More transport in Fermi Gas — Calculation of physical quantities

- Start with Boltzmann equation in relaxation time approximation:

$$\frac{\partial g}{\partial t} + \dot{\vec{r}}\cdot\frac{\partial g}{\partial \vec{r}} + \dot{\vec{k}}\cdot\frac{\partial g}{\partial \vec{k}} = -\frac{1}{\tau}\left(g - g^{(0)}\right)$$

- Calculate conductivity: $\vec{\nabla} T = 0$, $\vec{B} = 0$, $\vec{E} \neq 0$.

$$\left.\begin{array}{l} \bullet\ \text{Expect no spatial inhomogeneity}: \dfrac{\partial g}{\partial \vec{r}} = 0 \\ \bullet\ \text{Look for steady state}: \dfrac{\partial g}{\partial t} = 0. \end{array}\right\} \Rightarrow g \text{ depends only on } \vec{k}.$$

Have:

$$-\frac{e}{\hbar}\vec{E}\cdot\frac{\partial g}{\partial \vec{k}} = -\frac{1}{\tau}\left(g - g^{(0)}\right)$$

→ • Approximation: Linearize in electric field:

$$-\frac{e}{\hbar}\vec{E}\cdot\frac{\partial g^{(0)}}{\partial \vec{k}} = -\frac{1}{\tau}\left(g - g^{(0)}\right)$$

$$\cancel{\Rightarrow -\frac{e}{\hbar}\vec{E}\cdot\frac{\partial g^{(0)}}{\partial \varepsilon}}$$

$$\Rightarrow -\frac{e}{\hbar}\left(\vec{E}\cdot\frac{\partial \varepsilon_{\vec{k}}}{\partial \vec{k}}\right)\overbrace{\frac{\partial g^{(0)}}{\partial \varepsilon}\bigg|_{\varepsilon=\varepsilon_{\vec{k}}}}^{=f'(\varepsilon_{\vec{k}})} = -\frac{1}{\tau}\left(g-g^{(0)}\right)$$

$$\Rightarrow \boxed{g-g^{(0)} = e\tau\left(\vec{E}\cdot\vec{v}_{\vec{k}}\right)f'(\varepsilon_{\vec{k}}) + \mathcal{O}(\vec{E}^2)}$$

Current density:

can insert this, since $\vec{J}$ vanishes in equilibrium

$$\vec{J} = 2\int\frac{d^3\vec{k}}{(2\pi)^3}(-e\vec{v}_{\vec{k}})\,g(\vec{k}) = 2\int\frac{d^3\vec{k}}{(2\pi)^3}(-e\vec{v}_{\vec{k}})\left(g(\vec{k})-g^{(0)}(\vec{k})\right)$$

$$= 2e^2\tau\int\frac{d^3\vec{k}}{(2\pi)^3}\,\vec{v}_k\left(\vec{E}\cdot\vec{v}_{\vec{k}}\right)\left[-f'(\varepsilon_{\vec{k}})\right]$$

Write: $J^i = \sigma^{ij}E^j$ (sum on $j = x, y, z$ implied)

$$\Rightarrow \sigma^{ij} = 2e^2\tau\int\frac{d^3\vec{k}}{(2\pi)^3}\,v^i_{\vec{k}}v^j_{\vec{k}}\left[-f'(\varepsilon_{\vec{k}})\right]$$

Zero temperature limit:

$$\sigma^{ij} = 2e^2\tau \int \frac{d^3\vec{k}}{(2\pi)^3}\, V^i_{\vec{k}} V^j_{\vec{k}}\, \delta(\varepsilon_{\vec{k}} - \varepsilon_F)$$

$$= 2e^2\tau \int \frac{d^3\vec{k}}{(2\pi)^3}\, \frac{\hbar^2}{m^2}\, k^i k^j\, \delta\left(\frac{\hbar^2 k^2}{2m} - \varepsilon_F\right)$$

$$\cancel{= \frac{4}{3}\frac{e^2\tau}{m}\delta_{ij} \int \frac{d^3\vec{k}}{(2\pi)^3}}$$

$$= \frac{2e^2\tau\,\delta_{ij}}{3} \int \frac{d^3\vec{k}}{(2\pi)^3}\, \frac{\hbar^2}{m^2}\, \vec{k}^2\, \delta\left(\frac{\hbar^2 k^2}{2m} - \varepsilon_F\right)$$

$$= \frac{ne^2\tau}{m}\,\delta_{ij}$$

- Can also express in terms of density of states ~~at~~ Fermi surface:

$$\sigma^{xx} = \cancel{\tfrac{4}{3}\,e^2\tau}\ \frac{2}{3}\frac{e^2\tau}{m}\,\varepsilon_F D(\varepsilon_F)$$

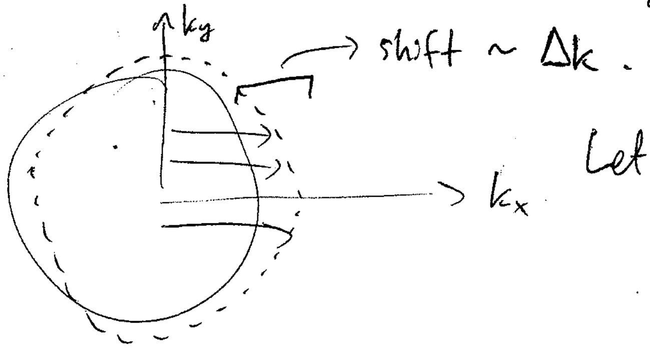
(Expression is True for free electron gas only)

- Typical resistivity of metal $\rho \sim 1\ \mu\Omega \cdot \text{cm}$

$$\Rightarrow \boxed{\tau \sim 10^{-13} - 10^{-14}\ \text{s}}$$

- Was linearizing in electric field a good approximation?

Distribution is shifted from its equilibrium value ...



Let's estimate Δk.

$$g(\vec{k}) = g^{(0)}(\vec{k}) + \left(\frac{\partial g^{(0)}}{\partial \vec{k}} \cdot \vec{E}\right)\frac{e\tau}{\hbar}$$

$$\sim g^{(0)}(\vec{k}) + \frac{\Delta g^{(0)}}{\Delta k} \cdot \Delta k$$

$$\Rightarrow \Delta k \sim \frac{e\tau|\vec{E}|}{\hbar} \approx 15\ \text{cm}^{-1} \quad \text{for}$$

$|\vec{E}| \sim 10^{-2}\ \text{V/cm}$ (a big field in a metal)

and $\tau \sim 10^{-12}\ \text{s}$

$$\frac{\Delta k}{k_F} \sim \frac{15\ \text{cm}^{-1}}{10^{8}\ \text{cm}^{-1}} \sim 1.5 \times 10^{-7} \longrightarrow$$ Very small shift! So linearizing is a good approximation.

Another perspective on transport: Drude model

- Consider average velocity $\vec{v}_{avg}$ of electrons.
- Drude model equation:

$$\boxed{\frac{d\vec{v}_{avg}}{dt} = -\frac{1}{\tau}\vec{v}_{avg} - \frac{e}{m}\left[\vec{E} + \frac{\vec{v}_{avg}}{c} \times \vec{B}\right]}$$

- τ is the characteristic time for velocity to decay

  ($\vec{v}_{avg}(t) = \vec{v}_{avg}(0)\, e^{-t/\tau}$ for zero external fields.)
- A & M derives this by defining the τ to be "mean free time" between collisions. (Chapter 1).
- Can also define mean-free path, the <u>distance</u> a typical electron travels between collisions. Since the velocity of a typical electron is $\sim v_F = p_F/m$, we have:

$$\ell = v_F \tau \sim 100\ \text{Å}$$

(for $v_F = 10^8$ cm/s & $\tau = 10^{-14}$ s — can be longer, or shorter)

- Leads to an important condition for validity of semiclassical transport: The wavelength λ of a typical electron should be much less than l, if we are to treat motion in between collisions classically. In fact if $\lambda \sim l$, it is not even obvious we should be thinking in terms of collisions as isolated events — the electron feels multiple impurities at a time. This all leads to the condition:

$$\boxed{k_F \, l \gg 1}$$

- For, say, room temperature Alkali or noble metals, roughly $k_F l \gtrsim 200$ from data in A&M. So semiclassical transport should be ok.

Current density in Drude model:

$$\vec{J} = -ne\vec{v}_{avg}$$

Drude conductivity:

Steady state $\Rightarrow \dfrac{d\vec{v}_{avg}}{dt} = 0$

$$\Rightarrow \vec{V}_{avg} = -\frac{e\tau}{m}\vec{E} \quad \Rightarrow \quad \vec{J} = \frac{ne^2\tau}{m}\vec{E}$$

$$\Rightarrow \sigma = \frac{ne^2\tau}{m} \longrightarrow$$ Same result as from Boltzmann equation!

Coincidence? **No!**

Derivation of Drude model from Boltzmann equation

~~Define $\vec{V}_{avg} = \int d^3k$~~

Define: $$\vec{V}_{avg}(\vec{r},t) = \frac{\int \frac{d^3\vec{k}}{4\pi^3}\, \vec{V}_k\, g(\vec{r},\vec{k},t)}{\int \frac{d^3\vec{k}}{4\pi^3}\, g(\vec{r},\vec{k},t)} = \frac{1}{n(\vec{r},t)} \int \frac{d^3k}{4\pi^3}\, \vec{V}_k\, g$$

Assume:
- Constant density $n(\vec{r},t) = n$
- Spatially homogeneous distribution: $\dfrac{\partial g}{\partial \vec{r}} = 0$.
- We shall linearize the Boltzmann equation in $\vec{E}$-field.

$$\frac{d}{dt}\vec{V}_{avg} = \frac{1}{n}\int \frac{d^3\vec{k}}{4\pi^3}\,\vec{V}_k\,\frac{\partial g}{\partial t}$$

$$= \frac{1}{n}\int \frac{d^3\vec{k}}{4\pi^3}\,\vec{V}_{\vec{k}}\left[-\frac{1}{\tau}\left(g - g^{(0)}\right) - \dot{\vec{k}}\cdot\frac{\partial g}{\partial \vec{k}}\right]$$

$$= -\frac{1}{\tau}\vec{V}_{avg} + \frac{e}{\hbar n}\int \frac{d^3\vec{k}}{4\pi^3}\,\vec{V}_k\left[\vec{E}\cdot\frac{\partial g^{(0)}}{\partial \vec{k}} + \left(\frac{\vec{V}_k}{c}\times\vec{B}\right)\cdot\frac{\partial g}{\partial \vec{k}}\right]$$

Evaluate:

$$\frac{e}{\hbar n}\int \frac{d^3\vec{k}}{4\pi^3}\,\vec{V}_k\left(\vec{E}\cdot\frac{\partial g^{(0)}}{\partial \vec{k}}\right)$$

$$= -\frac{eE_i}{\hbar n}\int \frac{d^3\vec{k}}{4\pi^3}\,\frac{\partial \vec{v}_k}{\partial k_i}\,g^{(0)}(\vec{k})$$ (integrate by parts)

$$= -\frac{e\cancel{\hbar}\vec{E}}{\cancel{\hbar}mn}\int \frac{d^3k}{4\pi^3}\,g^{(0)}(\vec{k}) = -\frac{e\vec{E}}{m}$$

Now evaluate:

$$\frac{e}{\hbar n c}\int \frac{d^3\vec{k}}{4\pi^3}\,\vec{V}_{\vec{k}}\left[\varepsilon_{ijl}(V_k)_i B_j \frac{\partial g}{\partial k_l}\right]$$

$$= -\frac{e}{\hbar n c}\int \frac{d^3\vec{k}}{4\pi^3}\, g \left\{ \frac{\partial \vec{V}_k}{\partial k_l}\left[\varepsilon_{ijl}(V_k)_i B_j\right] + \vec{V}_k \underbrace{\left[\cancel{\varepsilon_{ijl}\frac{\partial (V_k)^i}{\partial k_l} B_j}\right]}_{=\frac{1}{\hbar}\varepsilon_{ijl}\frac{\partial \varepsilon_k}{\partial k_i \partial k_l} B_j = 0} \right\}$$

$$= -\frac{e\cancel{\hbar}}{\cancel{\hbar} n c m}\int \frac{d^3\vec{k}}{4\pi^3}\left(\vec{V}_k \times \vec{B}\right) g(\vec{k})$$

$$= -\frac{e\cancel{n}}{mc}\,\vec{V}_{avg}\times\vec{B}$$

Putting it together:

$$\boxed{\frac{d}{dt}\vec{V}_{avg} = -\frac{1}{\tau}\vec{V}_{avg} - \frac{e}{m}\left(\vec{E} + \frac{1}{c}\vec{V}_{avg}\times\vec{B}\right)}$$

Transcport in a magnetic field

- Let $\vec{B} = B\hat{z}$; $\vec{E} = E_x\hat{x} + E_y\hat{y}$
- Look for steady state solution $\dfrac{d\vec{v}_{avg}}{dt} = 0$.
- Solution of form $\vec{v}_{avg} = V_x\hat{x} + V_y\hat{y}$. ($V_z = 0$, since that component will decay)
- Using $J_x = -nev_x$, $J_y = -nev_y$, result:

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} \equiv \overleftrightarrow{\sigma} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

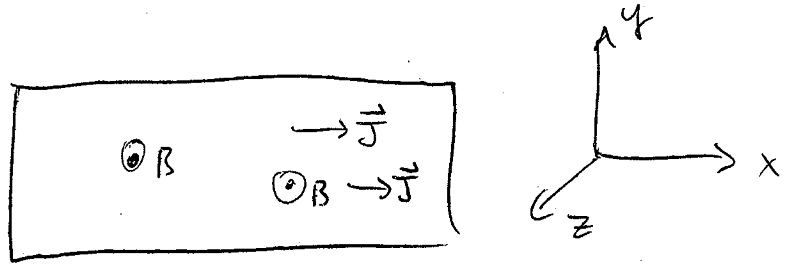
$$\sigma_{xx} = \sigma_{xy} = \frac{1}{1+(\omega_c\tau)^2}\,\frac{ne^2\tau}{m}$$

$\omega_c = \dfrac{eB}{mc}$ is "cyclotron frequency"! (It's the frequency of the circular motion of a free electron in a uniform magnetic field.)

$$\sigma_{xy} = -\sigma_{yx} = -\frac{(\omega_c\tau)}{1+(\omega_c\tau)^2}\,\frac{ne^2\tau}{m}$$

~~Invert to ge~~

Typical geometry:



• Fix $\vec{J}$ to flow in x-direction. Would like to know $\vec{E}$ given $\vec{J}$ ..

Invert to get resistivity:

$$\begin{pmatrix} E_x \\ E_y \end{pmatrix} = \overleftrightarrow{\rho} \begin{pmatrix} J_x \\ J_y \end{pmatrix} \qquad \overleftrightarrow{\rho} = \begin{pmatrix} \rho_{xx} & \rho_{xy} \\ \rho_{yx} & \rho_{yy} \end{pmatrix}$$

$$\rho_{xx} = \rho_{yy} = \frac{m}{ne^2\tau}$$ $\longrightarrow$ Same as in $B=0$. No "magnetoresistance" in free electron gas.

$$\rho_{yx} = -\rho_{xy} = -\frac{B}{nec}$$ $\longrightarrow$ Hall resistivity.

Hall coefficient: $R_H \equiv \dfrac{\rho_{yx}}{B} = -\dfrac{1}{nec}$

- In principle R_H gives a direct measure of electron density n, <u>within our free electron gas theory</u>.

- But in <u>some</u> materials, $R_H > 0$. For others $R_H < 0$ and consistent (mostly) with n (Alkali metals). For others $R_H < 0$ but only right order of magnitude. Also, in general R_H depends on magnetic field — values quoted in A&M are for large B. — this dependence not explained by free electron gas.

- <u>Other problems</u>: In general, ρ_{xx} does depend on B. (Magnetoresistance)

- <u>Big</u> problem: $\rho_{xx} = \dfrac{m}{ne^2\tau}$ expected to be finite even at $T=0$ (electrons are still scattered at $T=0$). For an insulator, need $I=0$ at $T=0$ $\rightarrow$ <u>infinite</u> scattering rate ... unphysical. We're therefore lacking a theory of insulating behavior...