Critical algebras and the Frattini congruence, II

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Abstract

We prove that any finite subdirectly irreducible algebra in a congruence modular variety with trivial Frattini congruence is critical. We also show that if $\mathbf{A}$ and $\mathbf{B}$ are critical algebras which generate the same congruence modular variety, then the variety generated by the proper sections of $\mathbf{A}$ equals the variety generated by the proper sections of $\mathbf{B}$.

1 Introduction

An algebra $\mathbf{C}$ is a *section* of $\mathbf{A}$ if there is a subalgebra $\mathbf{B} \leq \mathbf{A}$ and a surjective homomorphism $\varphi : \mathbf{B} \to \mathbf{C}$. $\mathbf{C}$ is a *proper* section unless $\mathbf{B} = \mathbf{A}$ and φ is an isomorphism. The class of proper sections of $\mathbf{A}$ is denoted $(\mathbf{HS} - 1)\mathbf{A}$. $\mathbf{A}$ is *critical* if it is finite and does not belong to the subvariety generated by its proper sections.

Problem 25 of H. Neumann's book [9] asks the following question: If $\mathbf{A}$ and $\mathbf{B}$ are critical groups generating the same variety, must the varieties generated by $(\mathbf{HS}-1)\mathbf{A}$ and $(\mathbf{HS}-1)\mathbf{B}$ be the same? I. D. Macdonald had previously shown in [6] that the answer is affirmative if $\mathbf{A}$ and $\mathbf{B}$ are p–groups. Assuming that $\mathcal{V}$ is generated by a critical p–group $\mathbf{C}$, Macdonald showed how to construct equations axiomatizing $\mathrm{var}((\mathbf{HS} - 1)\mathbf{C})$ relative to $\mathcal{V}$ solely from the fact that $\mathcal{V}$ is generated by a critical p–group. His construction of equations works the same way for $\mathbf{A}$ or $\mathbf{B}$ when $\mathcal{V} = \mathrm{var}(\mathbf{A}) = \mathrm{var}(\mathbf{B})$, so the result follows. Using properties of the Frattini subgroup, R. Bryant gave an affirmative answer to Neumann's question for arbitrary finite groups in [1]. The same question for other types of algebras is considered in the papers [10], [7], [11] and [5]. The last of these, by E. W. Kiss and S. M. Vovsi, subsumes the others with respect to this question. In their paper, Kiss and Vovsi prove that if $\mathbf{A}$ and $\mathbf{B}$ are critical algebras such that $\mathcal{V} := \mathrm{var}(\mathbf{A}) = \mathrm{var}(\mathbf{B})$ is congruence permutable, then the varieties generated by $(\mathbf{HS}-1)\mathbf{A}$ and $(\mathbf{HS}-1)\mathbf{B}$ are the same. Their proof is an elaboration of Bryant's argument. An interesting aspect of [5] is that the proof given seems to show that the 'correct' generalization of the Frattini subgroup is a Frattini congruence, not a Frattini subalgebra.

This note is a sequel to the Kiss–Vovsi paper. The first part proves, with simpler arguments, a far broader result than any of those mentioned. Let $\mathbf{A}$ and $\mathbf{B}$ be critical algebras generating the same variety. We give an easy necessary and sufficient condition for $(\mathbf{HS}-1)\mathbf{A}$ and $(\mathbf{HS} - 1)\mathbf{B}$ to generate the same variety. From these conditions it is possible to deduce that when $\mathbf{A}$ and $\mathbf{B}$ are simple then $(\mathbf{HS}-1)\mathbf{A}$ and $(\mathbf{HS}-1)\mathbf{B}$ do generate the same variety. Weak local modularity hypotheses on $\mathbf{A}$ and $\mathbf{B}$ suffice to force $(\mathbf{HS} - 1)\mathbf{A}$ and $(\mathbf{HS} - 1)\mathbf{B}$

to generate the same variety. It will follow that if $\mathcal{V} := \text{var}(\mathbf{A}) = \text{var}(\mathbf{B})$ is congruence modular, then $(\mathbf{HS}-1)\mathbf{A}$ and $(\mathbf{HS}-1)\mathbf{B}$ generate the same variety. This solves a problem raised in [5].

The Kiss–Vovsi definition of the Frattini congruence is re–introduced in Section 3, along with a concept from [2] which I call the *normalization* of a subdirectly irreducible algebra. If $\mathbf{A}$ is a finite subdirectly irreducible algebra with abelian monolith and $\mathbf{A}$ generates a congruence modular variety, then it turns out that $\mathbf{A}$ is isomorphic to its normalization iff its Frattini congruence is trivial. Any such algebra is critical.

2 Varieties Generated by Proper Sections

Let $(\mathbf{S}-1)\mathbf{A}$ denote the class of all proper subalgebras of $\mathbf{A}$ and $(\mathbf{H}-1)\mathbf{A}$ denote the class of all proper homomorphic images of $\mathbf{A}$. A finite algebra $\mathbf{A}$ is $\mathbf{S}$*–critical* or $\mathbf{H}$*–critical* if it is not a member of the variety generated by $(\mathbf{S}-1)\mathbf{A}$ or $(\mathbf{H}-1)\mathbf{A}$, respectively. Any critical algebra is both $\mathbf{S}$–critical and $\mathbf{H}$–critical and any $\mathbf{H}$–critical algebra is subdirectly irreducible.

The following simple lemma allows us to avoid introducing the Frattini congruence in our discussion of Neumann's problem.

LEMMA 2.1 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are* $\mathbf{S}$*–critical and generate the same variety, then* $(\mathbf{S}-1)\mathbf{A}$ *and* $(\mathbf{S}-1)\mathbf{B}$ *generate the same variety.*

Proof. Let $\mathcal{V}$ denote $\text{var}(\mathbf{A}) = \text{var}(\mathbf{B})$ and set $\mathcal{Q} = \mathbf{SP}(\mathbf{A}) \cap \mathbf{SP}(\mathbf{B})$. $\mathcal{Q}$ is a quasivariety which is contained in $\mathcal{V}$ and which contains the free algebras of $\mathcal{V}$. Since $\mathcal{V}$ is finitely generated, $\mathcal{Q}$ contains a finite relatively free algebra which generates $\mathcal{V}$. Choose $\mathbf{C}$ to be a member of $\mathcal{Q}$ which generates $\mathcal{V}$ and has least cardinality for this property.

Claim. $(\mathbf{S}-1)\mathbf{C}$ and $(\mathbf{S}-1)\mathbf{A}$ generate the same variety.

Proof of Claim. By the minimality hypothesis on $\mathbf{C}$, any proper subalgebra of $\mathbf{C}$ generates a proper subvariety of $\mathcal{V}$. Hence, no proper subalgebra of $\mathbf{C}$ has a homomorphism onto $\mathbf{A}$. Since all subalgebras of $\mathbf{C}$ belong to $\mathbf{SP}(\mathbf{A})$, it follows that $(\mathbf{S}-1)\mathbf{C} \subseteq \mathbf{SP}((\mathbf{S}-1)\mathbf{A})$. Hence $(\mathbf{S}-1)\mathbf{C}$ is contained in $\text{var}((\mathbf{S}-1)\mathbf{A})$. Conversely, since $\mathbf{C}$ generates $\mathcal{V}$, $\mathbf{C} \in \mathbf{SP}(\mathbf{A})$ and $(\mathbf{S}-1)\mathbf{A}$ doesn't generate $\mathcal{V}$, it follows that $\mathbf{C}$ has a homomorphism onto $\mathbf{A}$. If $h : \mathbf{C} \to \mathbf{A}$ is onto and $\mathbf{A}'$ is a proper subalgebra of $\mathbf{A}$, then $\mathbf{C}' := h^{-1}(\mathbf{A}')$ is a proper subalgebra of $\mathbf{C}$ which has a homomorphism onto $\mathbf{A}'$. Hence, $(\mathbf{S}-1)\mathbf{A}$ is contained in $\text{var}((\mathbf{S}-1)\mathbf{C})$. We get that $\text{var}((\mathbf{S}-1)\mathbf{A}) = \text{var}((\mathbf{S}-1)\mathbf{C})$.

It follows from the Claim that $\text{var}((\mathbf{S}-1)\mathbf{A}) = \text{var}((\mathbf{S}-1)\mathbf{C}) = \text{var}((\mathbf{S}-1)\mathbf{B})$. This proves the lemma. ∎

Lemma 2.1 corresponds to one half of Proposition 1 of [5] in the case that $\text{var}(\mathbf{A}) = \text{var}(\mathbf{B})$ is a congruence permutable variety. In the other half of Proposition 1 of [5] it is proved that when $\mathbf{A}$ and $\mathbf{B}$ are $\mathbf{S}$–critical algebras in a congruence permutable variety and $\text{var}(\mathbf{A}) = \text{var}(\mathbf{B})$, then $\mathbf{A}/\Phi_{\mathbf{A}} \cong \mathbf{B}/\Phi_{\mathbf{B}}$, where Φ denotes the Frattini congruence (defined in the next section). This other half of Proposition 1 can also be extended to arbitrary varieties by using the idea of the proof of Lemma 2.1. (In particular, if $\mathbf{A}$ and $\mathbf{B}$ are $\mathbf{S}$–critical algebras which generate the same variety and the Frattini congruence of $\mathbf{A}$ is trivial, then $\mathbf{A} \cong \mathbf{B}$.)

If an algebra $\mathbf{A}$ is critical, it is subdirectly irreducible. Denote its monolith by $\mu_{\mathbf{A}}$.

THEOREM 2.2 *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be critical algebras which generate the same variety. The inclusion*

$$\mathrm{var}((\mathbf{HS}-1)\mathbf{A}) \subseteq \mathrm{var}((\mathbf{HS}-1)\mathbf{B})$$

holds if and only if

$$\mathbf{A}/\mu_{\mathbf{A}} \in \mathrm{var}((\mathbf{HS}-1)\mathbf{B}).$$

PROOF. Notice that $(\mathbf{HS}-1)\mathbf{A} = \mathbf{H}[(\mathbf{S}-1)\mathbf{A}] \cup (\mathbf{H}-1)\mathbf{A}$. Lemma 2.1 guarantees the equality in

$$\mathbf{H}[(\mathbf{S}-1)\mathbf{A}] \subseteq \mathrm{var}((\mathbf{S}-1)\mathbf{A}) = \mathrm{var}((\mathbf{S}-1)\mathbf{B}) \subseteq \mathrm{var}((\mathbf{HS}-1)\mathbf{B}).$$

Therefore, the inclusion

$$\mathrm{var}((\mathbf{HS}-1)\mathbf{A}) \subseteq \mathrm{var}((\mathbf{HS}-1)\mathbf{B})$$

holds if and only if

$$(\mathbf{H}-1)\mathbf{A} \subseteq \mathrm{var}((\mathbf{HS}-1)\mathbf{B}).$$

But the condition $(\mathbf{H}-1)\mathbf{A} \subseteq \mathrm{var}((\mathbf{HS}-1)\mathbf{B})$ is equivalent to $\mathbf{A}/\mu_{\mathbf{A}} \in \mathrm{var}((\mathbf{HS}-1)\mathbf{B})$. ∎

It follows from this theorem that critical algebras $\mathbf{A}$ and $\mathbf{B}$ which generate the same variety have the property that

$$\mathrm{var}((\mathbf{HS}-1)\mathbf{A}) = \mathrm{var}((\mathbf{HS}-1)\mathbf{B})$$

if and only if $\mathbf{A}/\mu_{\mathbf{A}} \in \mathrm{var}((\mathbf{HS}-1)\mathbf{B})$ and $\mathbf{B}/\mu_{\mathbf{B}} \in \mathrm{var}((\mathbf{HS}-1)\mathbf{A})$.

COROLLARY 2.3 *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are simple critical algebras which generate the same variety, then* $(\mathbf{HS}-1)\mathbf{A}$ *and* $(\mathbf{HS}-1)\mathbf{B}$ *generate the same variety.* ∎

The Kiss–Vovsi result can be extended from congruence permutable varieties to congruence modular varieties by simply combining Theorem 2.2 with their Proposition 2. Let's see how to extend it further still.

If $\mathbf{A}$ is a finite algebra and $\alpha \prec \beta$ in $\mathrm{Con}\,(\mathbf{A})$, then the prime quotient $\langle \alpha, \beta \rangle$ will be called *modular* if the $\langle \alpha, \beta \rangle$–minimal sets are of type $\mathbf{2}$, $\mathbf{3}$ or $\mathbf{4}$ and these minimal sets have empty tails. If the $\langle \alpha, \beta \rangle$–minimal sets are of type $\mathbf{3}$ or $\mathbf{4}$ and these minimal sets have empty tails, then $\langle \alpha, \beta \rangle$ is *distributive*. Modular and distributive quotients have the following nice properties. Assume that $\mathbf{A}'$ is finite, $h : \mathbf{A}' \to \mathbf{A}$ is onto and that $\langle \alpha, \beta \rangle$ is a prime quotient of $\mathbf{A}$. Set $\alpha' = h^{-1}(\alpha)$ and $\beta' = h^{-1}(\beta)$. If $\langle \alpha, \beta \rangle$ is modular (distributive), then

- (i) $\langle \alpha', \beta' \rangle$ is modular (distributive), and
- (ii) there is a homomorphism of $\mathrm{Con}\,(\mathbf{A}')$ onto a modular (distributive) lattice which separates α' and β'.

In particular, it follows that if all prime quotients of $\mathbf{A}$ are modular (distributive), then Con $(\mathbf{A})$ is a modular (distributive) lattice. For stronger results, Theorems 8.5 and 8.6 of [3] can be rephrased to say that a locally finite variety $\mathcal{V}$ is congruence modular (distributive) if and only if all prime quotients of finite members of $\mathcal{V}$ are modular (distributive).

LEMMA 2.4 *Assume that* $\mathbf{A}$ *and* $\mathbf{B}$ *are critical algebras which generate the same variety and that* $\langle 0, \mu_{\mathbf{A}} \rangle$ *is distributive. Then* $\mathbf{A} \cong \mathbf{B}$ *and so* $\mathrm{var}((\mathbf{HS} - 1)\mathbf{A}) = \mathrm{var}((\mathbf{HS} - 1)\mathbf{B})$.

PROOF. If $\langle 0, \mu_{\mathbf{A}} \rangle$ is distributive and $\mathbf{A} \in \mathrm{var}(\mathbf{B})$, then a local version of Jónsson's Lemma proves that $\mathbf{A} \in \mathbf{HS}(\mathbf{B})$. Since $\mathbf{A} \notin (\mathbf{HS} - 1)\mathbf{B}$, it must be that $\mathbf{A} \cong \mathbf{B}$. ▮

For the next theorem, a congruence θ on a finite algebra is *hereditarily modular* if each prime quotient $\langle \alpha, \beta \rangle$ with $0 \leq \alpha \prec \beta \leq \theta$ is modular.

THEOREM 2.5 *Assume that* $\mathbf{A}$ *and* $\mathbf{B}$ *are critical algebras which generate the same variety and that* $\langle 0, \mu_{\mathbf{A}} \rangle$ *and* $\langle 0, \mu_{\mathbf{B}} \rangle$ *are modular. If every abelian congruence on either* $\mathbf{A}$ *or* $\mathbf{B}$ *is hereditarily modular, then* $\mathrm{var}((\mathbf{HS} - 1)\mathbf{A}) = \mathrm{var}((\mathbf{HS} - 1)\mathbf{B})$.

PROOF. Assume the hypotheses of the theorem, but that $\mathrm{var}((\mathbf{HS}-1)\mathbf{A}) \not\subseteq \mathrm{var}((\mathbf{HS}-1)\mathbf{B})$. By Lemma 2.4, it must be that $\mathrm{typ}(0, \mu_{\mathbf{A}}) = \mathrm{typ}(0, \mu_{\mathbf{B}}) = \mathbf{2}$. Since $\mathbf{A} \in \mathrm{var}(\mathbf{B})$, There is a finite algebra $\mathbf{C}$ which is a subdirect product of subdirectly irreducible algebras from $\mathbf{HS}(\mathbf{B})$ for which there is an onto homomorphism $h : \mathbf{C} \to \mathbf{A}$. Since $\mathbf{C}$ is a subdirect product of subdirectly irreducible algebras from $\mathbf{HS}(\mathbf{B})$, $\mathbf{C}$ has meet–irreducible congruences η_i, $i < n$, such that $\mathbf{C}/\eta_i \in \mathbf{HS}(\mathbf{B})$. For each i, let η_i^* denote the unique upper cover of η_i. Let $\delta = \ker\ h$ and let δ^* be its unique upper cover. Let θ be the least congruence on $\mathbf{C}$ such that $\mathbf{C}/\theta \in \mathrm{var}((\mathbf{HS} - 1)\mathbf{B})$. Clearly, $\theta \leq \eta_i^*$ for all i while $\theta \not\leq \eta_i$ holds if and only if $\mathbf{C}/\eta_i \cong \mathbf{B}$. By rearranging indices if necessary, we may assume that $\theta \not\leq \eta_i$ for $i < j$ and that $\theta \leq \eta_i$ for $j \leq i < n$. (This implies that $\mathbf{C}/\eta_i \cong \mathbf{B}$ for $i < j$, in which case $\langle \eta_i, \eta_i^* \rangle$ is modular of type $\mathbf{2}$.) Since

$$0_{\mathbf{B}} = \bigwedge_{i<n} \eta_i \leq \theta \leq (\bigwedge_{i<j} \eta_i^*) \wedge (\bigwedge_{j \leq i < n} \eta_i),$$

and $\mathrm{typ}(\eta_i, \eta_i^*) = \mathbf{2}$ for $i < j$, it must be that θ is abelian. If $\theta \leq \delta^*$, then

$$\mathbf{A}/\mu_{\mathbf{A}} \cong \mathbf{C}/\delta^* \in \mathbf{H}(\mathbf{C}/\theta) \subseteq \mathrm{var}((\mathbf{HS} - 1)\mathbf{B}).$$

But if this were so, then Theorem 2.2 would force $\mathrm{var}((\mathbf{HS}-1)\mathbf{A}) \subseteq \mathrm{var}((\mathbf{HS}-1)\mathbf{B})$ which contradicts the assumption in the first sentence of this proof. Hence $\theta \not\leq \delta^*$ and so $\delta^* < \delta \vee \theta$. Since θ is abelian, δ is meet–irreducible and $\langle \delta, \delta^* \rangle$ is modular of type $\mathbf{2}$, the interval $I[\delta, \delta \vee \theta]$ is abelian. It follows that $h(\delta \vee \theta)$ is an abelian congruence of $\mathbf{A}$. By hypothesis, $h(\delta \vee \theta)$ is hereditarily modular. Therefore, every prime quotient in the interval $I[\delta, \delta \vee \theta]$ of Con $(\mathbf{C})$ is modular.

The following comparabilities and non–comparabilities in Con $(\mathbf{C})$ have been established:

(*i*) $0 = \bigwedge_{i<n} \eta_i \leq \theta \leq \bigwedge_{i<n} \eta_i^*$.

(*ii*) $\theta \leq \eta_i$ for all $j \leq i < n$.

(*iii*) $\theta \not\leq \eta_i$ for $i < j$.

(*iv*) $\theta \not\leq \delta^*$.

Con $(\mathbf{C})$ has a homomorphism onto a modular lattice which separates all modular prime quotients. Such a homomorphism preserves all the comparabilities listed, of course. It also preserves the listed non–comparabilities, since $\langle \eta_i, \eta_i^* \rangle$ is modular for $i < j$ and every prime quotient in the interval $I[\delta, \delta \vee \theta]$ is modular. We may henceforth assume that Con $(\mathbf{C})$ is a modular lattice, as long as we depend only the comparabilities and non–comparabilities listed in this paragraph.

Let $\lambda = \bigwedge_{i<n} \eta_i^*$. By modularity, we have $(\eta_i \wedge \lambda) \prec \lambda$ whenever $\lambda \not\leq \eta_i$. Since $\bigwedge_{i<n} \eta_i = 0$, the zero congruence is a meet of lower covers of λ. Therefore, the interval $I[0, \lambda]$ is a complemented modular lattice; hence $I[0 \vee \delta, \lambda \vee \delta]$ is a complemented modular lattice. But δ is meet–irreducible, so we must have $\lambda \vee \delta \leq \delta^*$. Since $\theta \leq \lambda$, this gives us the contradiction that $\theta \leq \delta^*$. ∎

Theorem 2.5 solves the problem raised in [5] since, when $\mathrm{var}(\mathbf{A}) = \mathrm{var}(\mathbf{B})$ is congruence modular, then all prime quotients of $\mathbf{A}$ and $\mathbf{B}$ are modular.

We mention that there are critical algebras such that $\mathrm{var}(\mathbf{A}) = \mathrm{var}(\mathbf{B})$ but for which $\mathrm{var}((\mathbf{HS} - 1)\mathbf{A}) \neq \mathrm{var}((\mathbf{HS} - 1)\mathbf{B})$. Such examples can be easily constructed where $\mathrm{var}(\mathbf{A}) = \mathrm{var}(\mathbf{B})$ is a variety of G–sets.

3 Normalization

In this section we discuss a process called ‘normalization’ which converts a subdirectly irreducible algebra (in a congruence modular variety) into a better–behaved and related algebra. This process is described in [2], but not named.

We shall follow the notation of [2] except in the following cases: First, when R is a binary relation on S we will write $S \times_R S \times_R \cdots \times_R S$, with n factors, to denote the subset of S^n which consists of the tuples $(s_1, \ldots, s_n)$ with $(s_i, s_{i+1}) \in R$. If $\mathbf{A}$ is an algebra and α is a congruence, we use boldface notation $\mathbf{A} \times_\alpha \cdots \times_\alpha \mathbf{A}$ to indicate the subalgebra of $\mathbf{A}^n$ supported by $A \times_\alpha \cdots \times_\alpha A$. (This notation differs from [2] in the following way: what we write as $\mathbf{A} \times_\alpha \mathbf{A}$ is denoted by $\mathbf{A}(\alpha)$ in [2].) Next, we will write $\widehat{\mathbf{A}}$ in this section for something which is denoted $D(\mathbf{A})$ in [2]. Finally, if $\mathbf{A}$ is an algebra, $\mathbf{B}$ is a subalgebra and θ is a congruence on $\mathbf{A}$, then $\mathbf{B}^\theta$ denotes the subalgebra of $\mathbf{A}$ whose universe is $\{x \in A \mid \exists y \in B\ ((x, y) \in \theta)\}$. (This notion does not occur in [2].)

Definition 3.1 Assume that $\mathbf{A}$ is a subdirectly irreducible algebra with monolith μ. The *normalization* of $\mathbf{A}$ is defined as follows: if μ is nonabelian, then the normalization of $\mathbf{A}$ is $\mathbf{A}$; if μ is abelian, then the normalization of $\mathbf{A}$ is

$$\widehat{\mathbf{A}} := (\mathbf{A} \times_\mu \mathbf{A})/\Delta_{\mu,(0:\mu)}.$$

The congruence $(\mu_0 + \Delta_{\mu,(0:\mu)})/\Delta_{\mu,(0:\mu)}$ on $\widehat{\mathbf{A}}$ is denoted $\hat{\mu}$. $\mathbf{A}$ is *normal* if it is isomorphic to its normalization.

The next lemma summarizes those properties of the normalization which are proved in [2].

LEMMA 3.2 *Let $\mathcal{K}$ be a finite set of finite algebras and* $\mathbf{A}$ *be a subdirectly irreducible algebra. Assume that* $\mathrm{var}(\mathcal{K})$ *and* $\mathrm{var}(\mathbf{A})$ *are congruence modular. Let* μ *be the monolith of* $\mathbf{A}$.

(*i*) $\widehat{\mathbf{A}}$ *is a normal subdirectly irreducible algebra and* $\hat{\mu}$ *is its monolith.*

(*ii*) $\mathbf{A}/(0:\mu) \cong \widehat{\mathbf{A}}/(0:\hat{\mu})$.

(*iii*) $(0:\hat{\mu}) = \hat{\mu}$.

(*iv*) $\hat{\mu}$ *is the kernel of a retraction.*

(*v*) *If* $\mathbf{A} \in \mathrm{var}(\mathcal{K})$, *then* $\widehat{\mathbf{A}}$ *is isomorphic to the normalization of some subdirectly irreducible algebra in* $\mathbf{HS}(\mathcal{K})$. ∎

We will require the following technical lemma.

LEMMA 3.3 *Let* $\mathbf{A}$ *be a finite subdirectly irreducible algebra with abelian monolith* μ. *If* $\mathrm{var}(\mathbf{A})$ *is congruence modular, then* $|\widehat{A}| \leq |A|$ *with equality holding iff* $\mu = (0:\mu)$.

Sketch of Proof. Let $C_1, \ldots, C_m$ be an enumeration of the $(0:\mu)$–classes of $\mathbf{A}$. Each μ–class is a subset of some C_i and the different μ–classes in a single C_i have the same size. Therefore, the size of each C_i is determined by the size s_i and number n_i of μ–classes it contains. This implies that $|A| = \Sigma_{i=1}^m |C_i| = \Sigma_{i=1}^m n_i s_i$.

Now, in $\mathbf{A} \times_\mu \mathbf{A}$ we have $(0:\mu)_0 = (0:\mu_0)$ and that each $(0:\mu)_0$–class is of the form $C_i \times_\mu C_i$ for a uniquely chosen i. Each such $(0:\mu)_0$–class contains n_i different μ_0–classes and they are of size s_i^2. Using this, one calculates that

$$|A \times_\mu A| = \Sigma_{i=1}^m n_i s_i^2.$$

Each $(0:\mu)_0$–class C is a union of $\Delta_{\mu,(0:\mu)}$–classes and, by examining $\Delta_{\mu,(0:\mu)}$ on C, one can show that for a single μ_0–class $D \subseteq C$ we have that every element of C is $\Delta_{\mu,(0:\mu)}$–related to some element of D and that D intersects exactly s_i of the $\Delta_{\mu,(0:\mu)}$–classes. Hence, $|C/\Delta_{\mu,(0:\mu)}| = s_i$. This yields

$$|\widehat{A}| = \Sigma_{i=1}^m s_i \leq \Sigma_{i=1}^m n_i s_i = |A|.$$

Furthermore, equality holds iff each $n_i = 1$, which means exactly that $\mu = (0:\mu)$. ∎

LEMMA 3.4 *Let* $\mathbf{A}$ *be a subdirectly irreducible algebra which generates a congruence modular variety. If the monolith* μ *of* $\mathbf{A}$ *is abelian, then* $\mathbf{A}$ *is normal iff*

(*i*) $\mu = (0:\mu)$ *and*

(*ii*) μ *is the kernel of a retraction.*

PROOF. By Lemma 3.2, both (i) and (ii) of this lemma hold in the normalization of $\mathbf{A}$, hence in $\mathbf{A}$. We will argue that if (i) and (ii) hold, then $\mathbf{A}$ is isomorphic to its normalization.

Let $\rho : \mathbf{A} \to \mathbf{A}$ be a retraction of $\mathbf{A}$ with kernel μ. Define a homomorphism

$$\psi : \mathbf{A} \times_\mu \mathbf{A} \to \mathbf{A} \times_\mu \mathbf{A} \times_\mu \mathbf{A} : (a,b) \mapsto (a,b,\rho(a)) = (a,b,\rho(b)).$$

If $d(x,y,z)$ is a difference term for $\mathrm{var}(\mathbf{A})$, then by Proposition 5.7 we have that

$$d : \mathbf{A} \times_\mu \mathbf{A} \times_\mu \mathbf{A} \to \mathbf{A} : (a,b,c) \mapsto d(a,b,c)$$

is a homomorphism. The composite $d\psi : \mathbf{A} \times_\mu \mathbf{A} \to \mathbf{A}$ is easily checked to be a surjective homomorphism with kernel $\Delta_{\mu,\mu}$. Hence,

$$\mathbf{A} \cong (\mathbf{A} \times_\mu \mathbf{A})/\Delta_{\mu,\mu}.$$

Since $\mu = (0 : \mu)$, the algebra on the righthand side is the normalization of $\mathbf{A}$. This finishes the proof. ∎

Somewhat surprisingly, condition (i) of Lemma 3.4 is extraneous. That is, in a congruence modular variety, a subdirectly irreducible with abelian monolith is normal iff its monolith is the kernel of a retraction. This fact is a consequence of the following lemma (whose proof does not require congruence modularity).

LEMMA 3.5 *Let $\mathbf{A}$ be a subdirectly irreducible algebra with monolith μ. If μ is the kernel of a retraction, then $(0 : \mu) \leq \mu$.*

PROOF. Assume otherwise that μ is the kernel of the retraction $\rho : \mathbf{A} \to \mathbf{A}$ and that $\mu < (0 : \mu)$. Let $\mathbf{B} = \rho(\mathbf{A})$. Since $\ker\ \rho = \mu < (0 : \mu)$, there exists $(c,d) \in (0 : \mu)|_{\mathbf{B}} - 0_{\mathbf{B}}$. Choose $b \in B$ such that $|b/\mu| > 1$ and then pick $a \in A - B$ such that $(a,b) \in \mu$. Since

$$\mu = \mathrm{Cg}^{\mathbf{A}}(a,b) \leq \mathrm{Cg}^{\mathbf{A}}(c,d),$$

there is a Mal'cev chain connecting a to b by polynomial images of $\{c,d\}$. Since $b \in B$ and $a \notin B$, this implies the existence of a polynomial $p \in \mathrm{Pol}_1\mathbf{A}$ such that $p(c) = u \in B$ and $p(d) = v \notin B$ or the same with c and d interchanged. Assume that $p(x) = t^{\mathbf{A}}(x, w_1, \ldots, w_n)$ where t is a term and $w_i \in A$. Applying ρ to the equality $t^{\mathbf{A}}(c, w_1, \ldots, w_n) = u$ (and using $\rho(c) = c$, $\rho(u) = u$) yields

$$t^{\mathbf{A}}(\underline{c}, \rho(w_1), \ldots, \rho(w_n)) = t^{\mathbf{A}}(\rho(c), \rho(w_1), \ldots, \rho(w_n)) = \rho(u) = u = t^{\mathbf{A}}(\underline{c}, w_1, \ldots, w_n).$$

Now, using that $(c,d) \in (0 : \mu)$ and $(w_i, \rho(w_i)) \in \ker\ \rho = \mu$ we can change the $\underline{c}$ to d to get

$$t^{\mathbf{A}}(d, \rho(w_1), \ldots, \rho(w_n)) = t^{\mathbf{A}}(d, w_1, \ldots, w_n) = v.$$

But this is impossible! We have $d, \rho(w_i) \in B$, so

$$v = t^{\mathbf{A}}(d, \rho(w_1), \ldots, \rho(w_n)) = t^{\mathbf{B}}(d, \rho(w_1), \ldots, \rho(w_n)) \in B,$$

and yet we chose $v = p(d) \in A - B$. This ends the proof. ∎

The argument just given establishes a more general (and more technical) result than we claimed. Although we see no use for the more general result now, we include its statement for completeness: if $\mathbf{A}$ is an arbitrary algebra, $\rho : \mathbf{A} \to \mathbf{A}$ is an arbitrary retraction, $\mu = \ker\ \rho$, $\mathbf{B} = \rho(\mathbf{A})$ and $\theta = \mathrm{Cg}^{\mathbf{A}}((0 : \mu)|_{\mathbf{B}})$ (that is, θ is the extension of the contraction of $(0 : \mu)$), then $\mathbf{B}^{\theta} = \mathbf{B}$. In the case of the lemma, $\mathbf{A}$ is subdirectly irreducible with monolith μ. In this case, $\mu < (0 : \mu)$ implies $\mu \leq \theta$ which leads to the contradiction

$$\mathbf{A} = \mathbf{B}^{\mu} \leq \mathbf{B}^{\theta} = \mathbf{B}.$$

We connect the foregoing with the Frattini congruence and critical algebras.

Definition 3.6 If $\mathbf{A}$ is an algebra, $\mathbf{B}$ is a subalgebra of $\mathbf{A}$ and θ is a congruence on $\mathbf{A}$, then we say that $\mathbf{B}$ *contains* θ if $\mathbf{B}^{\theta} = \mathbf{B}$. Let $\Phi_{\mathbf{A}}$ be the join of all congruences θ which are contained in all maximal proper subalgebras of $\mathbf{A}$. $\Phi_{\mathbf{A}}$ is the *Frattini congruence* of $\mathbf{A}$.

It is easy to see that $\Phi_{\mathbf{A}}$ is the largest congruence contained in all maximal proper subalgebras of $\mathbf{A}$. Let's call a congruence θ on $\mathbf{A}$ *non-generating* if

$$\mathbf{B}^{\theta} = \mathbf{A} \Longrightarrow \mathbf{B} = \mathbf{A}$$

whenever $\mathbf{B}$ is a subalgebra of $\mathbf{A}$. It is straightforward to see that the Frattini congruence majorizes every non-generating congruence and, when $\mathbf{A}$ is finitely generated, the Frattini congruence is the largest non-generating congruence.

THEOREM 3.7 *Let $\mathbf{A}$ be a finite subdirectly irreducible algebra which generates a congruence modular variety. Assume that the monolith μ of $\mathbf{A}$ is abelian. The following implications hold among the conditions enumerated below: $(i) \iff (ii) \Longrightarrow (iii) \Longrightarrow (iv)$.*

- (i) $\mathbf{A}$ *has trivial Frattini congruence.*
- (ii) μ *is the kernel of a retraction.*
- (iii) $\mu = (0 : \mu)$.
- (iv) $\mathbf{A}$ *is critical.*

PROOF. We will argue that $(ii) \Longrightarrow (i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv)$.

Assume that (ii) holds. Since μ is the kernel of a retraction it cannot be a non-generating congruence. For suppose that $\rho : \mathbf{A} \to \mathbf{A}$ is a retraction with $\rho(\mathbf{A}) = \mathbf{B}$ and $\mu = \ker\ \rho$. Then $\mathbf{B}^{\mu} = \mathbf{A}$ even though $\mathbf{B} \neq \mathbf{A}$. Hence, when (ii) holds, the Frattini congruence is not above μ. This proves (i). (Here is a different argument: Since $\mathbf{A}$ is finite and μ is a minimal abelian congruence, Theorem 2.1 of [4] can be used to show that $\mathbf{B} := \rho(\mathbf{A})$ is a maximal subalgebra. $\mathbf{B}$ does not contain μ, so $\mu \not\leq \Phi_{\mathbf{A}}$.)

If (i) holds, then μ is not contained in some maximal proper subalgebra $\mathbf{B} \leq \mathbf{A}$. The congruence μ is abelian and not contained in $\mathbf{B}$, so Theorem 2.1 of [4] proves that $\mu|_{\mathbf{B}} = 0_{\mathbf{B}}$. Hence, $\mathbf{B}$ is a μ–transversal. It follows that μ is the kernel of a retraction onto $\mathbf{B}$.

The implication $(ii) \Longrightarrow (iii)$ follows from Lemma 3.4. We now prove that if $\mathbf{A}$ is subdirectly irreducible and $\mu = (0 : \mu)$, then $\mathbf{A} \notin \mathrm{var}((\mathbf{HS} - 1)\mathbf{A})$. For $\widehat{\mathbf{A}}$ equal to the normalization of $\mathbf{A}$, we have $\widehat{\mathbf{A}} \in \mathrm{var}(\mathbf{A})$. Therefore, to prove $\mathbf{A} \notin \mathrm{var}((\mathbf{HS}-1)\mathbf{A})$ it will suffice to prove that $\widehat{\mathbf{A}} \notin \mathrm{var}((\mathbf{HS}-1)\mathbf{A})$. Assume otherwise that $\widehat{\mathbf{A}} \in \mathrm{var}((\mathbf{HS}-1)\mathbf{A})$. From Lemma 3.2 (v), we get that $\widehat{\mathbf{A}}$ is the normalization of some subdirectly irreducible algebra $\mathbf{B} \in \mathbf{HS}((\mathbf{HS}-1)\mathbf{A}) = (\mathbf{HS}-1)\mathbf{A}$. But now (referring to Lemma 3.3), we have a cardinality problem:

$$|\widehat{A}| \leq |B| < |A| = |\widehat{A}|.$$

(The equality $|A| = |\widehat{A}|$ follows from the fact that $\mu = (0 : \mu)$.) This contradiction concludes the proof. ∎

The only implication in this proof which requires congruence modularity is $(iii) \Longrightarrow (iv)$.

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