A disjunction characterizing varieties with a weak difference term

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Disjunctions from Malcev conditions

Bjarni Jónsson said a variety $\mathcal{V}$ has distributive congruence lattices iff here exists ternary terms $p_0, \ldots, p_n$ which satisfy the identities

$$
\begin{array}{rcll}
p_0(xyz) & \approx & x & \\
p_n(xyz) & \approx & z & \\
p_i(xyx) & \approx & x & 0 \leq i \leq n \\
p_i(xxy) & \approx & p_{i+1}(xxy) & i \quad \text{even} \\
p_i(xyy) & \approx & p_{i+1}(xyy) & i \quad \text{odd}
\end{array}
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A very general finite axiomatizability result follows.

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Ross Willard said $\mathcal{V}$ is congruence meet-semidistributive iff there exist ternary terms $f_0, \ldots, f_n, g_1, \ldots, g_n$ such that

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- $\mathcal{V} \models \mathrm{SD}(\wedge)$ iff $\mathcal{V} \models \alpha \cap (\beta \circ \gamma) \subseteq \beta_m$

The congruences

For any $A \in \mathcal{K}$, the set of $\mathcal{K}$-congruences are $\mathrm{Con}_{\mathcal{K}}(A) = \{\alpha \in \mathrm{Con}(A) : A/\alpha \in \mathcal{K}\}$.

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Let $\alpha, \beta, \gamma \in \mathrm{Con}_{\mathcal{K}}(A)$, and define congruences β_m, $\gamma_m \in \mathrm{Con}_{\mathcal{K}}(A)$ inductively

$$\beta_0 = \beta, \gamma_0 = \gamma$$

$$\beta_{n+1} = \beta \vee^{\mathcal{K}} (\alpha \wedge \gamma_n) \quad \text{and} \quad \gamma_{n+1} = \gamma \vee^{\mathcal{K}} (\alpha \wedge \beta_n).$$

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Notice $\beta \leq \beta_1 \leq \beta_2 \leq \cdots$ and $\gamma \leq \gamma_1 \leq \gamma_2 \leq \cdots$.
Set

$$\beta_\infty = \bigcup_{n \in \omega} \beta_n \quad \text{and} \quad \gamma_\infty = \bigcup_{n \in \omega} \gamma_n$$

and note $\beta_\infty, \gamma_\infty \in \mathrm{Con}_{\mathcal{K}}(A)$.

A Disjunction

$W_n(x,y) := \bigvee_{i=1}^{n} [f_i(xxy) \approx g_i(xxy) \leftrightarrow f_i(xyy) \not\approx g_i(xyy)]$
$M_c(x,y) :=$
$[y \approx c(xxy) \wedge c(xxy) \approx c(yxx) \wedge c(yyx) \approx c(xyy) \wedge c(xyy) \approx x]$.

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Theorem

For any quasivariety $\mathcal{K}$ the following are equivalent:

1. *For any $A \in \mathcal{K}$ and $\alpha, \beta, \gamma \in Con_{\mathcal{K}}(A)$, $\alpha \wedge \beta = \alpha \wedge \gamma = 0_A$ implies $\alpha \wedge (\beta \circ \gamma) \subseteq \gamma \circ \beta$.*
2. *For the principle congruences $\alpha = \Theta(x,z)$, $\beta = \Theta(x,y)$, and $\gamma = \Theta(y,z)$ in $F_{\mathcal{K}}(x,y,z)$ there exists m such that $\alpha \cap (\beta \circ \gamma) \subseteq \gamma_m \circ \beta_m$.*
3. *There exists ternary terms $f_1, \ldots, f_n, g_1, \ldots, g_n, c$ such that $f_i(xyx) \approx g_i(xyx)$ and $\mathcal{K}$ satisfies the sentence*

$$\forall x \forall y \, [x \not\approx y \longrightarrow W_n(x,y) \vee M_c(x,y)].$$

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It is easy to see conditions in $W_n(x,y)$ and $M_c(x,y)$ cannot be satisfied by any interpretation by ternary projections.

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$x \gamma_m c(xyz) \beta_m z$

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$x\gamma_m c(xyz)\beta_m z$ implies $c(xyz)$ must be idempotent.

Varieties with a weak difference term

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1. $\mathcal{V} \models \alpha \cap (\beta \circ \gamma) \subseteq \gamma_m \circ \beta_m$.
2. *$\mathcal{V}$ has a weak difference term.*
3. *$\mathcal{V}$ satisfies a nontrivial idempotent Malcev condition which implies the abelian algebras are affine.*

$c(xyz)$ in $M_c(x, y)$ is your weak difference term

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2. *There exists ternary terms $f_1, \ldots, f_n, g_1, \ldots, g_n, c$ such that $f_i(xyx) \approx g_i(xyx)$ and*

$$\mathcal{V} \models \forall x \forall y \, [x \not\approx y \longrightarrow W_n(x, y) \vee M_c(x, y)] \, .$$

3. *$\mathcal{V}$ has an idempotent term which interprets as a malcev operation in abelian algebras; consequently, abelian algebras are affine.*

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The disjunction yields a proof which avoids the topic of quasi-affine or linear commutators, but you still need that righteous lemma.....you know the one.

Malcev or Willard?

Let $A \in \mathcal{V}$, $\alpha, \beta, \gamma \in \mathrm{Con}(A)$, and $a, b \in A$ such that $a \neq b$:

- If $(a,b) \in \alpha \cap (\beta \vee \gamma)$ and $A \models W_{\mathcal{V}}(a,b)$, then

$$\alpha \wedge \beta \neq 0_A \quad \text{or} \quad \alpha \wedge \gamma \neq 0_A.$$

- If $(a,b) \in \alpha \cap (\beta \vee \gamma) \setminus \delta$ where $\delta = \alpha \wedge \beta_\infty = \alpha \wedge \gamma_\infty$, then

$$a \,\delta\, c(abb) \,\delta\, c(bba) \quad \text{and} \quad b \,\delta\, c(baa) \,\delta\, c(aab).$$

- If $(a,b) \in \alpha \cap (\beta \vee \gamma)$ and $\alpha \wedge \beta = \alpha \wedge \gamma = 0_A$, then

$$A \models M_c(a,b) \wedge \neg W_{\mathcal{V}}(a,b).$$

We say (a,b) is a Malcev pair if $A \models M_c(a,b)$, and a Willard pair if $A \models W_n(a,b)$.

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- The homotopy theory says a minimal counterexample must have at least four elements.
- Then use pp-definition on the possible configurations and minimality.

no 3-cycles with two loops

Theorem

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$$f(aab) = g(aab) \leftrightarrow f(abb) \neq g(abb).$$

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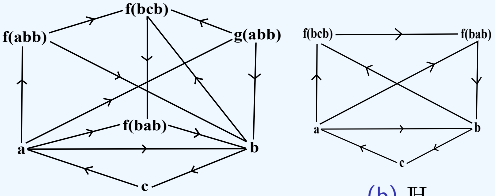

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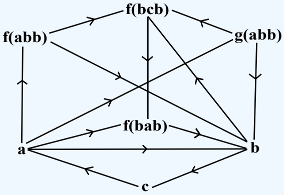

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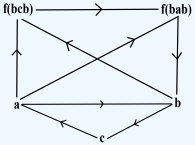

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- Either $f(abb) = a$ and $g(abb) = b$, $f(abb) = b$ and $g(abb) = a$. Any case, we consider $\mathbb{H}$.

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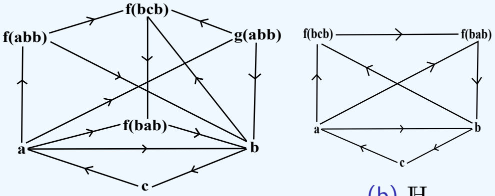

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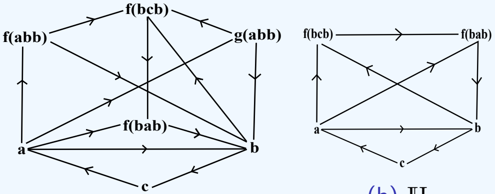

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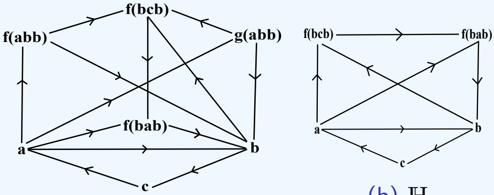

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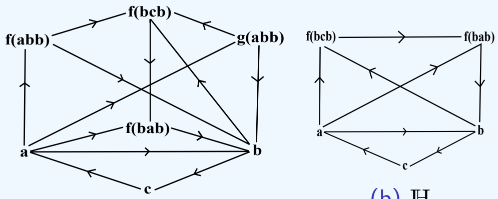

(a) $\mathbb{G}$ (b) $\mathbb{H}$

If not,

- Either $f(abb) = a$ and $g(abb) = b$, $f(abb) = b$ and $g(abb) = a$. Any case, we consider $\mathbb{H}$.
- Collapse onto the cycle creates a symmetric edge.
- There exists a vertex w such that $a \to w \to b$ and $w \to c$ (if not, reverse the edges)

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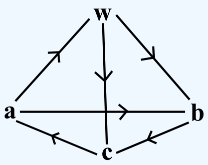

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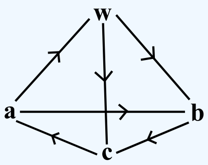

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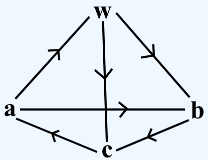

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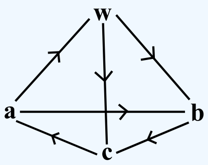

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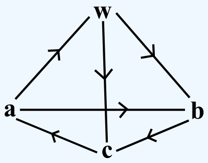

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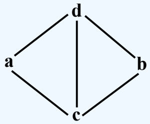

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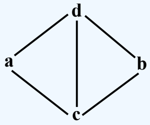

(So begins Hell and Nešetřil, Siggers, and Bulatov)

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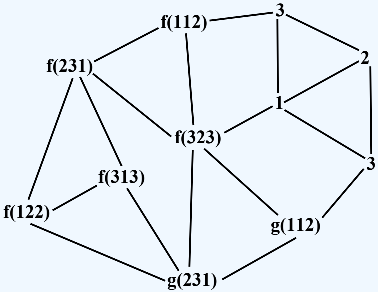

Figure: A leaf