Latest on Varieties of $\ell$-Groups, Unital $\ell$-Groups, and Related Things

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University of Colorado

Boulder 2013
History
Formal Logic

1847  Boolean Algebra

History
Formal Logic

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~1900  Quantum things
Formal Logic

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1917  Łukasiewicz Multi-Valued Logic
Formal Logic

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1917  Łukasiewicz Multi-Valued Logic

and  MV-Algebra

1958  Chang Completeness Theorem
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MV-Algebras ↔ Abelian Unital Lattice-Ordered Groups

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ΨMV-Algebras ↔ Unital Lattice-Ordered Groups

~ 2001

Pseudo Multi Valued
Lattice-ordered group $G$: $G$ is a group and a lattice, and for all $x, y, z \in G$

$$x(y \lor z) = (xy) \lor (xz) \text{ and } (y \lor z)x = (yx) \lor (zx)$$

($\ell$-group $G$)
Lattice-ordered group $G$: $G$ is a group and a lattice, and for all $x, y, z \in G$,

$$x(y \lor z) = (xy) \lor (xz) \quad \text{and} \quad (y \lor z)x = (yx) \lor (zx) \quad (\ell\text{-group } G)$$

Submitted papers to be discussed:

1. Darnel & Holland, More covers of the boolean variety of unital $\ell$-groups.
   (accepted by Algebra Universalis)

2. Darnel & Holland, Minimal non-metabelian varieties of $\ell$-groups which contain no nonabelian o-groups.
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1. **Unital lattice-ordered group $(G, u)$ (u\l-group):** $\ell$-group $G$
   With a chosen unit $u$ such that $e \leq u \in G$, and
   \[
   \forall g \in G, \exists n \in \mathbb{N}, u^{-n} \leq g \leq u^n
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1. **Unital lattice-ordered group** $(G, u)$ ($ul$-group): $\ell$-group $G$

   With a chosen unit $u$ such that $e \leq u \in G$, and

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   Varieties of $ul$-groups
Lattice-ordered group $G$: $G$ is a group and a lattice, and for all $x, y, z \in G$
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Lattice-ordered group $G$: $G$ is a group and a lattice, and for all $x, y, z \in G$

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Varieties of $u\ell$-groups ( = defined by a set of equations)

The unital $\ell$-groups in any variety of $\ell$-groups, and much more.
Lattice-ordered group $G$: $G$ is a group and a lattice, and for all $x, y, z \in G$

$x(y \lor z) = (xy) \lor (xz)$ and $(y \lor z)x = (yx) \lor (zx)$ \hspace{1cm} ($\ell$-group $G$)

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1. **Unital lattice-ordered group** $(G, u)$ (ul-group): $\ell$-group $G$

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   Varieties of ul-groups ($= \text{defined by a set of equations}$)

The unital $\ell$-groups in any variety of $\ell$-groups, and much more.

Examples: $\forall x$
Lattice-ordered group \( G \): \( G \) is a group and a lattice, and for all \( x, y, z \in G \)
\[ x(y \lor z) = (xy) \lor (xz) \quad \text{and} \quad (y \lor z)x = (yx) \lor (zx) \quad (\ell\text{-group } G) \]

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\[ \boxed{1. \quad \text{Unital lattice-ordered group } (G, u) \, (ul\text{-group})} \quad \ell\text{-group } G \]

With a chosen unit \( u \) such that \( e \leq u \in G \), and
\[ \forall g \in G, \exists n \in \mathbb{N}, u^{-n} \leq g \leq u^n \]

Varieties of \( ul \)-groups ( = defined by a set of equations)
The unital \( \ell \)-groups in any variety of \( \ell \)-groups, and much more.

Examples: \( \forall x \quad C : xu = ux \)
Lattice-ordered group \( G \): \( G \) is a group and a lattice, and for all \( x, y, z \in G \):
\[
x(y \lor z) = (xy) \lor (xz) \quad \text{and} \quad (y \lor z)x = (yx) \lor (zx)
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(\( \ell \)-group \( G \))

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   With a chosen unit \( u \) such that \( e \leq u \in G \), and  
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   Varieties of ul-groups ( = defined by a set of equations)
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Examples:
\[
\forall x \quad C: xu = ux \\
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Lattice-ordered group $G$: $G$ is a group and a lattice, and For all $x, y, z \in G$

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x(y \lor z) = (xy) \lor (xz) \text{ and } (y \lor z)x = (yx) \lor (zx)
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   Varieties of ul-groups (= defined by a set of equations)

The unital $\ell$-groups in any variety of $\ell$-groups, and much more.

Examples: \[ \forall x \]

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C : xu = ux
\]

\[
C_n : xu^n = u^n x
\]

\[
C_m \subseteq C_n \iff m | n
\]
Lattice-ordered group $G$: $G$ is a group and a lattice, and for all $x, y, z \in G$

$$x(y \vee z) = (xy) \vee (xz) \quad \text{and} \quad (y \vee z)x = (yx) \vee (zx) \quad (\ell\text{-group } G)$$

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   Varieties of $u\ell$-groups ($\equiv$ defined by a set of equations)

   The unital $\ell$-groups in any variety of $\ell$-groups, and much more.

   Examples: $\forall x$

   $$\mathcal{C} : xu = ux$$

   $$\mathcal{C}_n : xu^n = u^n x$$

   $$\mathcal{C}_m \subseteq \mathcal{C}_n \iff m | n$$

   $$\bigvee_{n \in \mathbb{N}} \mathcal{C}_n = \text{all } u\ell\text{-groups}$$
Lattice-ordered group $G$: $G$ is a group and a lattice, and For all $x, y, z \in G$
\[ x(y \vee z) = (xy) \vee (xz) \text{ and } (y \vee z)x = (yx) \vee (zx) \] (\text{$\ell$-group $G$})

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   The unital $\ell$-groups in any variety of $\ell$-groups, and much more.

   **Examples:**  \[ \forall x \]
   \[ \mathcal{C} : xu = ux \]
   \[ \mathcal{C}_n : xu^n = u^n x \]
   \[ \mathcal{C}_m \subseteq \mathcal{C}_n \iff m | n \]
   \[ \bigvee_{n \in \mathbb{N}} \mathcal{C}_n = \text{all ul-group} \]

   Boolean $\mathcal{B} = \text{Var}(\mathbb{Z}, 1) = \text{smallest proper variety.}$
Varieties of \( u \)-groups

\[
x(y \lor z) = (xy) \lor (xz) \quad \text{and} \quad (y \lor z)x = (yx) \lor (zx) \quad (\ell\text{-group } G)
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\mathcal{C}_m \subseteq \mathcal{C}_n \Leftrightarrow m | n
\]

\[
\lor n \in \mathbb{N} \mathcal{C}_n = \text{all } ul\text{-groups}
\]

Boolean \( B = \text{Var}(\mathbb{Z}, 1) = \text{smallest proper variety.} \)
Boolean Covers
Boolean Covers

\[
\begin{align*}
\forall \mathcal{A}^n & \\
\mathcal{A}^n & \\
\mathcal{A}^2 & \\
\mathcal{A} & \\
\mathcal{F}_{\alpha,u} & \\
\mathcal{M}_{u}^- & \\
\mathcal{M}_{u}^+ & \\
\mathcal{K}_2 & \\
\mathcal{K}_3 & \\
\mathcal{K}_5 & \\
\mathcal{K}_p & \\
\mathcal{K}_\infty & \\
\mathcal{E} & \\
\end{align*}
\]
Boolean Covers

\[ \forall \mathcal{A}^n \]

\[ \mathcal{A}^n \]

\[ \mathcal{A}^2 \]

\[ \mathcal{A} \]

\[ \mathcal{F}_{\alpha,u} \]

\[ \mathcal{M}_{u}^- \]

\[ \mathcal{M}_{u}^+ \]

\[ \mathcal{K}_2 \]

\[ \mathcal{K}_3 \]

\[ \mathcal{K}_5 \]

\[ \mathcal{K}_p \]

\[ \mathcal{K}_\infty \]

\[ \mathcal{E} \]

Komori 1981

\[ \mathcal{K}_p = \text{Var}(\mathbb{Z}, p) \]

\[ \mathcal{K}_\infty = \text{Var}(\mathbb{Z} \times \mathbb{Z}, (1, 0)) \]
Boolean Covers

\[ \mathcal{F}_{\alpha, u} \]

\[ \mathcal{M}^-_u, \mathcal{M}^+_u \]

\[ \mathcal{K}_2, \mathcal{K}_3, \mathcal{K}_5, \ldots, \mathcal{K}_p, \ldots, \mathcal{K}_\infty \]

\[ \mathcal{E} \]

Medvedev (H) 2005

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\[ \mathcal{K}_p = \text{Var}(\mathbb{Z}, p) \]

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Boolean Covers

\[ \bigvee A^n \]

\[ A^n \]

\[ A^2 \]

\[ A \]

\[ B \]

\[ F_{\alpha,u} \]

\[ M_u^- \]

\[ M_u^+ \]

\[ K_2 \]

\[ K_3 \]

\[ K_5 \]

\[ K_p \]

\[ K_\infty \]

\[ Var(\mathbb{Z}, p) \]

\[ Var(\mathbb{Z} \times \mathbb{Z}, (1, 0)) \]

Covering Layer

“Feil” (H) 2007

“Medvedev” (H) 2005

Komori 1981

Friday, April 12, 13
Boolean Covers

\[ F^\alpha \]

Covering Layer

\[ \forall A^n \]

\[ A^n \]

\[ A^2 \]

\[ A \]

\[ A \times A \]

\[ A \times A \]

\[ \mathcal{F}_{\alpha, u} \]

\[ \mathcal{M}_u^- \]

\[ \mathcal{M}_u^+ \]

\[ \mathcal{K}_2 \]

\[ \mathcal{K}_3 \]

\[ \mathcal{K}_5 \]

\[ \mathcal{K}_p \]

\[ \mathcal{K}_\infty \]

\[ \mathcal{E} \]

“Feil”

(\( H \))

2007

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\[ \mathcal{K}_p = \text{Var}(\mathbb{Z}, p) \]

\[ \mathcal{K}_\infty = \text{Var}(\mathbb{Z} \times \mathbb{Z}, (1, 0)) \]
Boolean Covers

\( \mathcal{F}_{\alpha,u} \)

\( \forall A^n \)

\( A^n \)

\( A^2 \)

\( A \)

\( \mathcal{K}_p = \text{Var}(\mathbb{Z}, p) \)

\( \mathcal{K}_\infty = \text{Var}(\mathbb{Z} \times \mathbb{Z}, (1, 0)) \)

“Feil” (H) 2007

“Medvedev” (H) 2005

Komori 1981

Friday, April 12, 13
Boolean Covers

\[ F_{\alpha, u} \]

\[ K_1 = \text{Var}(\mathbb{Z}, Z) \]

\[ K_p = \text{Var}(\mathbb{Z}, p) \]

\[ K_\infty = \text{Var}(\mathbb{Z} \times \mathbb{Z}, (1, 0)) \]

Medvedev (H) 2005

Feil (H) 2007

Covering Layer

Friday, April 12, 13
Boolean Covers

\[ \mathcal{F}_{\alpha, u} \]

\[ \mathcal{H} \mathcal{M}_{\alpha, u} \]

\[ \mathcal{K}_p = \text{Var}(\mathbb{Z}, p) \]

\[ \mathcal{K}_\infty = \text{Var}(\mathbb{Z} \times \mathbb{Z}, (1, 0)) \]

Feil (H) 2007

Medvedev (H) 2005

Komori 1981

Var(\mathbb{Z} \times \mathbb{Z}, (1, 0))
D-H covers of $\mathcal{B}$:
D-H covers of $\mathcal{B}$: $x << y \iff \forall n \in \mathbb{Z}, x^n < y$
D-H covers of $\mathcal{B}$: $x << y \Leftrightarrow \forall n \in \mathbb{Z}, x^n < y$

Let $s = (s_1, s_2, \ldots, s_i, \ldots)$, $s_i \in \{-1, +1\}$. 
D-H covers of \( B \): \( x << y \iff \forall n \in \mathbb{Z}, x^n < y \)

Let \( s = (s_1, s_2, \ldots, s_i, \ldots) \), \( s_i \in \{-1, +1\} \).

\( (F_s, u) \) is a totally ordered group with unit \( u \) and \( e < b << u \). If \( s_1 = +1 \) then \( b << b^u \), and if \( s_1 = -1 \) then \( b^u << b \).
D-H covers of $\mathcal{B}$: \[ x \ll y \iff \forall n \in \mathbb{Z}, \ x^n < y \]

Let \[ s = (s_1, s_2, \ldots, s_i, \ldots), \ s_i \in \{-1, +1\}. \]

$(F_s, u)$ is a totally ordered group with unit $u$ and $e < b \ll u$. If $s_1 = +1$ then $b \ll b^u$, and if $s_1 = -1$ then $b^u \ll b$.

Similarly, if $s_1 = -1$ and $s_2 = +1$ then $b^u \ll b$ and $b^u \ll (b^u)^b$, etc.
Let $s = (s_1, s_2, \ldots, s_i, \ldots)$, $s_i \in \{-1, +1\}$.

$(F_s, u)$ is a totally ordered group with unit $u$ and $e < b << u$. If $s_1 = +1$ then $b << b^u$, and if $s_1 = -1$ then $b^u << b$.

Similarly, if $s_1 = -1$ and $s_2 = +1$ then $b^u << b$ and $b^u << (b^u)^b$, etc.

**Theorem.**

Let $\mathcal{B}_s$ be the variety generated by $(F_s, u)$. Then $\mathcal{B}_s$ is a cover of the boolean variety $\mathcal{B}$, and if $s \neq t$ then $\mathcal{B}_s \neq \mathcal{B}_t$. 

$x << y \iff \forall n \in \mathbb{Z}, x^n < y$
D-H covers of $\mathcal{B}$: $x << y \Leftrightarrow \forall n \in \mathbb{Z}, x^n < y$

Let $s = (s_1, s_2, \ldots, s_i, \ldots)$, $s_i \in \{-1, +1\}$.

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Therefore, there are uncountably many of these.
D-H covers of $\mathcal{B}$: \[ x \ll y \iff \forall n \in \mathbb{Z}, x^n < y \]

Let \( s = (s_1, s_2, \ldots, s_i, \ldots) \), \( s_i \in \{-1, +1\} \).

\((F_s, u)\) is a totally ordered group with unit \( u \)
and \( e < b \ll u \). If \( s_1 = +1 \) then \( b \ll b^u \),
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Similarly, if \( s_1 = -1 \) and \( s_2 = +1 \) then \( b^u \ll b \) and \( b^u \ll (b^u)^b \), etc.

Theorem.

Let \( \mathcal{B}_s \) be the variety generated by \((F_s, u)\). Then \( \mathcal{B}_s \) is a cover of
the boolean variety \( \mathcal{B} \), and if \( s \neq t \) then \( \mathcal{B}_s \neq \mathcal{B}_t \).

Therefore, there are uncountably many of these. (Darnel–Holland)
D-H covers of $\mathcal{B}$: 

$x << y \iff \forall n \in \mathbb{Z}, x^n < y$

Let $s = (s_1, s_2, \ldots, s_i, \ldots), s_i \in \{-1, +1\}$.

$(F_s, u)$ is a totally ordered group with unit $u$
and $e < b << u$. If $s_1 = +1$ then $b << b^u$,
and if $s_1 = -1$ then $b^u << b$.

Similarly, if $s_1 = -1$ and $s_2 = +1$ then $b^u << b$ and $b^u << (b^u)^b$, etc.

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Let $\mathcal{B}_s$ be the variety generated by $(F_s, u)$. Then $\mathcal{B}_s$ is a cover of
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Therefore, there are uncountably many of these. (Darnel–Holland)

Are there more covers of $\mathcal{B}$ ?

* * * * * * * * * * * * * *
2. \( \mathcal{L} = \text{Var}(\text{Lattice-Groups}) \quad x = x \)
\( \mathcal{N} = \text{Normal Valued} \quad (x \vee e)(y \vee e) = ((x \vee e)(y \vee e)) \land ((y \vee e)^2(x \vee e)^2) \)
\( \mathcal{R} \quad (x \vee e) \land (z^{-1}(x \land e)^{-1}z) = e \) (subdirect product of totally ordered groups)
\( \mathcal{A} = \text{Abelian} \quad xy = yx \)
\( \mathcal{E} \quad x = e \)

\( \mathcal{L} \) \( \mathcal{N} \) \( \mathcal{H M}_s \) \( \mathcal{R} \) \( \mathcal{A}^3 \) \( \mathcal{A}^2 \) \( \mathcal{M}^- \) \( \mathcal{M}^0 \) \( \mathcal{M}^+ \) \( S_2 \) \( S_3 \) \( S_5 \) \( S_p \)

covering layer.
Let \( \mathcal{L} = \text{Var} \text{(Lattice-Groups)} \) and \( x = x \). 

\( \mathcal{N} = \text{Normal Valued} \quad (x \vee e)(y \vee e) = ((x \vee e)(y \vee e)) \land ((y \vee e)^2(x \vee e)^2) \)

\( \mathcal{R} \quad (x \vee e) \land (z^{-1}(x \wedge e)^{-1}z) = e \) (subdirect product of totally ordered groups)

\( \mathcal{A} = \text{Abelian} \quad xy = yx \)

\( \mathcal{E} \quad x = e \)

\( \mathcal{HM}_s \)

\( \mathcal{R} \)

\( \mathcal{A}^3 \)

\( \mathcal{A}^2 \)

\( \mathcal{A} \)

\( \mathcal{E} \)

covering layer.
2. \( \mathcal{L} = \text{Var}(\text{Lattice-Groups}) \quad x = x \)
\( \mathcal{N} = \text{Normal Valued} \quad (x \lor e)(y \lor e) = ((x \lor e)(y \lor e)) \land ((y \lor e)^2(x \lor e)^2) \)
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\[ \text{Scrimger (1975)} \]
\[ S_p = (\mathbb{Z}_1 \oplus \cdots \oplus \mathbb{Z}_p) \triangleright \mathbb{Z} \]
\[ p = \text{a prime number} \]
2.  \( \mathcal{L} = \text{Var(Lattice-Groups)} \quad x = x \)

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\( \mathcal{H}_M \)

\( \mathcal{R} \)

\( \mathcal{A} \)

\( \mathcal{M}^+ = (\cdots \oplus \mathbb{Z}_i \oplus \mathbb{Z}_{i+1} \oplus \cdots) \times \mathbb{Z} \)

\( \mathcal{S}_p = (\mathbb{Z}_1 \oplus \cdots \oplus \mathbb{Z}_p) \leftarrow \mathbb{Z} \)

\( p = \text{a prime number} \)

Medvedev (1977)

Scrimger (1975)

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Bergman (1984), two varieties; H. and Medvedev (1994), continuum many.
\[ \mathcal{L} = \text{Var}(\text{Lattice-Groups}) \quad x = x \]
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\[ A = \text{Abelian} \quad xy = yx \]
\[ E \quad x = e \]

\[ \mathcal{L} \quad \text{Var}(\text{Lattice-Groups}) \]

\[ \mathcal{N} \]

\[ \mathcal{M}_s \]

\[ \mathcal{R} \]

\[ A^3 \]

\[ A^2 \]

\[ A \]

\[ \mathcal{M}^- \quad \mathcal{M}^0 \quad \mathcal{M}^+ \quad S_2 \quad S_3 \quad S_5 \quad S_p \]

\[ H \xleftarrow{\sim} \mathbb{Z} \quad \text{Medvedev (1977)} \quad \text{Scrimger (1975)} \]

\[ \mathcal{M}^+ = (\cdots \oplus \mathbb{Z}_i \oplus \mathbb{Z}_{i+1} \oplus \cdots) \rtimes \mathbb{Z} \]

\[ H \text{ totally ordered} \quad \text{Bergman (1984), two varieties; H. and Medvedev (1994), continuum many.} \]

\[ S_p = (\mathbb{Z}_1 \oplus \cdots \oplus \mathbb{Z}_p) \rtimes \mathbb{Z} \quad p = \text{a prime number} \]
\[ \mathcal{L} = \text{Var}(\text{Lattice-Groups}) \quad x = x \]
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More?

\[ \mathcal{M}^- \quad \mathcal{M}^0 \quad \mathcal{M}^+ \quad S_2 \quad S_3 \quad S_5 \quad S_p \]

Feil \[ R_{\alpha} \leftrightarrow \mathbb{Z} \quad (1980) \]

(file) \[ R_{\alpha} \subset \mathbb{R} \quad \alpha \in \mathbb{R} \]

covering layer.

H. and Medvedev (1994), continuum many.

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\[ \mathcal{M}^+ = (\cdots \oplus \mathbb{Z}_i \oplus \mathbb{Z}_{i+1} \oplus \cdots) \ltimes \mathbb{Z} \]

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\[ \begin{align*}
\mathcal{L} & \quad \text{Var}(\text{Lattice-Groups}) \\
\mathcal{N} & \\
\mathcal{H}\mathcal{M}_s & \\
\mathcal{R} & \\
A^3 & \\
A^2 & \\
? & \\
\mathcal{M}^- & \mathcal{M}^0 & \mathcal{M}^+ & S_2 & S_3 & S_5 & \cdots & S_p \\
H & \xleftarrow{\times} \mathbb{Z} & H \text{ totally ordered} & \text{Medvedev (1977)} & \epsilon & \text{Scrimger (1975)} & S_p = (\mathbb{Z}_1 \oplus \cdots \oplus \mathbb{Z}_p) \xleftarrow{\times} \mathbb{Z} & p = \text{a prime number} \\
& & Bergman (1984), \text{two varieties;} \ H. \text{ and Medvedev (1994), continuum many}.\end{align*} \]
2. $\mathcal{L} = \text{Var}(\text{Lattice-Groups}) \quad x = x$
$\mathcal{N} = \text{Normal Valued} \quad (x \lor e)(y \lor e) = ((x \lor e)(y \lor e)) \land ((y \lor e)^2(x \lor e)^2)$
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$\mathcal{L}$

$\mathcal{N}$

$\mathcal{R}$

$\mathcal{A}^3$

$?\quad ?$

$\mathcal{A}^2$

$\mathcal{M}^-$

$\mathcal{M}^0$

$\mathcal{M}^+$

$\mathcal{S}_2$

$\mathcal{S}_3$

$\mathcal{S}_5$

$\mathcal{S}_p$

$\mathcal{H}\mathcal{M}_s$

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Medvedev (1977)

$H \leftrightarrow \mathbb{Z}$

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Scrimger (1975)

$S_p = (\mathbb{Z}_1 \oplus \cdots \oplus \mathbb{Z}_p) \leftrightarrow \mathbb{Z}$

$p = \text{a prime number}$

Friday, April 12, 13
Metabelian $\ell$-group $V$: $\mathcal{A}^2$
Metabelian $\ell$-group $V$: $A^2$

There exists a normal abelian convex $\ell$-subgroup $A \subseteq V$ such that $V/A$ is abelian.
Metabelian \( \ell \)-group \( V \): \( A^2 \)

There exists a normal abelian convex \( \ell \)-subgroup \( A \subseteq V \) such that \( V/A \) is abelian.

The collection \( A^2 \) of all metabelian \( \ell \)-groups is a variety with \( A \subset A^2 \).
**Metabelian \( \ell \)-group \( V \): \( \mathcal{A}^2 \)**

There exists a normal abelian convex \( \ell \)-subgroup \( A \subseteq V \) such that \( V/A \) is abelian.

The collection \( \mathcal{A}^2 \) of all metabelian \( \ell \)-groups is a variety with \( A \subset \mathcal{A}^2 \).

\[ \mathcal{A}^2 \]

minimal non-abelian

= covering layer of \( A \)
Metabelian $\ell$-group $V$: $A^2$

There exists a normal abelian convex $\ell$-subgroup $A \subseteq V$ such that $V/A$ is abelian.

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Covering layer of $A$ under $A^2$:

minimal non-abelian
$= \text{covering layer of } A$

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Medvedev varieties $\mathcal{M}^-, \mathcal{M}^0, \mathcal{M}^+$
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**Metabelian** $\ell$-group $V$: $\mathcal{A}^2$

There exists a normal abelian convex $\ell$-subgroup $A \subseteq V$ such that $V/A$ is abelian.

The collection $\mathcal{A}^2$ of all metabelian $\ell$-groups is a variety with $A \subset \mathcal{A}^2$.

A variety $\mathcal{V}$ is *minimally non-metabelian* if there is no variety $\mathcal{W}$ with $\mathcal{V} \cap \mathcal{A}^2 \subset \mathcal{W} \subset \mathcal{V}$.

- Medvedev varieties $\mathcal{M}^-, \mathcal{M}^0, \mathcal{M}^+$
- Scrimger varieties $\mathcal{S}_p$, $p$ prime.

Covering layer of $A$ under $\mathcal{A}^2$: minimal non-abelian $=\text{covering layer of } A$
Metabelian $\ell$-group $V$: $A^2$

There exists a normal abelian convex $\ell$-subgroup $A \subseteq V$ such that $V/A$ is abelian.

The collection $A^2$ of all metabelian $\ell$-groups is a variety with $A \subset A^2$.

A variety $V$ is minimally non-metabelian if there is no variety $W$ with $V \cap A^2 \subset W \subset V$.

$U$ is minimal non-metabelian if for all $W \subset U$, $W \subseteq A^2$.

Covering layer of $A$ under $A^2$: Medvedev varieties $M^-, M^0, M^+$

Scrimger varieties $S_p$, $p$ prime.

minimal non-abelian = covering layer of $A$
There exists a normal abelian convex $\ell$-subgroup $A \subseteq V$ such that $V/A$ is abelian.

The collection $A^2$ of all metabelian $\ell$-groups is a variety with $A \subset A^2$.

A variety $V$ is \textit{minimally non-metabelian} if there is no variety $W$ with $V \cap A^2 \subset W \subset V$.

$U$ is \textit{minimal non-metabelian} if for all $W \subset U$, $W \subset A^2$.

If $U$ is minimal non-metabelian, it is minimally non-metabelian.
The Scrimger $\ell$-groups:
The Scrimger $\ell$-groups:

$$S_n = \mathbb{Z} \times \sum_{i=0}^{n-1} \mathbb{Z} = \mathbb{Z} \times \sum_{i=0}^{n-1} H$$
The Scrimger \( \ell \)-groups:

\[
S_n = \mathbb{Z} \times \sum_{i=0}^{n-1} \mathbb{Z} = \mathbb{Z} \times \sum_{i=0}^{n-1} H
\]

For \( a, b \in S_n \)

\[
ab = (a', (a_0, a_1, \ldots, a_{n-1}))(b', (b_0, b_1, \ldots, b_{n-1}))
\]

\[
= (a' + b', (a_0 + b_{0-a'}, a_1 + b_{1-a'}, \ldots, a_{n-1} + b_{n-1-a'}))
\]
The Scrimger $\ell$-groups:

\[ S_n = \mathbb{Z} \times \sum_{i=0}^{n-1} \mathbb{Z} = \mathbb{Z} \times \sum_{i=0}^{n-1} H \]

For \( a, b \in S_n \)

\[ ab = (a', (a_0, a_1, \ldots, a_{n-1}))(b', (b_0, b_1, \ldots, b_{n-1})) = (a' + b', (a_0 + b_{0-a'}, a_1 + b_{1-a'}, \ldots, a_{n-1} + b_{n-1-a'})) \]

A useful representation of \((k, (m_0, m_1, \ldots, m_{n-1})) \in S_n\) is:
The Scrimger $\ell$-groups:

\[ S_n = \mathbb{Z} \times \sum_{i=0}^{n-1} \mathbb{Z} = \mathbb{Z} \times \sum_{i=0}^{n-1} H \]

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A useful representation of \((k, (m_0, m_1, \ldots, m_{n-1})) \in S_n\) is:

\[ k \]

\[ m_0 \quad m_1 \quad \cdots \quad m_{n-1} \]
The Scrimger $\ell$-groups:

$$S_n = \mathbb{Z} \times \sum_{i=0}^{n-1} \mathbb{Z} = \mathbb{Z} \times \sum_{i=0}^{n-1} H$$

For $a, b \in S_n$

$$ab = (a', (a_0, a_1, \ldots, a_{n-1}))(b', (b_0, b_1, \ldots, b_{n-1}))$$

$$= (a' + b', (a_0 + b_{0-a'}, a_1 + b_{1-a'}, \ldots, a_{n-1} + b_{n-1-a'}))$$

A useful representation of $(k, (m_0, m_1, \ldots, m_{n-1})) \in S_n$ is:

$$S_n \in \mathcal{A}^2$$

is metabelian.
The Scrimger $\ell$-groups:

\[ S_n = \mathbb{Z} \rtimes \sum_{i=0}^{n-1} \mathbb{Z} = \mathbb{Z} \rtimes \sum_{i=0}^{n-1} H \]

For \( a, b \in S_n \)

\[ ab = (a', (a_0, a_1, \ldots, a_{n-1}))(b', (b_0, b_1, \ldots, b_{n-1})) = (a' + b', (a_0 + b_{0-a'}, a_1 + b_{1-a'}, \ldots, a_{n-1} + b_{n-1-a'})) \]

A useful representation of \((k, (m_0, m_1, \ldots, m_{n-1})) \in S_n\) is:

\[ S_n \in A^2 \text{ is metabelian.} \]

An extension is \(S_{m,n}\) with \(H\) in \(S_n\) replaced by \(S_m\).
The Scrimger $\ell$-groups:

$$S_n = \mathbb{Z} \rtimes \sum_{i=0}^{n-1} \mathbb{Z} = \mathbb{Z} \rtimes \sum_{i=0}^{n-1} H$$

For $a, b \in S_n$

$$ab = (a', (a_0, a_1, \ldots, a_{n-1}))(b', (b_0, b_1, \ldots, b_{n-1}))$$

$$= (a' + b', (a_0 + b_0 - a', a_1 + b_1 - a', \ldots, a_{n-1} + b_{n-1} - a'))$$

A useful representation of $(k, (m_0, m_1, \ldots, m_{n-1})) \in S_n$ is:

$$S_n \in \mathcal{A}^2$$ is metabelian.

An extension is $S_{m,n}$ with $H$ in $S_n$ replaced by $S_m$.

$$S_{m,n} \in \mathcal{A}^3$$
The Scrimger $\ell$-groups:

\[ S_n = \mathbb{Z} \times \sum_{i=0}^{n-1} \mathbb{Z} = \mathbb{Z} \times \sum_{i=0}^{n-1} H \]

For \( a, b \in S_n \)

\[ ab = (a', (a_0, a_1, \ldots, a_{n-1}))(b', (b_0, b_1, \ldots, b_{n-1})) = (a' + b', (a_0 + b_{0-a'}, a_1 + b_{1-a'}, \ldots, a_{n-1} + b_{n-1-a'})) \]

A useful representation of \((k, (m_0, m_1, \ldots, m_{n-1})) \in S_n\) is:

\[ S_n \in A^2 \text{ is metabelian.} \]

An extension is \( S_{m,n} \) with \( H \) in \( S_n \) replaced by \( S_m \).

\[ S_{m,n} \in A^3 \]

\[ \text{Var}(S_n) = S_n \text{ and Var}(S_{m,n}) = S_{m,n} \]
Theorem. Let $p, q$ be positive prime integers. Then $\mathcal{S}_{p,q}$ is a minimal non-metabelian variety.
Theorem. Let $p, q$ be positive prime integers. Then $\mathcal{S}_{p,q}$ is a minimal non-metabelian variety.

So, $\mathcal{S}_{p,q}$ covers $\mathcal{S}_{p,q} \cap A^2$.  

(D - H)
Theorem. Let \( p, q \) be positive prime integers. Then \( \mathcal{S}_{p,q} \) is a minimal non-metabelian variety.

So, \( \mathcal{S}_{p,q} \) covers \( \mathcal{S}_{p,q} \cap \mathcal{A}^2 \). \hfill (D - H)

Theorem. If \( p \) and \( q \) are distinct positive prime integers, then \( \mathcal{S}_{p,q} \cap \mathcal{A}^2 \) is the Scrimger variety \( \mathcal{S}_{pq} \). \hfill (D - H)
Theorem. Let $p, q$ be positive prime integers. Then $\mathcal{S}_p,q$ is a minimal non-metabelian variety. So, $\mathcal{S}_p,q$ covers $\mathcal{S}_p,q \cap A^2$.  

(D - H)

Theorem. If $p$ and $q$ are distinct positive prime integers, then $\mathcal{S}_p,q \cap A^2$ is the Scrimger variety $S_{pq}$.  

(D - H)

Theorem. The family $\{\mathcal{S}_p,q : p, q \text{ positive prime integers}\}$ is a countable infinite set of minimal non-metabelian $\ell$-group varieties which contain no nonabelian o-groups.  

(D - H)
$H_{1,n} = S_n$, a Scrimger $\ell$-group of width $n$ and shift by 1.
\[ H_{1,n} = S_n, \text{ a Scrimger } \ell\text{-group of width } n \text{ and shift by 1.} \]

In general, \( H_{r,s} \) is a generalized Scrimger \( \ell \)-group of width \( rs \) and shift by \( r \).
$H_{1,n} = S_n$, a Scrimger $\ell$-group of width $n$ and shift by 1.

In general, $H_{r,s}$ is a generalized Scrimger $\ell$-group of width $rs$ and shift by $r$.

$M_{n,r,s}$ is the definition of $S_n$ with $H$ replaced by $H_{r,s}$. 
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$M_{n,r,s}$ is the definition of $S_n$ with $H$ replaced by $H_{r,s}$.

$M_{n,r,s} = \mathbb{Z} \times \sum_{i=0}^{n-1} H_{r,s}$
$H_{1,n} = S_n$, a Scrimger $\ell$-group of width $n$ and shift by 1.

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$$M_{n,r,s} = \mathbb{Z} \times \sum_{i=0}^{n-1} H_{r,s}$$

$\mathcal{M}_{n,r,s}$ is the variety generated by $M_{n,r,s}$.
\[ H_{1,n} = S_n, \] a Scrimger \( \ell \)-group of width \( n \) and shift by 1.

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\[ M_{n,r,s} \text{ is the definition of } S_n \text{ with } H \text{ replaced by } H_{r,s}. \]

\[ M_{n,r,s} = \mathbb{Z} \rightarrow \sum_{i=0}^{n-1} H_{r,s} \]

\[ M_{n,r,s} \text{ is the variety generated by } M_{n,r,s}. \]

**Theorem.** Let \( p \) be a positive prime integer and \( k \) any positive integer.
\[ H_{1,n} = S_n, \text{ a Scrimger } \ell\text{-group of width } n \text{ and shift by 1.} \]

In general, \( H_{r,s} \) is a generalized Scrimger \( \ell\text{-group of width } rs \text{ and shift by } r. \)

\[ M_{n,r,s} \text{ is the definition of } S_n \text{ with } H \text{ replaced by } H_{r,s}. \]

\[ M_{n,r,s} = \mathbb{Z} \times \sum_{i=0}^{n-1} H_{r,s} \]

\( M_{n,r,s} \) is the variety generated by \( M_{n,r,s}. \)

**Theorem.** Let \( p \) be a positive prime integer and \( k \) any positive integer.

The varieties \( M_{p,p,p^k} \) are minimal non-metabelian varieties.
$H_{1,n} = S_n$, a Scrimger $\ell$-group of width $n$ and shift by 1.

In general, $H_{r,s}$ is a generalized Scrimger $\ell$-group of width $rs$ and shift by $r$.

$M_{n,r,s}$ is the definition of $S_n$ with $H$ replaced by $H_{r,s}$.

\[ M_{n,r,s} = \mathbb{Z} \rightarrow \sum_{i=0}^{n-1} H_{r,s} \]

$\mathcal{M}_{n,r,s}$ is the variety generated by $M_{n,r,s}$.

**Theorem.** Let $p$ be a positive prime integer and $k$ any positive integer.

The varieties $\mathcal{M}_{p,p,p^k}$ are minimal non-metabelian varieties.

Like $\mathfrak{S}_{p,q}$ they contain no nonabelian o-groups.
$H_{1,n} = S_n$, a Scrimger $\ell$-group of width $n$ and shift by 1.

In general, $H_{r,s}$ is a generalized Scrimger $\ell$-group of width $rs$ and shift by $r$.

$M_{n,r,s}$ is the definition of $S_n$ with $H$ replaced by $H_{r,s}$.

$M_{n,r,s} = \mathbb{Z} \rightarrow \sum_{i=0}^{n-1} H_{r,s}$

$\mathcal{M}_{n,r,s}$ is the variety generated by $M_{n,r,s}$.

**Theorem.** Let $p$ be a positive prime integer and $k$ any positive integer.

The varieties $\mathcal{M}_{p,p,p^k}$ are minimal non-metabelian varieties.

Like $\mathcal{S}_{p,q}$ they contain no nonabelian o-groups.

Every minimal non-metabelian variety which contains no nonabelian o-groups must be either $\mathcal{S}_{p,q}$ or $\mathcal{M}_{p,p,p^k}$. (D - H)
$H_{1,n} = S_n$, a Scrimger $\ell$-group of width $n$ and shift by 1.

In general, $H_{r,s}$ is a generalized Scrimger $\ell$-group of width $rs$ and shift by $r$.

$M_{n,r,s}$ is the definition of $S_n$ with $H$ replaced by $H_{r,s}$.

$$M_{n,r,s} = \mathbb{Z} \times \sum_{i=0}^{n-1} H_{r,s}$$

$\mathcal{M}_{n,r,s}$ is the variety generated by $M_{n,r,s}$.

**Theorem.** Let $p$ be a positive prime integer and $k$ any positive integer.

The varieties $\mathcal{M}_{p,p,p^k}$ are minimal non-metabelian varieties.

Like $\mathcal{S}_{p,q}$ they contain no nonabelian $o$-groups.

Every minimal non-metabelian variety which contains no nonabelian $o$-groups must be either $\mathcal{S}_{p,q}$ or $\mathcal{M}_{p,p,p^k}$.  

(D - H)
Questions
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The known covers of the abelian variety of $\ell$-groups are:
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    Scrimger \( S_p, p \) a prime    (Scrimger, 1975)
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The known covers of the abelian variety of \(\ell\)-groups are:

Scrimger \(S_p\), \(p\) a prime \hspace{1cm} (Scrimger, 1975)

\(M^{-}, M^{0}, M^{+}\) \hspace{1cm} (Medvedev, 1977)

\(H_r \cong \mathbb{Z}\), continuum many \hspace{1cm} (Holland, Medvedev, 1994)

Is that all?
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The known covers of the abelian variety of $\ell$-groups are:

- Scrimger $S_p$, $p$ a prime  (Scrimger, 1975)
- $\mathcal{M}^-, \mathcal{M}^0, \mathcal{M}^+$  (Medvedev, 1977)
- $H_r \leftarrow \mathbb{Z}$, continuum many  (Holland, Medvedev, 1994)

Is that all?

Every known cover of $\mathcal{B}$ is generated by a totally ordered ul-group. of the form $H \leftarrow \mathbb{Z}$.
Questions

The known covers of the abelian variety of $\ell$-groups are:

- Scrimger $S_p$, $p$ a prime (Scrimger, 1975)
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- $H_r \rightarrow \times \mathbb{Z}$, continuum many (Holland, Medvedev, 1994)

Is that all?

Every known cover of $B$ is generated by a *totally* ordered $ul$-group of the form $H \rightarrow \times \mathbb{Z}$.

Is that all?
Questions

The known covers of the abelian variety of \( \ell \)-groups are:

- Scrimger \( \mathcal{S}_p \), \( p \) a prime \hspace{1cm} (Scrimger, 1975)
- \( \mathcal{M}^{-}, \mathcal{M}^{0}, \mathcal{M}^{+} \) \hspace{1cm} (Medvedev, 1977)
- \( H_r \leftarrow \mathbb{Z} \), continuum many \hspace{1cm} (Holland, Medvedev, 1994)

Is that all?

Every known cover of \( \mathcal{B} \) is generated by a \textit{totally} ordered \( \mathcal{U}\ell \)-group.

Is that all?

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Some References


5. Darnel and Holland, *Minimal non-metabelian varieties of \( \ell \)-groups which contain no nonabelian o-groups*, submitted.
u ℓ-groups  \[ \Psi \text{MV-algebras} \]
$u\ell$-groups

$\Psi\text{MV-algebras}$
$u \ell$-groups

$\Psi$MV-algebras
- u ℓ-groups
- ?

ΨMV-algebras
- ?

Come on, guys. Cooperate!
u ℓ-groups

ΨMV-algebras

Thank You!!

Come on, guys. Cooperate!