ_0. g is the interconversion rate (or interaction strength).

The limit $g \to 0$ is very attractive theoretically. Indeed, these are now noninteractive bosons and fermions.

Let us understand this limit at zero temperature. Even though these are noninteracting particles, the fermion and boson numbers are not separately conserved. Only the combination

$$\hat{N} = \sum_{\mathbf{k},\sigma}\ \hat{a}^\dagger_{\mathbf{k},\sigma}\hat{a}_{\mathbf{k},\sigma} + 2\sum_{\mathbf{p}} \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}} \tag{1.2}$$

which is conserved.

In the spirit of the grand canonical ensemble we need to minimize

$$\hat{H} - \mu\hat{N} = \sum_{\mathbf{k},\sigma} \left(\frac{k^2}{2m} - \mu\right)\ \hat{a}^\dagger_{\mathbf{k},\sigma}\hat{a}_{\mathbf{k},\sigma} + \sum_{\mathbf{p}} \left(\epsilon_0 - 2\mu + \frac{p^2}{4m}\right) \hat{b}^\dagger_{\mathbf{p}}\hat{b}_{\mathbf{p}} \tag{1.3}$$

Define ϵ_F to be the Fermi energy of the fermions if bosons are not present. Then we find three regimes:

1) $\epsilon_0 > 2\epsilon_F$. No bosons, all particles are fermions. $\mu = \epsilon_F$.

2) $0 < \epsilon_0 < 2\epsilon_F$. Some particles are bosons, and some particles are fermions. The number of fermions is such that the fermion Fermi energy is $\epsilon_0/2$. Consequently, $\mu = \epsilon_0/2$.

3) $\epsilon_0 < 0$. All particles are bosons. $\mu = \epsilon_0/2 < 0$.

It goes without saying that the bosons, when present, form a noninteracting Bose condensate.

This is summarized on pictures shown on Figs. 1 and 2.

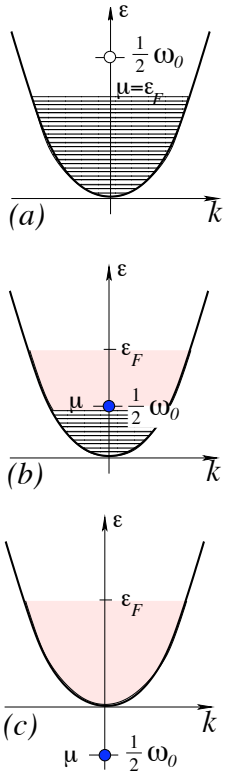

Figure 1: An illustration of the BCS-BEC crossover in the limit of a vanishing resonance width $g \to 0$. The evolution with detuning ϵ_0 is illustrated, with (a) the BCS regime of $\epsilon_0 > 2\epsilon_F$, where particles are all free atoms forming a Fermi sea, (b) the crossover regime of $0 < \epsilon_0 < 2\epsilon_F$, where a fraction of atoms between ϵ_0 and ϵ_F have converted into BEC of bosonic molecules, and (c) the BEC regime of $\epsilon_0 < 0$, where only Bose-condensed molecules are present.

1.2 The meaning of the model

The model, at nonzero g, looks transparent, but to understand the meaning of the interactions in this model, one needs to study how 2 fermions in vacuum would scatter. We compute the 2-body scattering amplitude. This is an exact calculation, not an approximation. We find

$$f(p) = \frac{1}{-\frac{1}{a} + r_0 \frac{p^2}{2} - ip}, \tag{1.4}$$

where

$$a = \frac{mg^2}{4\pi\omega_0}, \; r_0 = -\frac{8\pi}{m^2 g^2}, \; \omega_0 = \epsilon_0 - \frac{g^2 m\Lambda}{2\pi^2}. \tag{1.5}$$

Here Λ is the momentum cutoff (the upper limit of the summation in the interaction term of (1.1)), or in other words, the maximum momenta $\mathbf{k}$ and $-\mathbf{k}$ two fermions would have in

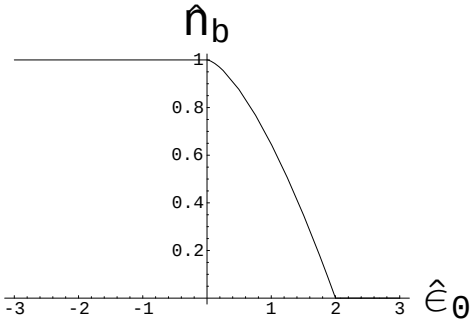

Figure 2: The normalized density of bosonic molecules $\hat{n}_b = 2n_b/n$, vs the normalized detuning $\hat{\epsilon}_0 = \epsilon_0/\epsilon_F$ in the limit of a vanishing resonance width, $g \to 0$.

order to still form a molecule. At higher relative momenta the fermions will pass through each other without interacting.

We see that the scattering length a turns into infinity when ω_0 is 0. This is the value of detuning at which bound states of fermions (molecules) are about to start to form. It corresponds to

$$a = \infty \ \rightarrow \ \epsilon_0 = \frac{g^2 m \Lambda}{2\pi^2}. \tag{1.6}$$

Thus the detuning is still very positive when bound states of fermions start to form. At lower value of ϵ_0 there are real bound states. This distinction between ω_0 and ϵ_0 is of course absent in the absence of interactions, such as in (1.3).

Notice that at weak interactions, $g \to 0$, $r_0 \to -\infty$.

To further understand the meaning of the interactions, it is useful to study the poles of this scattering amplitude. The analysis has been thoroughly discussed in the Ann. Phys. paper referred to on the front page of the notes.

1.3 Solution at nonzero but small g

Small g means the problem can be attacked by the mean field theory. We define a parameter

$$\gamma = \frac{g^2}{\sqrt{\epsilon_F}} \tag{1.7}$$

which has to be small for g to be small. This is equivalent to demanding that

$$|r_0| \sim \frac{1}{g^2} \gg \frac{1}{\sqrt{\epsilon_F}} \sim l, \tag{1.8}$$

where l is the mean separation between the particles and r_0 was defined in (1.5). Thus r_0 is large and negative.

To use mean field theory we assume that the bosons Bose condensed. We replace

$$\hat{b}_{\mathbf{p}} \to \sqrt{V} B \delta_{\mathbf{p},0} \tag{1.9}$$

where B is the condensate density. This gives us a reduced Hamiltonian

$$\hat{H}_{\text{reduced}} = \sum_{\mathbf{k},\sigma} \left(\frac{k^2}{2m} - \mu \right) \hat{a}^\dagger_{\mathbf{k},\sigma} \hat{a}_{\mathbf{k},\sigma} + (\epsilon_0 - 2\mu) B^* B + \sum_{\mathbf{k}} g \left(B \, \hat{a}^\dagger_{\mathbf{k},\uparrow} \hat{a}^\dagger_{-\mathbf{k},\downarrow} + B^* \, \hat{a}_{-\mathbf{k},\downarrow} \hat{a}_{\mathbf{k},\uparrow} \right), \tag{1.10}$$

One needs to minimize this at given B over all possible fermion configurations, and calculate the ground state energy $E(B)$. Then one needs to minimize $E(B)$ with respect to B. This is a standard procedure because (1.10) looks like the usual BCS Hamiltonian studied in superconductivity. Once $E(B)$ is found, the condition $dE/dB^* = 0$ becomes

$$\epsilon_0 - 2\mu = \frac{g^2}{2} \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{\left(\frac{p^2}{2m} - \mu \right)^2 + g^2 B^* B}} \tag{1.11}$$

This equation (called the gap equation in the theory of superconductivity) has two unknowns, μ and B (assuming that ϵ_0 is given). The second equation which relates them is the equation that $\left\langle \hat{N} \right\rangle = N$, where the operator $\hat{N}$ is introduced in (1.2), and N is the number of initial fermions in the systems. It can be shown that this equation takes the form

$$\frac{N}{V} = 2B^* B + \int \frac{d^3k}{(2\pi)^3} \left[1 - \frac{\frac{k^2}{2m} - \mu}{\sqrt{\left(\frac{p^2}{2m} - \mu \right)^2 + g^2 B^* B}} \right]. \tag{1.12}$$

Solving these two equations gives the boson density $n = B^* B$ as a function of ω_0. The solution looks very much like the one shown on Fig 2, except the "corners" have been rounded and ϵ_0 got replaced by ω_0. Now even if $\omega_0 > 2\epsilon_F$, there are some (very small) number of bosons present, while remaining fermions form a BCS superconductor, described by (1.10). Even at $\omega_0 < 0$, there are still some fermions present which also form a superconductor, however at negative chemical potential. μ approximately follows $\omega_0/2$, if $\omega_0 \leq 2\epsilon_F$, and $\mu = \epsilon_F$ if $\omega_0 > 2\epsilon_F$. This is just like in the noninteracting case, except μ follows ω_0 instead of ϵ_0 and it does so approximately (deviating from it by an amount which goes to 0 if g goes to 0) and not exactly.

Notice that this solution works for all ω_0, as long as g is small (or $|r_0| \gg l$). It works even when $a \to \infty$.

1.4 Large g or small r_0 limit

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If g is large, it can be shown that mean field theory still works at very large ω_0 where (1.1) describes a BCS superconductor, and at very large negative ω_0 where it describes a BEC of weakly interacting molecules. When ω_0 is close to zero, this problem is not solvable except numerically. To elucidate further this limit, we observe that physically large g means that $g \to \infty$, while a is kept fixed. In other words, according to (1.5), ϵ_0 should also be taken into infinity at the same time so that the ratio ϵ_0/g remains fixed. In this limit, the two channel model becomes equivalent to a one channel model. This can be shown, for example, in the following way. Let's write the equations of motion for $\hat{b}_{\mathbf{p}}$ which follow from (1.1)

$$i\dot{\hat{b}}_{\mathbf{p}} = [\hat{H}, \hat{b}_{\mathbf{p}}] = \epsilon_0 \hat{b}_{\mathbf{p}} + \frac{g}{\sqrt{V}} \sum_{\mathbf{k}} \hat{a}_{\frac{\mathbf{p}}{2}-\mathbf{k},\downarrow} \hat{a}_{\frac{\mathbf{p}}{2}+\mathbf{k},\uparrow} \tag{1.13}$$

At large g and large ϵ_0, the time derivate can be neglected, to give

$$\hat{b}_{\mathbf{p}} = -\frac{g}{\epsilon_0 \sqrt{V}} \sum_{\mathbf{k}} \hat{a}_{\frac{\mathbf{p}}{2}-\mathbf{k},\downarrow} \hat{a}_{\frac{\mathbf{p}}{2}+\mathbf{k},\uparrow}. \tag{1.14}$$

Substituting it back into the Hamiltonian, we find

$$\hat{H} = \sum_{\mathbf{k},\sigma=\uparrow,\downarrow} \frac{k^2}{2m}\, \hat{a}^\dagger_{\mathbf{k},\sigma} \hat{a}_{\mathbf{k},\sigma} - \frac{\lambda}{V} \sum_{\mathbf{p},\mathbf{k}} \hat{a}^\dagger_{\frac{\mathbf{p}}{2}+\mathbf{k},\uparrow} \hat{a}^\dagger_{\frac{\mathbf{p}}{2}-\mathbf{k},\downarrow} \hat{a}_{\frac{\mathbf{p}}{2}-\mathbf{k},\downarrow} \hat{a}_{\frac{\mathbf{p}}{2}+\mathbf{k},\uparrow} \tag{1.15}$$

Here

$$\lambda = \frac{g^2}{\epsilon_0}, \tag{1.16}$$

which remains constant in this limit, and can be expressed in terms of the scattering length a (given by (1.5)).

(1.15) is the Hamiltonian for a one channel model, describing the fermions with short ranged featureless attractive interactions. It describes the crossover between BCS and BEC, as the attractive interactions strength gets stronger. Thus this Hamiltonian can be solved in the far BCS limit (small λ), and far BEC limit (large λ), but hard to study in the intermediate regime. It is completely equivalent to the two-channel problem if g is large.

This can be summarized by the diagram shown on Fig 3. It shows when, depending on the values of the scattering length a, r_0 and the particle separation $l \sim n^{-1/3}$, the problem can be solved perturbatively and when it cannot.

Finally, we remark that the regime of large g is usually called *broad* resonance, while small g is *narrow* resonance. The origin of this term is the following: g controls the ratio

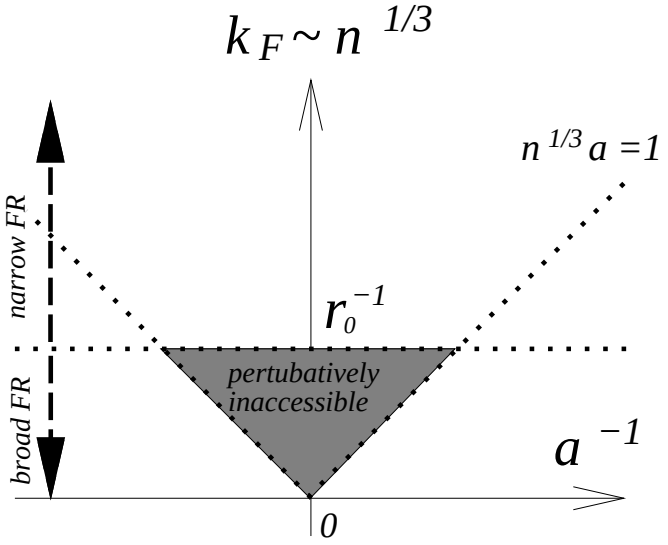

Figure 3: Illustration of perturbatively accessible and inaccessible (grey) regions in the inverse particle spacing vs inverse scattering length, $n^{1/3}$–a^{-1} plane around a Feshbach resonace, where a diverges. Note that outside the grey region, even for a broad Feshbach resonance there is a small parameter that is either the gas parameter or Feshbach resonance coupling, or both, and hence the system can be analyzed perturbatively.
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of a to ω_0, according to (1.5). The experimentalists tune ω_0 by changing magnetic field H, so that $\omega_0 \sim \mu_B \Delta H$. Here μ_B is the Bohr magneton, and ΔH is the deviation of the magnetic field from the "resonance". They then plot $1/a$ as a function of ΔH. The bigger g, the more slowly a^{-1} changes with changing ΔH or ω_0. Thus the plot of a^{-1} vs ΔH becomes more broad.

Typically most of the experiments are carried out in the regime of a broad resonance. One reason is that reducing the particle density always takes us into the regime of a broad resonance, corresponding to $l \gg |r_0|$. It is easy experimentally to reduce the density (while increasing it may lead to instabilities and is problematic).

2 Lecture 2: p-wave resonances

2.1 p-wave 2-channel model

If all fermions are identical (in the identical atomic or spin state), they cannot interact via s-wave interactions. The minimal coupling is p-wave, and the corresponding resonant

model takes form

$$\hat{H} = \sum_{\mathbf{k}} \frac{k^2}{2m}\, \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \sum_{\mathbf{p},\alpha} \left(\epsilon_0 + \frac{p^2}{4m} \right) \hat{b}^\dagger_{\mathbf{p},\alpha} \hat{b}_{\mathbf{p},\alpha} + \frac{g}{\sqrt{V}} \sum_{\mathbf{k},\mathbf{p}} k_\alpha \left(\hat{b}_{\mathbf{p},\alpha}\, \hat{a}^\dagger_{\mathbf{k}+\frac{\mathbf{p}}{2}} \hat{a}^\dagger_{-\mathbf{k}+\frac{\mathbf{p}}{2}} + \hat{b}^\dagger_{\mathbf{p},\alpha}\, \hat{a}_{-\mathbf{k}+\frac{\mathbf{p}}{2}} \hat{a}_{\mathbf{k}+\frac{\mathbf{p}}{2}} \right). \tag{2.1}$$

Now the molecuesl have structure, expressed by the index α in the creation operator $\hat{b}^\dagger$. They have angular momentum (spin) 1.

The interactions are weak when g is small. The corresponding dimensionless parameter is

$$\gamma = g^2 \sqrt{\epsilon_F} \ll 1. \tag{2.2}$$

This parameter is small at low density. Thus experiments (when they are done) will typically be in the regime of weak interactions, which can be understood using mean field theory.

Why would one want to study these p-wave systems?

1. The transition between BCS and BEC is a transition, not a crossover. Indeed, in the mean field theory we will replace $\hat{b}_{\mathbf{p},\alpha} \to \sqrt{V} B_\alpha \delta_{\mathbf{p},0}$ and plug it back into the Hamiltonian. The resulting superconductor Hamiltonian has a spectrum (this is derived in the BCS theory)

$$E = \sqrt{\left(\frac{p^2}{2m} - \mu \right)^2 + B_\alpha B^*_\beta\, p_\alpha p_\beta}. \tag{2.3}$$

If $\mu > 0$, we will have gapless excitations with the appropriate choice of p_α (such that $p^2/(2m) = \mu$ and $B_\alpha p_\alpha = 0$). If $\mu < 0$, there are no gapless excitations. Conclusion: there is a phase transition at $\mu = 0$. As we saw in the previous lecture, varying ϵ_0 changes μ and makes it go through zero, thus allowing to observe the transition.

2. The condensate density B_α is a vector. Choosing different directions of this vector chooses different phases of such p-wave condensate. Thus there is a possibility of observing different phases and transitions between them in addition to the BCS-BEC transition.

3. If we are in 2 dimensional space, and if $B_\alpha = u_\alpha + i v_\alpha$ where $v_\alpha u_\alpha = 0$, then this leads to a superfluid which is called a chiral p-wave 2D superconductor. Such a superfluid, if $\mu > 0$, has excitations which obey nonabelian statistics. Observing such excitations would be a milestone achievement.

Notice that in 2D, g is dimensionless, so in 2D it is not as straightforward to see if mean field theory works. It depends on how large g is regardless of particle separation.

2.2 The meaning of the model

As before, we need to compute the 2-body (two fermion) scattering amplitude to understand the meaning of the model. This is given by

$$f(p) = \frac{p^2}{-\frac{1}{v} + k_0 \frac{p^2}{2} - ip^3}. \tag{2.4}$$

Here

$$v^{-1} = -\frac{6\pi}{mg^2}(\epsilon_0 - c_1), \; k_0 = -\frac{12\pi}{m^2 g^2}(1 + c_2). \tag{2.5}$$

Here

$$c_1 = \frac{m}{9\pi^2}\Lambda^3 g^2, \; c_2 = \frac{m^2}{3\pi^2} g^2 \Lambda. \tag{2.6}$$

Here v is called the scattering volume and can be controlled via changing detuning ϵ_0 making it go through infinity in a manner similar to the scattering length in case of s-wave resonances. k_0 is a parameter equivalent to r_0 in the case of s-wave, but having dimensions of momentum.

We see that at small momenta ip^3 can be completely neglected, and the poles of the scattering amplitude are given by

$$\frac{p^2}{2m} = \frac{1}{mk_0 v}. \tag{2.7}$$

At positive v, this is the bound state (with negative energy) (remember that $k_0 < 0$!). At negative v, this is a resonance (a long lived bound state with positive energy, which eventually decays back into nonbound fermions). $-v^{-1} \sim \epsilon_0$, thus we tune between these two regimes with the detuning, as before.

2.3 Phases of the condensates

Once the condensation takes place, $\hat{b}_{\mathbf{p},\alpha} \to \sqrt{V} B_\alpha \delta_{\mathbf{p},0}$. B_α is a complex vector and can take two different forms.

We can always write

$$B_\alpha = u_\alpha + iv_\alpha \tag{2.8}$$

where u_α and v_α are two real vectors. Moreover, they can always be chosen orthogonal to each other (changing the angle between them is equivalent to multiplying B by a phase, which is not obvious, but can be checked by explicit algebra).

If $v = 0$, then

$$B_\alpha = u_\alpha \tag{2.9}$$

and is real. This is called the p_x phase of the condensate (or sometimes called the polar phase). This describes the molecules whose angular momentum has projection zero on the axis formed by u_α.

If $v \neq 0$, let us consider $u = v$ (their lengths coincide). Then

$$B_\alpha = u_\alpha + iv_\alpha \tag{2.10}$$

and can never be real. This is called the chiral, or ferromagnetic, or $p_x + ip_y$ phase of the superconductor. This describes molecules whose angular momentum has projection $+1$ on the axis parallel to $\vec{u} \times \vec{v}$.

By choosing u to point along the x-direction, and v along the y-direction, the two situations are described by

$$B_\alpha = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} : \ p_x, \ B_\alpha = \begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} : \ p_x + ip_y. \tag{2.11}$$

The important question is which one is realized in the 2-channel model.

2.4 Mean field theory and the choice of the phase

To understand which situation is realized in a system governed by the 2-channel model, we implement the mean field scheme, just as in the s-wave case (except in this case this scheme is reliable as long as the system is dilute enough). We substitute $\hat{b}_{\mathbf{p},\alpha} \to \sqrt{V} B_\alpha \delta_{\mathbf{p},0}$ to find by analogy with (1.10)

$$\hat{H}_{\text{reduced}} = \sum_{\mathbf{k}} \left(\frac{k^2}{2m} - \mu \right) \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}} + \sum_\alpha \left(\epsilon_0 - 2\mu + \frac{p^2}{4m} \right) B^*_\alpha B_\alpha + \frac{g}{\sqrt{V}} \sum_{\mathbf{k}} k_\alpha \left(B_\alpha \ \hat{a}^\dagger_{\mathbf{k}} \hat{a}^\dagger_{-\mathbf{k}} + B^*_\alpha \ \hat{a}_{-\mathbf{k}} \hat{a}_{\mathbf{k}} \right). \tag{2.12}$$

Here the conserved particle number is now

$$\hat{N} = \sum_{\mathbf{k}} a^\dagger_{\mathbf{k}} a_{\mathbf{k}} + \sum_{\mathbf{p},\alpha} b^\dagger_{\mathbf{p},\alpha} b_{\mathbf{p},\alpha}. \tag{2.13}$$

We now need to find the ground state energy $E(B_\alpha)$ and minimize it with respect to B_α. In contrast with the s-wave case, there could be now different minima and maxima of E, so one needs to look for a global minimum. A calculation to look for the ground state energy of (2.12) gives the following expression

$$\frac{E}{1 + c_2} = \left(u^2 + v^2 \right) \left[\omega_0 - 2\mu + a_1 \ln \left\{ a_0 \left(u + v \right) \right\} \right] + a_1 \frac{u^3 + v^3}{u + v} + a_2 \left[\left(u^2 + v^2 \right)^2 + \frac{1}{2} \left(u^2 - v^2 \right)^2 \right]. \tag{2.14}$$

Here it is assumed that $u_\alpha v_\alpha = 0$, that is, the two vectors are perpendicular, and u, v are their respective lengths. a_0, a_1, and a_2 are three coefficients which can be expressed in terms of g, ϵ_F, and μ, and whose explicit form is unimportant. The only important information about it is that a_1 is an increasing function of μ if $\mu > 0$. If $a_1 < 0$, then a_1 vanishes.

Now we distinguish two possible regimes. In the BCS regime, ω_0 is large positive. We expect that u and v are going to be small. If so, then the quartic terms with a_2 coefficient in front can be neglected. The large ω_0 will be compensated by the large logarithm if u and v are small. We need to minimize the term proportional to a_1. It can be shown that the minimum is achieved if $u = v$. Thus in the BCS regime $u = v$ and the phase is $p_x + ip_y$.

As ω_0 is decreased, μ decreases with it (recall the discussion at the beginning of Lecture 1. Then we enter the BCS regime. In this regime, a_1 becomes zero (when $\mu < 0$). Then the $\omega_0 - 2\mu$ term is balanced by the quartic term, while the terms proportional to a_1 vanish. Now we need to minimize the quartic terms proportional to a_2. Obviously this is achieved if $u = v$ (since that's when $u^2 - v^2 = 0$). Thus in the BEC regime the appropriate phase is also $p_x + ip_y$. One way to interpret it is to say that the interactions between the molecules, described by the quartic term, are ferromagnetic and their angular momenta want to align.

This shows that the BCS-BEC condensate described by the 2-channel model is always in the chiral $p_x + ip_y$ phase.

This argument goes through in 2D as well (assuming that the mean field theory applies). Thus when confined to 2D, this BCS-BEC condensate will form a chiral 2D p-wave superconductor, which has non-Abelian excitations at $\mu > 0$. The idea is now to tune μ by tuning ϵ_0 so that $\mu > 0$ but not too large so that to maximize the gap (and allow for stable quasiparticles with which to do quantum computations using their non-Abelian statistics).

2.5 Stability of the p-wave superfluid

This section follows the paper J. Levinsen, N. Cooper, V. Gurarie, PRL 99, 210402 (2007). Since it deals with estimates, this section features explicit $\hbar$, which was set to 1 previously.

The experiments done to try to create p-wave molecules showed that they are unstable, with the lifetime of the order of $2ms$. This so far prevented the experimentalists from creating a p-wave superfluid in the lab. Let us understand why this is so.

The main process which leads to decay of the molecules is the following. All of the

atoms interact with attractive but relatively short ranged van der Waals interactions which, at very short distances of $R_e \approx 2.5nm$, lead to formation of a strongly bound molecule with a very large binding energy (of the order of $\hbar^2/(mRe^2)$. Here $R_e \sim \hbar/\Lambda$, where Λ is the cutoff introduced earlier. These are not the same as the weakly bound molecules discussed here. Once these strongly bound molecules are formed, large amount of energy is released in the form of a nearby atom which flies out carrying away this energy. This leads to the destruction of the molecules nearby.

We can estimate the rate for this process in the following way. Suppose two weakly bound molecules collide and one of them becomes a strongly bound molecule while the other decays and its constituent atoms fly out carrying away the large binding energy of the strongly bound molecule. The rate of this process is given by

$$\Gamma = nv\sigma_{\rm in}. \tag{2.15}$$

Here n is the density, v is the molecular velocity, and $\sigma_{\rm in}$ is the inelastic cross section of such a collapse process. Suppose the weakly bound molecules have size l, and an average separation between the particles is also a (we are close to resonance where all distances are roughly the same). Then we can estimate the inelastic cross section by

$$\sigma_{\rm in} \sim l^{d-1} \frac{l}{v} \frac{\hbar}{mR_e^2} P. \tag{2.16}$$

Here l^{d-1} is the elastic cross section, d is the dimensionality of space, l/v is the time the molecules spend close by, and $\hbar/(mR_e^2)$ is a characteristic rate of the collapse (the only parameter of dimension of rate which depends on R_e only). Finally, P is the probability that as two molecules of size l collide, at least 3 of the four atoms which formed these two molecules will find themselves at a distance R_e to be able to carry out the collapse process.

Estimating P is not obvious since the atoms are all strongly interacting. If atoms were not interacting at all, then $P \sim (R_e/l)^{2d}$. Indeed, the probability that one atom approaches the other is $(R_e/l)^d$, and that has to be multiplied by the probability that the third atom approaches the first two. However, it was demonstrated by Petrov, Salomon, and Shlyapnikov, that strongly interacting fermions have a wave function which goes as $\psi \sim r^\gamma$, where r represents coordinates of all three particles (rescaling their coordinate by a factor of λ rescales the wave function by a factor of λ^γ). γ is an exponent which has to be calculated separately for each case. Then

$$P \sim \left(\frac{R_e}{l}\right)^{2d+2\gamma}. \tag{2.17}$$

This is obtained by integrating $|\psi|^2$ over the coordinates of the two particles with the third one being kept fixed. Putting it all together using $n \sim 1/l^d$ gives

$$\Gamma \sim \frac{\hbar}{ml^2} \left(\frac{R_e}{l}\right)^{2d+2\gamma-2}. \tag{2.18}$$

Noting that $\hbar/(ml^2)$ is of the order of $\epsilon_F/\hbar$, we find

$$\Gamma \sim \frac{\epsilon_F}{\hbar} \left(\frac{R_e}{l}\right)^{2d+2\gamma-2}. \tag{2.19}$$

In order to have enough time to observe the resonant superfluids before they decay, we need, at least, to have $\Gamma \ll \epsilon_F/\hbar$.

Now we distinguish several cases:

1. 3D s-wave ferimons. γ was calculated by Petrov, Salomon, and Shlyapnikov, to be $\gamma \approx -0.22$. This gives

$$\Gamma \sim \frac{\epsilon_F}{\hbar} \left(\frac{R_e}{l}\right)^{2d+2\gamma-2} = \frac{\epsilon_F}{\hbar} \left(\frac{R_e}{l}\right)^{3.56}. \tag{2.20}$$

Typically $R_e/l \sim 1/200$. This gives

$$\Gamma \sim \frac{\epsilon_F}{\hbar} \left(\frac{1}{200}\right)^{3.56} \ll \frac{\epsilon_F}{\hbar}. \tag{2.21}$$

The decay is indeed very slow. The system has plenty of time to be created and observed before it decays.

2. 3D s-wave bosons. γ was calculated by Efimov, to be $\gamma = -2$. Then

$$\Gamma \sim \frac{\epsilon_F}{\hbar}. \tag{2.22}$$

This is very fast. The system will decay before anything can be observed with it.

3. 3D p-wave fermions. Unlike the s-wave case, this needs to be rethought. The p-wave molecules are very small because their wave function behaves as $1/r^2$ (in general, the angular momentum s wave function goes as $1/r^{s+1}$ according to the standard results in quantum mechanics). Thus the probability of seeing atoms close by goes as $1/r^4$, which is not integrable over d^3r at small r. Thus the molecule has a typical size R_e, and not l. This leads to $\sigma_{\text{in}} \sim R_e^2 (R_e/v)\hbar/(mR_e^2)$ and

$$\Gamma = nv\sigma_{\text{in}} = \frac{\hbar}{ml^2}\frac{R_e}{l} \sim \frac{\epsilon_F}{\hbar}\frac{R_e}{l}. \tag{2.23}$$

This is better than 3D s-wave bosons, but not as good as $3D$ s-wave fermions. The decay rate can be estimated as (taking into account that $\epsilon_F/\hbar \sim 10\text{KHz}$ in a typical experiment)

$$\Gamma \sim 10\text{KHz} \cdot \frac{1}{200} \sim 1/(20ms). \tag{2.24}$$

Thus the lifetime of the molecules is expected to be 20ms, which is about 10 times larger than observed. Yet it is known that this formula tends to overestimate the lifetime, presumably due to some numerical factor of the order of 10 neglected in its derivation.

The origin of the relatively short lifetime of the p-wave molecules is the following. In the s-wave case, a molecule decays if a third atom approaches it (to take away its binding energy so that it could collapse into a deeply bound state). But the third atom is identical to one of the atoms which made up the molecule and the Pauli principle will prevent it from coming close.

In case of p-wave, a molecule has an angular momentum, say $+1$. A third atom will approach this molecule with the angular momentum -1, thus beating the Pauli principle. Then the molecule will collapse and the atom will carry away its energy.

Therefore, new fresh ideas are needed if a stable p-wave superfluid is to be created.