Transport of Band Electrons in Magnetic Field

• To start, let's understand the geometry of motion in both $\vec{r}$-space and $\vec{k}$-space. Next, we will discuss the Hall effect.

EOM: $\rightarrow$ Single band.

$$\dot{\vec{r}} = \vec{v}(\vec{k}) = \frac{1}{\hbar}\frac{\partial \varepsilon}{\partial \vec{k}}$$

$$\hbar \dot{\vec{k}} = -e\vec{E} - \frac{e}{c}\vec{v}\times\vec{B}$$

Write: $\vec{B} = B\hat{B}$ $\rightarrow$ unit vector, and consider:

$$\hat{B}\times\hbar\dot{\vec{k}} = -\frac{eB}{c}\left[\vec{v} - \hat{B}(\vec{v}\cdot\hat{B})\right] - e\hat{B}\times\vec{E}$$

$$\equiv -\frac{eB}{c}\dot{\vec{r}}_\perp + \frac{eB}{c}\vec{w}$$

Here: $\dot{\vec{r}}_\perp$ is component of $\dot{\vec{r}}$ $\perp$ to $\vec{B}$,

and $\vec{w} \equiv c\frac{E}{B}\hat{E}\times\hat{B}$ $\quad (\vec{E} = E\hat{E})$.

~~Solve for~~

Solve for $\dot{\vec{r}}_\perp$ and ~~th~~ integrate with respect to time, to get:

$$\cancel{\vec{r}_\perp(t) = \vec{r}_\perp(0) =}$$

$$\boxed{\vec{r}_\perp(t) = \vec{r}_\perp(0) - \frac{\hbar c}{eB}\left(\hat{B} \times \left(\vec{k}(t) - \vec{k}(0)\right)\right) + \vec{w}\, t}$$

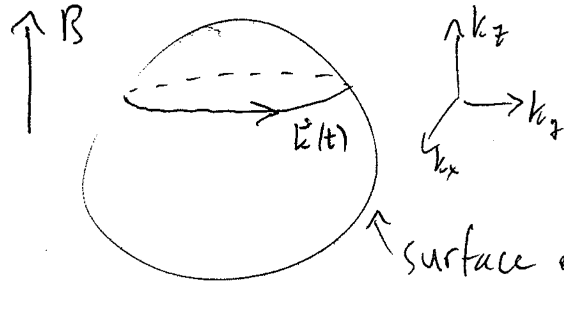
- When $\vec{E} = 0$, $\vec{w} = 0$ and the r-space orbit is given simply in terms of k-space orbit.
- For $\vec{E} \neq 0$, $\vec{w}$ is a <u>drift velocity</u>.

<u>Now, focus on $\vec{E} = 0$</u>

Recall free electrons, then $\vec{k}(t)$ has a circular orbit.

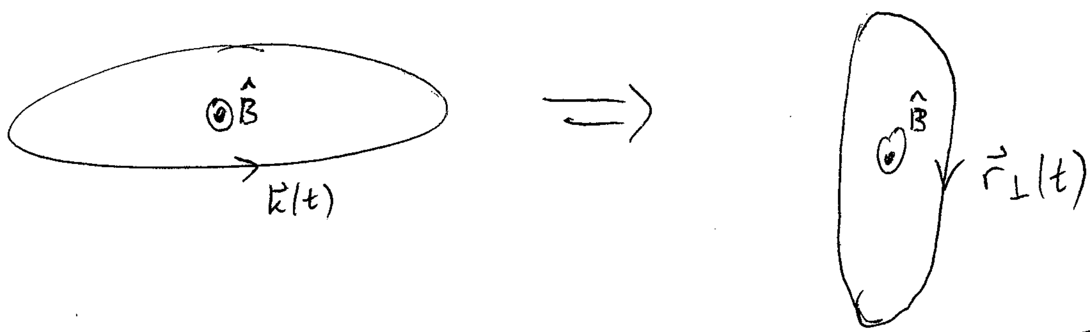
$$\hbar \dot{\vec{k}} = -\frac{e}{\hbar c} \frac{\partial \varepsilon}{\partial \vec{k}} \times \vec{B} \implies$$

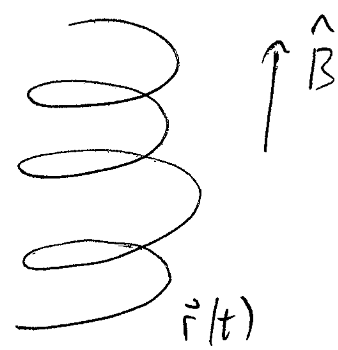
$\vec{k}(t)$ lies on surface of constant energy, $\perp$ to $\vec{B}$



surface of energy ε

- In this case, expression for $\vec{r}_\perp(t)$ tells us that ~~Thus~~ the $\vec{r}_\perp$-orbit is just the $\vec{k}$-space orbit, rotated 90° about the $\hat{B}$-axis





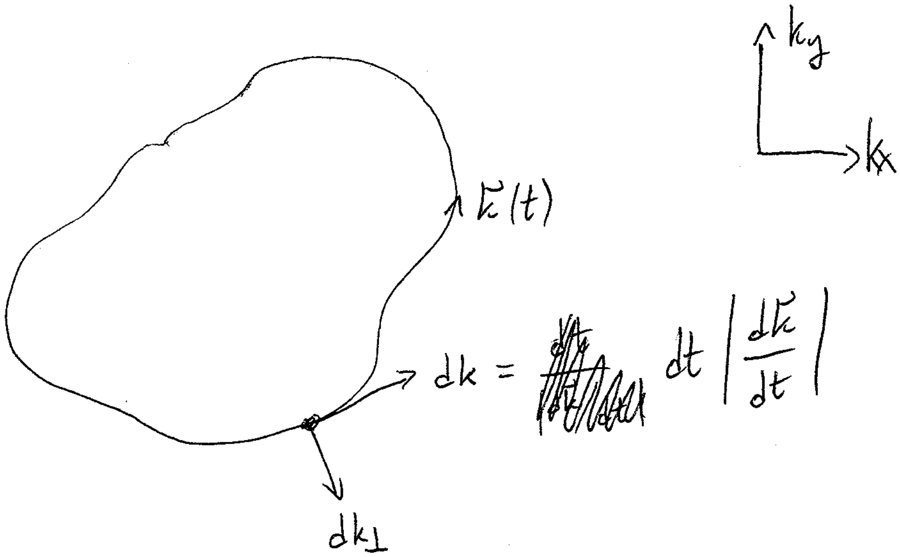
- In general, trajectory is roughly a helix. However, motion along the $\hat{B}$-direction is more complicated than for free electrons, since $\vec{v}(\vec{k})\cdot\hat{B}$ is not a constant.

- If we consider the projection into the plane $\perp$ to $\vec{B}$, the motion is periodic.

- What's the period T?

  - Will be important for a variety of effects that can directly measure the shape of the Fermi surface. We'll cover these soon.

Look at $\vec{k}$-space orbit, in plane $\perp$ to $\vec{B}$: $\longrightarrow$ Assume $\vec{B} \parallel \hat{z}$.



$$T = \int dt = \int \frac{dk}{|d\vec{k}/dt|} = \frac{\hbar^2 c}{eB} \int \frac{dk\, dk_\perp}{d\varepsilon}$$

$$\left|\frac{d\vec{k}}{dt}\right| = \frac{e}{\hbar^2 c} \left| \frac{\partial \varepsilon}{\partial \vec{k}} \times \vec{B} \right| = \frac{eB}{\hbar^2 c} \left| \overbrace{\left(\frac{\partial \varepsilon}{\partial \vec{k}}\right)_\perp}^{\text{Component } \perp \text{ to } \vec{B}} \right| = \frac{eB}{\hbar^2 c} \left| \frac{d\varepsilon}{dk_\perp} \right|$$

Now, $\int dk\, dk_\perp = dA \longrightarrow$

dA is area of this region.

$$\Rightarrow \boxed{T = \frac{\hbar^2 c}{eB}\frac{dA}{d\varepsilon} = \frac{\hbar^2 c}{eB}\frac{\partial}{\partial \varepsilon}A(\varepsilon, k_z)}$$

derivative taken w/ k_z held fixed.

- Nice geometrical result → will be useful later.

For free electrons, recall $T = \frac{2\pi}{\omega_c} = \frac{2\pi m c}{eB}$

ω_c ↳ "cyclotron frequency"

Often one writes, $T = \frac{2\pi m^* c}{eB}$, defining

"cyclotron effective mass" $m^*(\varepsilon, k_z) = \frac{\hbar^2}{2\pi}\frac{\partial}{\partial \varepsilon}A(\varepsilon, k_z)$

(NB: In general this is not same as other effective masses.)

Hall Effect (in large $\vec{B}$-field)

Suppose $\vec{E} \perp \vec{B}$, then ...

$$\hbar \dot{\vec{k}} = -e\left[\vec{E} + \frac{1}{\hbar c}\frac{\partial \varepsilon}{\partial \vec{k}} \times \vec{B}\right]$$

Can write: $\underbrace{\vec{E} = -\frac{1}{c}\vec{w} \times \vec{B}}_{\text{(Only true for } \vec{E} \perp \vec{B}.)}$, where $\vec{w}$ is drift velocity defined earlier.

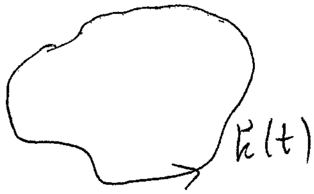
$$\Rightarrow \hbar \dot{\vec{k}} = -\frac{e}{\hbar c}\left[\frac{\partial \varepsilon}{\partial \vec{k}} - \hbar \vec{w}\right] \times \vec{B}$$

$$= -\frac{e}{\hbar c}\frac{\partial \bar{\varepsilon}}{\partial \vec{k}} \times \vec{B}, \quad \text{where } \bar{\varepsilon}(\vec{k}) = \varepsilon(\vec{k}) - \hbar \vec{w} \cdot \vec{k}.$$

$\Rightarrow$ $\vec{k}$-space orbit is same as for $\vec{E} = 0$, but with new energy $\bar{\varepsilon}(\vec{k})$. $\rightarrow$ Moves on constant $\bar{\varepsilon}$-surface, $\perp$ to $\vec{B}$.

- For small $\vec{E}$, $\vec{w}$ is small, $\varepsilon \approx \bar{\varepsilon}$. $\Rightarrow$ Expect $\vec{k}$-space orbits aren't changed *much*

In particular, if $\vec{k}(t)$ is a closed curve like



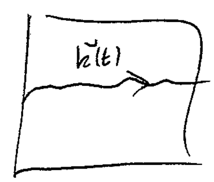
when $\vec{E}=0$, expect it will still be for $\vec{E}\neq 0$.
(For generic orbits.)

- We found before:

$$\vec{r}_\perp(t) = \vec{r}_\perp(0) - \frac{\hbar c}{eB}\hat{B}\times\left(\vec{k}(t)-\vec{k}(0)\right) + \vec{w}t.$$

- Suppose $\vec{k}(t)$ is a closed orbit as above,

NB:



Open orbits also possible → leads to "open" trajectories in $\vec{r}$-space.

then over times long compared to oscillation period T,

$$\langle\vec{r}_\perp(t)\rangle \approx \vec{r}_\perp(0) + \vec{w}t \Rightarrow \langle\vec{v}_\perp\rangle \approx \vec{w}$$

- If we include scattering, need $T \ll \tau$ for this to hold.

Since $T \sim \frac{2\pi}{\omega_c}$, roughly need $\omega_c\tau \gg 1 \Rightarrow$ Large B-field.

- Suppose our band has all occupied orbits <u>closed</u>, then what's the current?

$$\vec{J} = -e\int \frac{d^3k}{4\pi^3}\,\vec{v}(\vec{k})\,g(\vec{k})$$

$$= -en\,\frac{\int \frac{d^3k}{4\pi^3}\,\vec{v}(\vec{k})\,g(\vec{k})}{\underbrace{\int \frac{d^3k}{4\pi^3}\,g(\vec{k})}_{=n,\text{ density of electrons in the band}}} = -ne\langle\vec{v}\rangle_{\text{occupied}} = -ne\langle\vec{v}_\perp\rangle_{\text{occupied}} = -ne\vec{w}$$

- This translates into: $\boxed{R_H = -\dfrac{1}{nec}} \to$ Same as free electron result!

- Next, suppose all <u>empty</u> orbits are closed, then:

$$\vec{J} = +e\int \frac{d^3k}{4\pi^3}\,\vec{v}(\vec{k})\,[1-g(\vec{k})] = +en_h\,\frac{\int \frac{d^3k}{4\pi^3}\,\vec{v}(\vec{k})\,[1-g(\vec{k})]}{\underbrace{\int \frac{d^3k}{4\pi^3}\,[1-g(\vec{k})]}_{=n_h,\text{ density of holes}}}$$

$$= +n_h e\langle\vec{v}\rangle_{\text{empty}} = +n_h e\langle\vec{v}_\perp\rangle_{\text{empty}} = +n_h e\vec{w}.$$

→ This winds up giving: $\boxed{R_H = +\frac{1}{n_h e c}}$

• If band ~~[illegible]~~ doesn't satisfy these conditions (e.g., it has both empty and occupied orbits that are <u>open</u>), then form of R_H is <u>not</u> ~~[illegible]~~ simple.

Suppose many bands contribute to $\vec{J}$, all satisfying our closed orbit conditions. Then their contributions to $\vec{J}$ add together, and $R_H = -\frac{1}{n_{eff} e c}$, n_{eff} is total density (with holes counted negative).

Question: What about $\vec{J} \parallel \vec{E}$? (Longitudinal resistivity.)

- We're working in large-B limit, so expect

$$\vec{J} = \underbrace{\vec{J}_{\perp E}}_{\text{found} \sim \frac{1}{B}} + \vec{J}_{\parallel E}$$

Did not find in large-B limit, so expect $\sim \frac{1}{B^2}$.

- Thus gives $\sigma \sim \begin{pmatrix} a/B^2 & b/B \\ -b/B & a/B^2 \end{pmatrix}$

invert

$$\Rightarrow \rho \sim \begin{pmatrix} a/b^2 & -B/b \\ \underbrace{B/b}_{= R_H \cdot B} & a/b^2 \end{pmatrix}$$

$\rightarrow \rho_{xx}, \rho_{yy}$ saturate to B-independent value