Finitely decidable varieties admitting type 1 are residually finite

Ralph McKenzie and Matthew Smedberg

Vanderbilt University
Department of Mathematics

13 April 2013

The Finite Decidability Problem

Let $\mathcal{V}$ be a variety (usually locally finite) in a finite language. We say $\mathcal{V}$ is *decidable* if its first-order theory is, and *finitely decidable* if the theory of $\mathcal{V}_{\text{fin}}$ is decidable.

The Finite Decidability Problem

Let $\mathcal{V}$ be a variety (usually locally finite) in a finite language. We say $\mathcal{V}$ is *decidable* if its first-order theory is, and *finitely decidable* if the theory of $\mathcal{V}_{\mathrm{fin}}$ is decidable.

Decidable and finitely decidable varieties are rare and structurally constrained. For example,

The Finite Decidability Problem

Let $\mathcal{V}$ be a variety (usually locally finite) in a finite language. We say $\mathcal{V}$ is *decidable* if its first-order theory is, and *finitely decidable* if the theory of $\mathcal{V}_{\text{fin}}$ is decidable.

Decidable and finitely decidable varieties are rare and structurally constrained. For example,

Fact

- *If* $\mathbf{A}$ *has any congruence covers of the lattice or semilattice types, or*

then every variety containing $\mathbf{A}$ *is finitely undecidable.*

The Finite Decidability Problem

Let $\mathcal{V}$ be a variety (usually locally finite) in a finite language. We say $\mathcal{V}$ is *decidable* if its first-order theory is, and *finitely decidable* if the theory of $\mathcal{V}_{\text{fin}}$ is decidable.

Decidable and finitely decidable varieties are rare and structurally constrained. For example,

Fact

- *If* **A** *has any congruence covers of the lattice or semilattice types, or*
- *If any boolean- or affine-type minimal sets in* **A** *have nonempty tails, or*

then every variety containing **A** *is finitely undecidable.*

The Finite Decidability Problem

Let $\mathcal{V}$ be a variety (usually locally finite) in a finite language. We say $\mathcal{V}$ is *decidable* if its first-order theory is, and *finitely decidable* if the theory of $\mathcal{V}_{\text{fin}}$ is decidable.

Decidable and finitely decidable varieties are rare and structurally constrained. For example,

Fact

- *If* $\mathbf{A}$ *has any congruence covers of the lattice or semilattice types, or*
- *If any boolean- or affine-type minimal sets in* $\mathbf{A}$ *have nonempty tails, or*
- *If* $\mathbf{A}$ *is a subdirectly irreducible finite algebra with two incomparable nonabelian congruences,*

then every variety containing $\mathbf{A}$ *is finitely undecidable.*

Fact (continued)

If **A** *is a finite algebra*

- *and* **A** *has a solvable congruence which is nonabelian, or*

then every variety containing **A** *is finitely undecidable.*

Fact (continued)

If **A** *is a finite algebra*

- *and* **A** *has a solvable congruence which is nonabelian, or*
- **A** *is subdirectly irreducible with boolean monolith and also has a cover of type 1 or 2, or*

then every variety containing **A** *is finitely undecidable.*

Fact (continued)

If **A** *is a finite algebra*

- *and* **A** *has a solvable congruence which is nonabelian, or*
- **A** *is subdirectly irreducible with boolean monolith and also has a cover of type 1 or 2, or*
- **A** *is SI with type 2 monolith and has also a cover of type 1, or vice versa,*

then every variety containing **A** *is finitely undecidable.*

Fact (continued)

If **A** *is a finite algebra*

- *and* **A** *has a solvable congruence which is nonabelian, or*
- **A** *is subdirectly irreducible with boolean monolith and also has a cover of type 1 or 2, or*
- **A** *is SI with type 2 monolith and has also a cover of type 1, or vice versa,*

then every variety containing **A** *is finitely undecidable.*

These facts (and many of a similar nature) were established for modular varieties in the 90s (see [Idziak 1997]). The results for nonmodular varieties are in most cases new.

Bounding Subdirect Irreducibles in $\mathcal{V}$

Theorem

Let $\mathcal{K}$ be a finite set of finite algebras, and suppose $\mathcal{V} = \mathrm{HSP}(\mathcal{K})$ is finitely decidable. Then there is a finite bound on the cardinalities of SI algebras in $\mathcal{V}$.

Bounding Subdirect Irreducibles in $\mathcal{V}$

Theorem

Let $\mathcal{K}$ be a finite set of finite algebras, and suppose $\mathcal{V} = \mathrm{HSP}(\mathcal{K})$ is finitely decidable. Then there is a finite bound on the cardinalities of SI algebras in $\mathcal{V}$.

Using familiar methods from the congruence-modular case, we show that

- every SI with boolean-type monolith belongs to $\mathrm{HS}(\mathcal{K})$;

Bounding Subdirect Irreducibles in $\mathcal{V}$

Theorem

Let $\mathcal{K}$ be a finite set of finite algebras, and suppose $\mathcal{V} = \mathrm{HSP}(\mathcal{K})$ is finitely decidable. Then there is a finite bound on the cardinalities of SI algebras in $\mathcal{V}$.

Using familiar methods from the congruence-modular case, we show that

- every SI with boolean-type monolith belongs to $\mathrm{HS}(\mathcal{K})$;
- there is a bound ($\sim$ quadruply exponential) on the affine-type SIs.

Bounding unary-type SIs in $\mathcal{V}$

So let $\mathbf{S} \in \mathcal{V}$ have monolith $\perp \overset{1}{\prec} \mu$.

Lemma

$\mathrm{Rad}(\mathbf{S})$ *is comparable to all congruences of* $\mathbf{S}$.

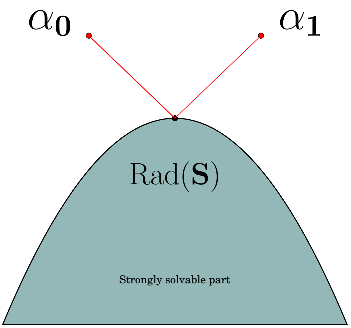
Lemma

$\mathrm{Rad}(\mathbf{S})$ *is meet-irreducible.*

Each of these is proved by contradiction: supposing the respective lemma were false, we construct a (relatively straightforward) semantic interpretation of some finitely undecidable class, usually graphs, into $\mathrm{HSP}(\mathbf{S})$.

Meet-irreducibility of the solvable radical

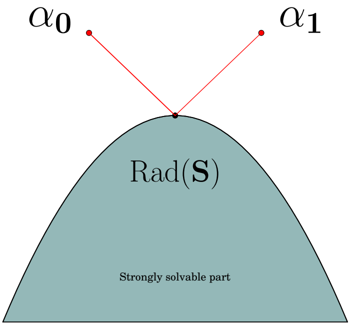
Goal: to semanticallly interprect a structure of the form $\langle I; E_0, E_1 \rangle$ (where the E_j are disjoint equivalence relations) into subpowers of $\mathbf{S}$.



Meet-irreducibility of the solvable radical

Goal: to semantically interprect a structure of the form $\langle I; E_0, E_1 \rangle$ (where the E_j are disjoint equivalence relations) into subpowers of **S**.

Let $\{0_j, 1_j\}$ be $(\mathrm{Rad}(\mathbf{S}), \alpha_j)$-minimal sets. Let $\mathbf{B} \leq \mathbf{S}^I$ consist of all **x** which are α_1-constant on E_1-blocks and vice versa.

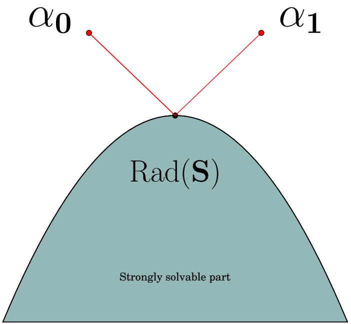


Meet-irreducibility of the solvable radical

Goal: to semantically interprect a structure of the form $\langle I; E_0, E_1\rangle$ (where the E_j are disjoint equivalence relations) into subpowers of $\mathbf{S}$.

Let $\{0_j, 1_j\}$ be $(\mathrm{Rad}(\mathbf{S}), \alpha_j)$-minimal sets. Let $\mathbf{B} \leq \mathbf{S}^I$ consist of all $\mathbf{x}$ which are α_1-constant on E_1-blocks and vice versa.

Using a failure of $\mathrm{C}(\mu, \{0_j, 1_j\}; \perp_S)$, and some tricks from tame congruence theory,



we reconstruct the original structure $\langle I; E_0, E_1\rangle$ in a first-order way from $\mathbf{B}$.

Since $\mathrm{Rad}(\mathbf{S})$ is meet-irreducible, we know that its index cannot exceed the maximum size of a boolean-type SI in $\mathcal{V}$.

Theorem

$\mathrm{Rad}(\mathbf{S})$ *is strongly abelian.*

Proof.

Long! □

Takeaway idea: Subalgebra generation (and congruence generation) can frequently be proven to be "sparse" in some useful sense, when the generators are chosen so that they are almost constant modulo a strongly abelian congruence (such as the monolith).

Sparse subalgebra generation: Example

Fix a $(\perp, \mu)$-minimal set U, and say we are working to semantically interpret a graph $\langle V, E \rangle$ into a power of $\mathbf{S}$. Let $I = \{v^+, v^- : v \in V\}$. Define a subalgebra

$$\Delta \subseteq \mathbf{B} \leq \mathbf{S}^I$$

with generators those $\mathbf{x} \in U^I$ such that for some $v \in V$,

$$\begin{cases} x^{v^+} \equiv_\mu x^{v^-} \\ x^{w^+} = x^{w^-} \equiv_\mu x^{v^+} & \text{for all other } w \in V \end{cases}$$

We claim that $B \cap U^I$ consists of just the generators and no more.

Sparse subalgebra generation: Example II

Claim

$B \cap U^I$ consists of just the generators and no more.

Proof: let $\mathbf{y} = \mathbf{f}(\mathbf{x}_1, \ldots, \mathbf{x}_k) \in U^I$, where $f : \mathbf{S}^k \to U$ is a polynomial operation acting in $\mathbf{B}$ coordinatewise. Let C_j be the μ-class where $\mathbf{x}_j$ lives; then on $C_1 \times \cdots \times C_k$, f is essentially unary; say it depends on $\mathbf{x}_1$, which has its spike at $v_1 \in V$. Then $y^{v_1^+} \equiv_\mu y^{v_1^-}$, and for all $w \neq v_1$,

$$x_1^{w^+} = x_1^{w^-} \qquad \text{and} \qquad x_j^{w^+} \equiv_\mu x_j^{w^-}$$

so that

$$y^{w^+} = f(x_1^{x^+}, \ldots, x_k^{w^+}) = f(x_1^{x^-}, \ldots, x_k^{w^-}) = y^{w^-}$$

Sparse congruence generation

This sparseness allows us to construct some very involved semantic interpretations; for example, we may declare a congruence Θ on an algebra $\mathbf{B}$ as above to identify elements coding the endpoints of an edge of the graph $\langle V, E\rangle$; we use the sparseness to show that no vertices coding non-edges are made congruent as a consequence.

Bounding $\mathrm{Rad}(\mathbf{S})$-blocks

Say $\mathrm{Rad}(\mathbf{S})$ has index ℓ and some fixed monolith pair $c \neq d$. Since $\mathrm{Rad}(\mathbf{S})$ is strongly abelian,

Lemma

For any polynomial $t(v_0, \vec{v}_1, \ldots, \vec{v}_\ell)$, there exist subsets of each variable set $\vec{v}_i$, of size no more than $\log |\mathbf{F}_{\mathcal{V}}(2+\ell)|$, such that for all $\mathrm{Rad}(\mathbf{S})$-blocks $B_1, \ldots, B_\ell$, the mapping

$$A \times \vec{B}_1 \times \cdots \times \vec{B}_\ell \to A$$

induced by t depends only on v_0 and the indicated subsets.

Because of the Lemma, terms $f(v_0) = t(v_0, \vec{s})$ of bounded arity suffice to send exactly one of any unequal elements $x_1 \neq x_2$ to c.

Bounding $\mathrm{Rad}(\mathbf{S})$-blocks, II

Consider a fixed $\mathrm{Rad}(\mathbf{S})$-block B, and to each $b \in B$ associate the set of terms $t(v_0, v_1, \ldots, v_k)$, with k bounded as described in the last slide, such that for some $p_1, \ldots, p_k$ from the appropriate $\mathrm{Rad}(\mathbf{S})$-blocks, $t(b, \vec{p}) = c$.

Claim

This is an injective map from B to subsets of $\mathbf{F}_{\mathcal{V}}(1 + k)$

For if not, we get a failure of the strong term condition

$$c = t(b_1, \vec{p}_1) = t(b_2, \vec{p}_2) \text{ but } t(b_2, \vec{p}_1) \neq c$$

This contradiction completes the proof.

Open Problems

Problem

Are tails of minimal sets of type 1 always empty in FD varieties?

Problem

Do finitely decidable, locally finite varieties have definable principal congruences? Definable principal subcongruences? Definable principal solvable congruences?

Problem

In a finite algebra $\mathbf{A}$ *in a finitely decidable variety, must every congruence permute with* $\mathrm{Rad}(\mathbf{A})$*? With* $\mathrm{Rad}_u(\mathbf{A})$*?*

Thank you!