Varieties of Generalized Hoops and Integral GBL-algebras

Peter Jipsen

Chapman University, Orange, California

AMS Section Meeting, April 13, 2013, Boulder Colorado

Generalized Hoops

Generalized hoops were first studied by Bosbach [1969, 70] and the name **hoop** was introduced by Büchi and Owen [1975].

A **generalized hoop** $(A, \cdot, 1, \backslash, /)$ is a residuated partially ordered monoid in which

$$x \leq y \iff \exists u(x = uy) \iff \exists v(x = yv).$$

I.e. the monoid is **naturally ordered**, hence **integral**: $x \leq 1$

Residuated means: $xy \leq z \iff y \leq x\backslash z \iff x \leq z/y$

Two simple identities

$$\frac{\left(\frac{x}{y}\right)}{z} =$$

Two simple identities

$$\frac{\left(\frac{x}{y}\right)}{z} = \frac{1}{z}\left(\frac{x}{y}\right) =$$

Two simple identities

$$\frac{\left(\frac{x}{y}\right)}{z} = \frac{1}{z}\left(\frac{x}{y}\right) = \frac{x}{zy}$$

Two simple identities

$$\frac{\left(\frac{x}{y}\right)}{z} = \frac{1}{z}\left(\frac{x}{y}\right) = \frac{x}{zy}$$

$$(x/y)/z =$$

Two simple identities

$$\frac{\left(\frac{x}{y}\right)}{z} = \frac{1}{z}\left(\frac{x}{y}\right) = \frac{x}{zy}$$

$$(x/y)/z = x/(zy)$$

Two simple identities

$$\frac{(\frac{x}{y})}{z} = \frac{1}{z}(\frac{x}{y}) = \frac{x}{zy}$$

$$(x/y)/z = x/(zy)$$

$$x\backslash(y\backslash z) =$$

Two simple identities

$$\frac{\left(\frac{x}{y}\right)}{z} = \frac{1}{z}\left(\frac{x}{y}\right) = \frac{x}{zy}$$

$$(x/y)/z = x/(zy)$$

$$x\backslash(y\backslash z) = (y$$

Two simple identities

$$\frac{\left(\frac{x}{y}\right)}{z} = \frac{1}{z}\left(\frac{x}{y}\right) = \frac{x}{zy}$$

$$(x/y)/z = x/(zy)$$

$$x \backslash (y \backslash z) = (yx)$$

Two simple identities

$$\frac{(\frac{x}{y})}{z} = \frac{1}{z}(\frac{x}{y}) = \frac{x}{zy}$$

$$(x/y)/z = x/(zy)$$

$$x\backslash(y\backslash z) = (yx)\backslash z$$

Other simple identities

$\frac{x}{x} = 1$ (true in **integral** residuated monoids) $\quad 1y = y$

Therefore $\quad \frac{x}{x}y = y$

Another **Basic** identity: $(x/y)y = (y/x)x$

NOT true in residuated monoids, but an axiom of hoops.

Equivalent to $x \leq y \implies x = (x/y)y$

Equivalent to **naturally ordered**: $x \leq y \implies \exists u(x = uy)$

A lemma

If $y = (x/x)y$ and $x/(y \cdot z) = ((x/z)/y)$ and $(x/y)y = (y/x)x$ then $\cdot$ is **associative**.

Proof: $x(yz) = [((xy)z)/((xy)z)](x(yz))$

A lemma

If $y = (x/x)y$ and $x/(y \cdot z) = ((x/z)/y)$ and

$(x/y)y = (y/x)x$ then $\cdot$ is **associative**.

Proof: $x(yz) = [((xy)z)/((xy)z)](x(yz))$

$= [(((xy)z)/z)/(xy)](x(yz))$

A lemma

If $y = (x/x)y$ and $x/(y \cdot z) = ((x/z)/y)$ and

$(x/y)y = (y/x)x$ then $\cdot$ is **associative**.

Proof: $x(yz) = [((xy)z)/((xy)z)](x(yz))$

$= [(((xy)z)/z)/(xy)](x(yz))$

$= [((((xy)z)/z)/y)/x](x(yz))$

A lemma

If $y = (x/x)y$ and $x/(y \cdot z) = ((x/z)/y)$ and

$(x/y)y = (y/x)x$ then $\cdot$ is **associative**.

Proof: $x(yz) = [((xy)z)/((xy)z)](x(yz))$

$= [(((xy)z)/z)/(xy)](x(yz))$

$= [((((xy)z)/z)/y)/x](x(yz))$

$= [(((xy)z)/(yz))/x](x(yz))$

A lemma

If $y = (x/x)y$ and $x/(y \cdot z) = ((x/z)/y)$ and

$(x/y)y = (y/x)x$ then $\cdot$ is **associative**.

Proof: $x(yz) = [((xy)z)/((xy)z)](x(yz))$

$= [(((xy)z)/z)/(xy)](x(yz))$

$= [((((xy)z)/z)/y)/x](x(yz))$

$= [(((xy)z)/(yz))/x](x(yz))$

$= [((xy)z)/(x(yz))](x(yz))$

A lemma

If $y = (x/x)y$ and $x/(y \cdot z) = ((x/z)/y)$ and

$(x/y)y = (y/x)x$ then $\cdot$ is **associative**.

Proof: $x(yz) = [((xy)z)/((xy)z)](x(yz))$

$= [(((xy)z)/z)/(xy)](x(yz))$

$= [((((xy)z)/z)/y)/x](x(yz))$

$= [(((xy)z)/(yz))/x](x(yz))$

$= [((xy)z)/(x(yz))](x(yz))$

$= [(x(yz))/((xy)z)]((xy)z) =$

A lemma

If $y = (x/x)y$ and $x/(y \cdot z) = ((x/z)/y)$ and

$(x/y)y = (y/x)x$ then $\cdot$ is **associative**.

Proof: $x(yz) = [((xy)z)/((xy)z)](x(yz))$

$= [(((xy)z)/z)/(xy)](x(yz))$

$= [((((xy)z)/z)/y)/x](x(yz))$

$= [(((xy)z)/(yz))/x](x(yz))$

$= [((xy)z)/(x(yz))](x(yz))$

$= [(x(yz))/((xy)z)]((xy)z) =$ reverse steps to get $= (xy)z$

Equational basis for generalized hoops

$x1 = x$

$x/x = 1 = x\backslash x$

$x/(yz) = (x/z)/y \qquad y\backslash(z\backslash x) = (zy)\backslash x$

$(x/y)y = (y/x)x = y(y\backslash x)$

Generalized hoops are also called **pseudo hoops**

Note: The term $(x/y)y$ defines a binary operation that is **commutative** and **idempotent** ($(x/x)x = 1x = x$).

A meet-semilattice term

Lemma: $(x/y)y$ is associative, hence written as $x \wedge y$. It is a meet since $x \leq y \iff 1 = y/x \iff x = (x/y)y$

Proof. $(x \wedge y) \wedge z = (((x/y)y)/z)z$

A meet-semilattice term

Lemma: $(x/y)y$ is associative, hence written as $x \wedge y$. It is a meet since $x \leq y \iff 1 = y/x \iff x = (x/y)y$

Proof. $(x \wedge y) \wedge z = (((x/y)y)/z)z = (z/(x/y)y)(x/y)y$

A meet-semilattice term

Lemma: $(x/y)y$ is associative, hence written as $x \wedge y$. It is a meet since $x \leq y \iff 1 = y/x \iff x = (x/y)y$

Proof. $(x \wedge y) \wedge z = (((x/y)y)/z)z = (z/(x/y)y)(x/y)y$

$= ((z/y)/(x/y))(x/y)y$

A meet-semilattice term

Lemma: $(x/y)y$ is associative, hence written as $x \wedge y$. It is a meet since $x \leq y \iff 1 = y/x \iff x = (x/y)y$

Proof. $(x \wedge y) \wedge z = (((x/y)y)/z)z = (z/(x/y)y)(x/y)y$

$= ((z/y)/(x/y))(x/y)y$

$= ((x/y)/(z/y))(z/y)y$

A meet-semilattice term

Lemma: $(x/y)y$ is associative, hence written as $x \wedge y$. It is a meet since $x \leq y \iff 1 = y/x \iff x = (x/y)y$

Proof. $(x \wedge y) \wedge z = (((x/y)y)/z)z = (z/(x/y)y)(x/y)y$

$= ((z/y)/(x/y))(x/y)y$

$= ((x/y)/(z/y))(z/y)y$

$= (x/(z/y)y)(z/y)y$

A meet-semilattice term

Lemma: $(x/y)y$ is associative, hence written as $x \wedge y$. It is a meet since $x \leq y \iff 1 = y/x \iff x = (x/y)y$

Proof. $(x \wedge y) \wedge z = (((x/y)y)/z)z = (z/(x/y)y)(x/y)y$

$= ((z/y)/(x/y))(x/y)y$

$= ((x/y)/(z/y))(z/y)y$

$= (x/(z/y)y)(z/y)y = x \wedge (z \wedge y) = x \wedge (y \wedge z)$

Multiplication distributes over meet

[Galatos ~04] Generalized hoops satisfy $(x \wedge y)z = xz \wedge yz$

Preliminary: $xz \leq xz \implies x \leq xz/z$

Multiplication distributes over meet

[Galatos ~04] Generalized hoops satisfy $(x \wedge y)z = xz \wedge yz$

Preliminary: $xz \leq xz \implies x \leq xz/z$ hence $xz \leq (xz/z)z$

Multiplication distributes over meet

[Galatos ~04] Generalized hoops satisfy $(x \wedge y)z = xz \wedge yz$

Preliminary: $xz \leq xz \implies x \leq xz/z$ hence $xz \leq (xz/z)z$

$xz/z \leq xz/z \implies (xz/z)z \leq xz$

Multiplication distributes over meet

[Galatos ~04] Generalized hoops satisfy $(x \wedge y)z = xz \wedge yz$

Preliminary: $xz \leq xz \implies x \leq xz/z$ hence $xz \leq (xz/z)z$

$xz/z \leq xz/z \implies (xz/z)z \leq xz$ therefore $xz = (xz/z)z$

Multiplication distributes over meet

[Galatos ~04] Generalized hoops satisfy $(x \wedge y)z = xz \wedge yz$

Preliminary: $xz \leq xz \implies x \leq xz/z$ hence $xz \leq (xz/z)z$

$xz/z \leq xz/z \implies (xz/z)z \leq xz$ therefore $xz = (xz/z)z$

Now $(x \wedge y)z \leq xz \wedge yz$ always holds since $\cdot$ is order-preserving

Multiplication distributes over meet

[Galatos ~04] Generalized hoops satisfy $(x \wedge y)z = xz \wedge yz$

Preliminary: $xz \leq xz \implies x \leq xz/z$ hence $xz \leq (xz/z)z$

$xz/z \leq xz/z \implies (xz/z)z \leq xz$ therefore $xz = (xz/z)z$

Now $(x \wedge y)z \leq xz \wedge yz$ always holds since $\cdot$ is order-preserving

$xz \wedge yz = (xz/yz)yz = ((xz/z)/y)yz$

Multiplication distributes over meet

[Galatos ~04] Generalized hoops satisfy $(x \wedge y)z = xz \wedge yz$

Preliminary: $xz \leq xz \implies x \leq xz/z$ hence $xz \leq (xz/z)z$

$xz/z \leq xz/z \implies (xz/z)z \leq xz$ therefore $xz = (xz/z)z$

Now $(x \wedge y)z \leq xz \wedge yz$ always holds since $\cdot$ is order-preserving

$xz \wedge yz = (xz/yz)yz = ((xz/z)/y)yz$

$= (y/((xz)/z))(xz/z)z$

Multiplication distributes over meet

[Galatos ~04] Generalized hoops satisfy $(x \wedge y)z = xz \wedge yz$

Preliminary: $xz \leq xz \implies x \leq xz/z$ hence $xz \leq (xz/z)z$

$xz/z \leq xz/z \implies (xz/z)z \leq xz$ therefore $xz = (xz/z)z$

Now $(x \wedge y)z \leq xz \wedge yz$ always holds since $\cdot$ is order-preserving

$xz \wedge yz = (xz/yz)yz = ((xz/z)/y)yz$

$= (y/((xz)/z))(xz/z)z = (y/((xz)/z))xz$

Multiplication distributes over meet

[Galatos ~04] Generalized hoops satisfy $(x \wedge y)z = xz \wedge yz$

Preliminary: $xz \leq xz \implies x \leq xz/z$ hence $xz \leq (xz/z)z$

$xz/z \leq xz/z \implies (xz/z)z \leq xz$ therefore $xz = (xz/z)z$

Now $(x \wedge y)z \leq xz \wedge yz$ always holds since $\cdot$ is order-preserving

$xz \wedge yz = (xz/yz)yz = ((xz/z)/y)yz$

$= (y/((xz)/z))(xz/z)z = (y/((xz)/z))xz$

$\leq (y/x)xz = (y \wedge x)z$

Hoops and GBL-algebras

Commutative generalized hoops are called **hoops**

In this case $x/y = y\backslash x$ usually written as $y \to x$

If we expand the signature of generalized hoops with $\vee$

and add lattice identities then we get **integral GBL-algebras**

Add **bottom** 0, **commutativity**, and $(x \to y) \vee (y \to x) = 1$

get Hajek's **Basic Logic algebras**

Includes BA, Heyting algebras, MV-algebras, GA, PA

Open Problem: Is the equational theory of integral GBL-algebras decidable?

Finite generalized hoops

Finite GH are **reducts** of integral GBL-algebras

[J. & Montagna 06] Finite GBL-algebras are commutative

Hence finite GH are commutative

[J. & Montagna 09] Finite GBL-algebras are **poset products**

of **Wajsberg chains** $W_n = (\{0, a^{n-1}, \ldots, a^3, a^2, a, 1\}, \cdot, 1, \to)$

A poset product is a subalgebra of a direct product over a partially ordered index set

Poset products

For **bounded** GH or GBL-algebras C_i indexed by a poset P

$$\prod_{\mathbf{P}} C_i = \{f \in \prod_{i \in P} C_i : \forall i > j \in P \ \ (f(i) \neq 0 \implies f(j) = 1)\}$$

The operations $\wedge, \vee, \cdot$ are defined pointwise and the bounds are the constant functions $\mathbf{0}, \mathbf{1}$. The residuals are given by

$$(f \backslash g)(i) = \begin{cases} f(i) \backslash g(i) & \text{if } f(j) \leq g(j) \text{ for all } j < i \\ 0 & \text{otherwise} \end{cases}$$

$$(g / f)(i) = \begin{cases} g(i) / f(i) & \text{if } f(j) \leq g(j) \text{ for all } j < i \\ 0 & \text{otherwise.} \end{cases}$$

If the poset is **linear** we get an **ordinal sum** of the factors

If the poset is an **antichain**, we get the **direct product**

If the factors are **Boolean algebras**, get a **Heyting algebra**

Can build **all** finite GH and GBL-algebras: **pick a finite poset** P

Pick a positive integer n_i for each $i \in P$

Get **all** finite GH and GBL-algebras uniquely up to isomorphism

The algebra is **subdirectly irreducible** iff poset has a **top**

Generalized hoops are **congruence distributive** [Botur, Dvurečenskij, Kowalski 2012]

Can construct lattice of **finitely generated subvarieties**

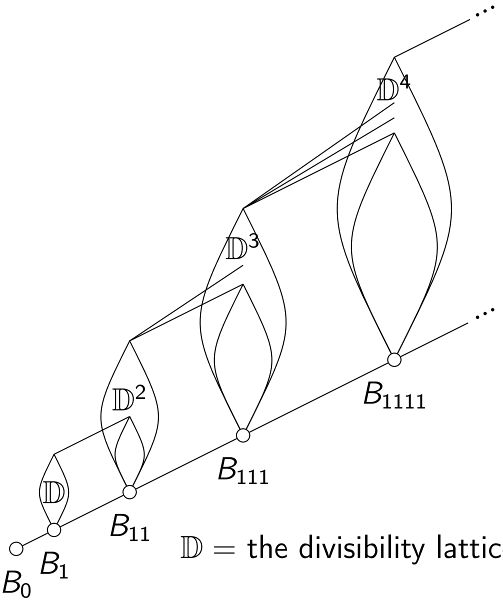
W_m is a subalgebra of W_n iff $m|n$

Therefore the varieties $V(W_n)$, ordered by inclusion, form the divisibility lattice $\mathbb{D}$

The lattice of all finitely generated subvarieties of Wasjberg hoops is isomorphic to the downset lattice of $\mathbb{D}$ [Komori 81]

Theorem. The poset of **finitely generated join irreducible BL-varieties** is isomorphic to $\mathbb{D}^* = \bigcup_{n=0}^{\infty} \mathbb{D}^n$ with the order on $\mathbb{D}^*$ extending the pointwise divisibility order on each component as follows: The order relation $(a_1, \dots, a_m) \leq (b_1, \dots, b_n)$ is a **covering relation** if and only if either

- $m = n$ and $(b_1, \dots, b_n) = (a_1, \dots, a_{i-1}, pa_i, a_{i+1}, \dots, a_n)$ for some prime p and a unique $i \leq n$, or
- $m + 1 = n$ and $(b_1, \dots, b_n) = (a_1, \dots, a_{i-1}, 1, a_i, \dots, a_m)$ for some $i \in \{2, \dots, n\}$



$\mathbb{D}$ = the divisibility lattice on $\mathbb{Z}^+$

SAT-solvers

SAT stands for *satisfiability* of Boolean formulas

Given a Boolean formula φ with propositional variables $p_1, \ldots, p_n$

decide if there is an assignment $h : \{p_1, \ldots, p_n\} \to \{T, F\}$ such that

h extended homomorphically to all formulas makes $h(\varphi) = T$

SAT was the first problem proved to be NP-complete

i.e., there is a nondeterministic Turing machine that decides SAT in polynomial time and every other problem that can be decided in nondeterministic polynomial time has a polynomial time reduction to a SAT problem

SMT-solvers

SMT stands for *satisfiability modulo theories*

Combines SAT-solving with other decision procedures for fragments of first-order logic and arithmetic

SMT-solvers were developed in computer science for static analysis of programs

Input is a (limited) choice of a decidable theory and a list of Boolean combinations of atomic formulas in the signature of this theory

Quantifier-free decidable theories

QF_LRA **quantifier free linear real number arithmetic** with $+, -, <, =$

e.g. **not**$(0 > x + y$ **or** $x + y > 5)$ **and** $(x + x - y - y = 1)$

QF_RA is like QF_LRA but also allows multiplication, division

SMT-solvers decide if there exists an assignment of real numbers to the variables in the list of formulas such that all the formulas are true in $\mathbb{R}$; return assignment if it exists

How SMT-solvers work

Basic idea: replace atomic formulas by Boolean variables, call a SAT-solver

if the Boolean formulas are **not satisfiable**, return **F**

else use each possible Boolean assignment to generate a list of linear atomic formulas and call a **Linear Programming package**

if an assignment is found, return it, but if none of the Boolean assignments work, return **F**

SMT-solver input for abelian ℓ-groups

Easy, the variety of abelian ℓ-groups is generated by $(\mathbb{R}, \min, \max, +, -, 0)$

SMT_LIB2 is a standard LISP-like language for SMT-solver input

```
;Testing abelian l-group equations in SMT
(set-logic QF_LRA)
(define-fun wedge ((x Real) (y Real)) Real (ite (> x y) y x))
(define-fun vee ((x Real) (y Real)) Real (ite (> x y) x y))
(declare-const x Real)
(declare-const y Real)
(assert (> (vee (+ x x) (+ y y)) (+ (vee x y) (vee x y))))
```
; test if $(x + x) \vee (y + y) \leq (x \vee y) + (x \vee y)$ is an identity
```
(check-sat)
```

SMT-solver input for infinitely-valued logics

The idea of using SMT-solvers for logics based on intervals of the real numbers is from the following paper:

C. Ansótegui, M. Bofill, F. Manyà and M. Villaret, ***Building automated theorem provers for infinitely-valued logics with satisfiability modulo theory solvers***, in Proceedings, IEEE 42nd International Symposium on Multiple-Valued Logic. ISMVL 2012, 25–30.

They give examples of SMT-LIB2 code for **Lukasiewicz logic** and **product logic**

SMT-solver input for MV-algbras

The variety of MV-algebras is $HSP(([0,1], \wedge, \vee, \cdot, 1, 0, \to))$

```
;Testing MV-algebra equations in SMT
(set-logic QF_LRA)
(define-fun wedge ((x Real) (y Real)) Real (ite (> x y) y x))
(define-fun vee ((x Real) (y Real)) Real (ite (> x y) x y))
(define-fun oplus ((x Real) (y Real)) Real (wedge (+ x y) 1))
(define-fun cdot ((x Real) (y Real)) Real (vee (- (+ x y) 1) 0))
(define-fun neg ((x Real)) Real (- 1 x))
(define-fun to ((x Real) (y Real)) Real (wedge 1 (- (+ 1 y) x)))
(declare-const x Real) (assert (<= 0 x)) (assert (<= x 1))
(declare-const y Real) (assert (<= 0 y)) (assert (<= y 1))
(assert (< (to (vee (cdot x x) (cdot y y)) (cdot (vee x y) (vee
x y))) 1))
```
; test if $(x^2 \vee y^2) \to (x \vee y)^2 < 1$ is satisfiable
```
(check-sat)
```

Other standard Basic Logic algebras

For Gödel algebras redefine fusion as min(x,y).

(define-fun cdot ((x Real) (y Real)) Real (ite (> x y) y x))

For product algebras use

(define-fun cdot ((x Real) (y Real)) Real (ite (> x y) y x))
(declare-const x Real) (assert (<= x 0));
(declare-const x Real) (assert (<= x 0));

and do a translation to the formula that adds an extra variable z (for bottom)

replacing variable x by $x \vee z$ and subterms $s \cdot t$ by $s \cdot t \vee z$

Prop 7.4 in Galatos, Tsinakis (2005) Generalized MV-algebras

Checking identities in BL-algebras

To decide propositional basic logic with an SMT-solver requires the following result of Agliano Montagna 2003 (see also Aguzzoli and Bova 2010).

Theorem

Let $A_n = \bigoplus_{i=0}^{n}[0,1]$ *be the ordinal sum of* $n+1$ *unit-interval MV-algebras, and let* $\mathcal{V}_n$ *be the variety generated by all n-generated BL-algebras. Then* $\mathcal{V}_n = HSP(A_n)$*, hence an n-variable BL-identity holds in* A_n *if and only if it holds in all BL-algebras.*

By constructing the algebra A_n of the above result within the SMT language, one obtains an effective means of checking n-variable BL-identities.

Checking identities in BL-algebras

The universe for A_n is taken to be the interval $[0, n+1]$
The definition of fusion and implication are

$$x \cdot y = \begin{cases} \max(x + y - 1 - \lfloor y \rfloor, \lfloor x \rfloor) & \text{if } \lfloor x \rfloor = \lfloor y \rfloor \\ \min(x, y) & \text{otherwise} \end{cases}$$

$$x \to y = \begin{cases} n+1 & \text{if } x \leq y \\ y & \text{if } \lfloor y \rfloor < \lfloor x \rfloor \\ \min(1 + y - x + \lfloor x \rfloor, 1 + \lfloor y \rfloor) & \text{otherwise} \end{cases}$$

A straightforward SMT-LIB2 implementation of these operations uses $n+1$ cases, so the formula does become long even for small values of n

Below we give the implementations for $n = 1$ and $n = 2$, which can be used to check 1-variable and 2-variable BL-identities

Checking identities in BL-algebras

$n = 1$:

```
(define-fun cdot ((x Real) (y Real)) Real (ite (and (< x 1) (<
y 1)) (vee (- (+ x y) 1) 0) (ite (and (>= x 1) (>= y 1)) (vee
(- (+ x y) 2) 1) (wedge x y) ) ) )
```

```
(define-fun to ((x Real) (y Real)) Real (ite (<= x y) 2 (ite
(and (>= x 1) (< y 1)) y (wedge 1 (- (+ 1 y) x)) ) ) )
```

$n = 2$:

```
(define-fun cdot ((x Real) (y Real)) Real (ite (and (< x 1) (<
y 1)) (vee (- (+ x y) 1) 0) (ite (and (>= x 1) (< x 2) (>= y
1) (< y 2)) (vee (- (+ x y) 2) 1) (ite (and (>= x 2) (>= y
2)) (vee (- (+ x y) 3) 2) (wedge x y)) )))
```

```
(define-fun to ((x Real) (y Real)) Real (ite (<= x y) 3 (ite
(and (< x 1) (< y 1)) (+ (- 1 x) y) (ite (and (<= 1 x) (< x
2) (<= 1 y) (< y 2)) (+ (- 2 x) y) (ite (and (<= 2 x) (<= 2
y) ) (+ (- 3 x) y) y)))))
```

Automating the translation

A Python program is used to parse a LaTeX BL-algebra identity

A SMT-LIB2 file is generated using $\cdot$ and $\to$ of A_n

The python program then calls an SMT-solver with the file as input

The result is analyzed and the truth value is returned

If the identity fails, an assignment in $[0, n]$ can be obtained

Demo

Some References

P. Agliano and F. Montagna, *Varieties of BL-algebras I: general properties*, Journal of Pure and Applied Algebra, 181 (2003), 105–129

S. Aguzzoli and S. Bova, *The free n-generated BL-algebra*, Annals of Pure and Applied Logic 161 (2010), 1144–1170

C. Ansótegui, M. Bofill, F. Manyà and M. Villaret, *Building automated theorem provers for infinitely-valued logics with satisfiability modulo theory solvers*, in Proceedings, IEEE 42nd International Symposium on Multiple-Valued Logic. ISMVL 2012, 25–30

M. Botur, A. Dvurečenskij and T. Kowalski, *On normal-valued basic pseudo hoops*, Soft Computing, 16(4) (2012), 635–644

N. Galatos and C. Tsinakis, *Generalized MV-algebras*, Journal of Algebra, 283(1) (2005), 254–291

P. Jipsen and F. Montagna, *The Blok-Ferreirim theorem for normal GBL-algebras and its application*, Algebra Universalis, 60 (2009), 381–404

Thank You

BLAST 2013, August 5-9, Chapman University, Orange, CA