

Algebraic closure of some generalized convex sets

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AFFINE SPACES AND CONVEX SETS

1. Real affine spaces

Given a vector space (or a module) A over a subfield (or a subring) R of \mathbb{R} :

An **affine space** A **over** R (or **affine R -space**) is the algebra

$$\left(A, \sum_{i=1}^n x_i r_i \mid \sum_{i=1}^n r_i = 1 \right).$$

This algebra is equivalent to

$$(A, P, \underline{R}),$$

where

$$\underline{R} = \{ \underline{f} \mid f \in R \}$$

and

$$xy\underline{f} := x(1 - f) + yf =: \underline{f}(x, y),$$

and P is the Mal'cev operation

$$xyzP := x - y + z =: P(x, y, z).$$

The class $\underline{\underline{R}}$ of all affine R -spaces is a variety.

The variety $\underline{\underline{R}}$ satisfies the **entropic** identities

$$xyp \ ztp \ q = xzq \ ytq \ p$$

for all $p, q \in R$.

Abstractly, $\underline{\underline{R}}$ is defined as the class of idempotent entropic Mal'cev algebras $(A, P, \underline{\underline{R}})$ with a ternary Mal'cev operation P and binary operations \underline{r} for each $r \in R$, satisfying the identities:

$$xy\underline{0} = x = yx\underline{1},$$

$$xyp \ xyq \ \underline{r} = xy \ \underline{pqr},$$

$$xyp \ xyq \ xy\underline{r} \ P = xy \ \underline{pqr}P$$

for all $p, q, r \in R$.

The variety $\underline{\underline{R}}$ also satisfies the **cancellation laws**

$$(xyp = xzp) \rightarrow y = z$$

for all invertible $p \in R$.

2. Convex sets and barycentric algebras

Let F be a subfield of \mathbb{R} , $I^o(F) :=]0, 1[\subset F$ and $I(F) := [0, 1] \subset F$.

Convex subsets of affine F -spaces (or **F -convex sets**) are $\underline{I}^o(F)$ -subreducts $(A, \underline{I}^o(F))$ of affine F -spaces.

The class $\mathcal{C}v(F)$ of F -convex sets generates the variety $\mathcal{B}(F)$ of **F -barycentric algebras**.

Theorem The class $\mathcal{C}v(F)$ and the quasivariety $\mathcal{C}(F)$ of cancellative F -barycentric algebras coincide. $\mathcal{C}v(F)$ is a minimal subquasivariety of the variety $\mathcal{B}(F)$.

3. Intervals of the F -line

The algebra (F, \underline{F}) is called an F -**line**, and intervals of the F -line are closed bounded intervals considered as $\underline{I}^o(F)$ -algebras.

Proposition The following conditions are equivalent for any non-trivial subalgebra $(A, \underline{I}^o(F))$ of $(F, \underline{I}^o(F))$:

- (a) $(A, \underline{I}^o(F))$ is a closed interval of $(F, \underline{I}^o(F))$;
- (b) $(A, \underline{I}^o(F))$ is isomorphic to $(I(F), \underline{I}^o(F))$;
- (c) $(A, \underline{I}^o(F))$ is generated by two (distinct) elements;
- (d) $(A, \underline{I}^o(F))$ is a free algebra on two free generators in the quasivariety $\mathcal{C}(F)$ and in the variety $\mathcal{B}(F)$.

4. R -convex sets

Now assume that R is a principal ideal subdomain of \mathbb{R} such that $\mathbb{Z} \subset R \subseteq \mathbb{R}$.

The algebra (R, P, \underline{R}) is called an R -**line**.

Let $I^o(R) :=]0, 1[\subset R$ and $I(R) := [0, 1] \subset R$.

Intervals of (R, P, \underline{R}) are defined as in the case $R = F$.

Note that not all intervals of the line (R, P, \underline{R}) are isomorphic to the unit interval $(I(R), \underline{I^o(R)})$, and not all are generated by its endpoints.

However $(I(R), \underline{I^o(R)})$ is generated by the endpoints and is free on two generators, in the quasivariety and the variety it generates.

Algebraic R -convex subsets of affine R -spaces are $I^o(R)$ -subreducts $(A, \underline{I^o(R)})$ of faithful affine R -spaces.

Geometric R -convex sets of affine R -spaces R^n are the intersections of \mathbb{R} -convex subsets of \mathbb{R}^n with the subspace R^n .

If R is a field, both concepts coincide.

If not, then the algebraic and geometric definitions of R -convex sets do not coincide.

Proposition The class $\mathcal{C}v(R)$ of $\underline{I}^o(R)$ -subreducts of faithful affine R -spaces is a (minimal) quasivariety containing the class of geometric R -convex sets.

$\mathcal{C}v(R)$ **does not** coincide with the quasivariety of cancellative members of the variety generated by $\underline{I}^o(R)$ -subreducts of affine R -spaces.

MODES

An algebra (A, Ω) is a **mode** if it is

- **idempotent:**

$$x \dots x \omega = x,$$

for each n -ary $\omega \in \Omega$, and

- **entropic:**

$$\begin{aligned} & (x_{11} \dots x_{1n} \omega) \dots (x_{m1} \dots x_{mn} \omega) \varphi \\ &= (x_{11} \dots x_{m1} \varphi) \dots (x_{1n} \dots x_{mn} \varphi) \omega. \end{aligned}$$

for all $\omega, \varphi \in \Omega$.

Affine R -spaces, R -convex sets and their sub-reducts are modes.

ALGEBRAIC CLOSURES OF GEOMETRIC R -CONVEX SETS

From now on, R is a principal ideal subdomain of \mathbb{R} such that $\mathbb{Z} \subset R \subseteq \mathbb{R}$, and $I^o(R)$ contains an invertible element s .

All R -convex sets $(C, \underline{I}^o(R))$ are assumed to be geometric subsets of an affine R -space A isomorphic to (R^k, P, \underline{R}) for some $k = 1, 2, \dots$

For $(a, b) \in C \times C$,

$\langle a, b \rangle$ denotes the $\underline{I}^o(R)$ -subalgebra generated by a and b , and

$\langle a, b \rangle^o := \langle a, b \rangle \setminus \{a, b\}$.

1. Algebraic s -closures

The pair (a, b) is called **s -eligible**, if for each $x \in \langle a, b \rangle^o$ there is a $y \in C$ with $b = xys$.

$E_s(C)$ denotes the set of s -eligible pairs of $(C, \underline{I}^o(R))$.

Lemma The set $E_s(C)$ forms a subalgebra of $(A \times A, \underline{I}^o(R))$.

Lemma Let $(a, b) \in C \times C$. Then (a, b) is an s -eligible pair of $(C, \underline{I}^o(R))$ if and only if $\underline{xb1/s} \in C$ for each $x \in \langle a, b \rangle^o$.

An R -convex subset $(C, \underline{I}^o(R))$ of an affine R -space A is called **algebraically s -closed** if for each s -eligible pair $(a, b) \in C \times C$, there is a $c \in C$ such that $b = acs$.

Proposition An R -convex subset $(C, \underline{I}^o(R))$ of an affine R -space A is algebraically s -closed if and only if $\underline{ab1/s} \in C$ for each s -eligible pair $(a, b) \in C \times C$.

Let

$$\overline{C}_s := \{\underline{ab1/s} \mid (a, b) \in E_s(C)\}.$$

The set \overline{C}_s is called the **algebraic s -closure** of $(C, \underline{I}^o(R))$.

Lemma The s -closure \overline{C}_s of an R -convex subset $(C, \underline{I}^o(R))$ of an affine R -space A is a subalgebra of $(A, \underline{I}^o(R))$.

Lemma Let s and t be any two invertible elements of $I^o(R)$. Then \overline{C}_s and \overline{C}_t coincide.

2. Algebraic closures

The s -closure \overline{C}_s of C will be called the **algebraic closure** or simply the **closure** of C , and will be denoted by \overline{C} .

Proposition Let C be a k -dimensional geometric convex subset of the affine R -space R^k . Then its closure \overline{C} is also a geometric k -dimensional convex subset of R^k , and it coincides with the convex hull $\text{conv}_R(\overline{C})$ of \overline{C} .

Proposition The following hold for the closures \overline{B} and \overline{C} of R -convex subsets $(C, \underline{I}^o(R))$ and $(B, \underline{I}^o(R))$ of an affine R -space R^k .

- (a) $C \leq \overline{C}$;
- (b) If $(B, \underline{I}^o(R)) \leq (C, \underline{I}^o(R))$,
then $(\overline{B}, \underline{I}^o(R)) \leq (\overline{C}, \underline{I}^o(R))$;
- (c) $\overline{\overline{C}} = \overline{C}$.

ALGEBRAIC AND OTHER CLOSURES

Consider an affine R -space (A, P, \underline{R}) . Define the following relation \sim_s on the set $A \times A$:

$$(a_1, b_1) \sim_s (a_2, b_2) \text{ iff } a_1 b_2 \underline{s} = a_1 a_2 \underline{s} b_1 \underline{s}.$$

Lemma (a) The relation \sim_s is a congruence relation of the affine R -space $(A \times A, P, \underline{R})$.

(b) The mapping

$$\varphi : A \rightarrow (A \times A)^{\sim_s} ; a \mapsto (a, a)^{\sim_s}$$

is an embedding of affine R -spaces.

(c) The relation \sim_s is a congruence relation of $\underline{I}^o(R)$ -subreducts of $(A \times A, P, \underline{R})$, in particular of each R -convex set $(C \times C, \underline{I}^o(R))$.

Lemma Let $(A, \underline{I}^o(R))$ be the $\underline{I}^o(R)$ -reduct of an affine R -space (A, P, \underline{R}) . Then

$$(E_s(A), \underline{I}^o(R))^{\sim_s} \cong (A, \underline{I}^o(R))$$

.

1. Algebraic closures and aiming congruences

The congruence \sim_s of $(C \times C, \underline{I}^o(R))$ is called the **aiming congruence**.

Proposition Let $(C, \underline{I}^o(R))$ be an R -convex subset of an affine R -space (A, P, \underline{R}) . Then

$$(\overline{C}_s, \underline{I}^o(R)) \cong (E_s(C), \underline{I}^o(R)) \sim_s.$$

Corollary The following conditions are equivalent for a k -dimensional geometric R -convex subset C of the affine R -space R^k , where $k = 1, 2, \dots$, and an invertible element $s \in I^o(R)$:

- (a) $(C, \underline{I}^o(R))$ is algebraically closed,
- (b) $(C, \underline{I}^o(R)) \cong (\overline{C}, \underline{I}^o(R))$,
- (c) $(C, \underline{I}^o(R)) \cong (E_s(C), \underline{I}^o(R)) \sim_s$.

2. Algebraic and topological closures

We consider the usual Euclidean topology on \mathbb{R}^k , and R^k as a topological subspace of \mathbb{R}^k . Its closed (open) sets are simply closed (open) subsets of \mathbb{R}^k intersected with R^k .

For a geometric convex subset C of R^k , let C_R^{tc} be its topological closure in R^k , and $C_{\mathbb{R}}^{tc}$ its topological closure in \mathbb{R}^k .

Theorem Let $(C, \underline{I}^o(R))$ be a k -dimensional geometric convex subset of an affine R -space (R^k, P, \underline{R}) . Then the algebraic closure \overline{C} of C and the topological closure C_R^{tc} of C in R^k coincide:

$$\overline{C} = C_R^{tc}.$$

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