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Congruence semimodular varieties I: Locally finite varieties

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Dedicated to Bjarni Jónsson on the occasion of his 70th birthday

1. Introduction

A lattice $\mathbf{L}$ is **semimodular** if for all $x, y \in L$ the implication $x \wedge y \prec x \rightarrow y \prec x \vee y$ holds. The notion of semimodularity arose from the investigation of closed subsets of a set under a closure operator, C, which satisfies the following *exchange principle*:

$$x \notin C(X) \quad \text{and} \quad x \in C(X \cup \{y\}) \quad \text{implies} \quad y \in C(X \cup \{x\}).$$

The lattice of closed subsets of a set under such a closure operator is semimodular. Perhaps the best known example of a closure operator satisfying the exchange principle is the closure operator on a vector space $\mathbf{W}$ where for $X \subseteq W$ we let $C(X)$ equal the span of X. The lattice of C-closed subsets of W is isomorphic to $\mathrm{Con}(\mathbf{W})$ in a natural way; indeed, if $Y \subseteq W \times W$ and $\mathrm{Cg}(Y)$ denotes the congruence on $\mathbf{W}$ generated by Y, then the closure operator Cg satisfies the exchange principle. For another example, let C denote the closure operator on the set $A \times A$ where $C(X)$ equals the equivalence relation generated by $X \subseteq A^2$. This closure operator satisfies the exchange principle, so the lattice of all equivalence relations on A is semimodular. Equivalently, the congruence lattice of any set is semimodular.

The preceding examples suggest to us that semimodularity may be a natural congruence condition worth investigating. Research into varieties of algebras with modular congruence lattices had led to the development of a deep structure theory for them. We wonder: how much of the structure involved in congruence modular varieties exists for congruence semimodular varieties? How much more diversity is permitted? This paper may be considered to be an attack on the former question

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while the sequel to this paper, *Congruence semimodular varieties II: regular varieties*, is an attack on the latter question.

In this paper we examine locally finite congruence semimodular (CSM) varieties. The first two sections are fairly introductory, although there is material in Section 2 that is used later and can be found nowhere else. In Section 3 we exhibit natural congruences on Con(**A**) for any finite **A** generating a CSM variety. These natural congruences play a role similar to that played by the commutator for algebras in a congruence modular variety. As an application we show that a strong version of Jónsson's Lemma extends to locally finite CSM varieties. In Section 4 we find that these congruences induce natural congruences on the subvariety lattice of any locally finite CMS variety. Section 5 exploits some of Dilworth's lattice-theoretic results on decompositions in semimodular lattices. This section is on subdirect representations of CSM algebras. Sections 6, 7 and 8 concern special topics. Section 6 shows that it is fairly easy to locate the relatively distributive and relatively modular subquasivarieties of a CSM variety. Section 7 concerns the congruence extension property and shows, for example, that a variety with the CEP is CSM if and only if the 5-generated free algebra is CSM. Section 8 proves that a locally finite, CSM variety whose members have atomistic congruence lattices decomposes as a varietal product of strongly abelian variety and an affine variety.

The sequel to this paper examines regular CSM varieties. In that paper we introduce a natural way to quasi-order the universe of any algebra. Regular CSM varieties are characterized in terms of this quasi-order. A consequence of this result is that a regular CSM variety can contain no nontrivial congruence modular subvarieties. Although there is visible structure in the algebras which generate regular CSM varieties, this structure is quite different than the kind of structure found in congruence modular varieties.

The variety of sets is semimodular, so the results of [21] imply that congruence semimodularity implies no nontrivial Mal'cev condition. This means that some of the most useful techniques of universal algebra are not applicable to CSM varieties. Correspondingly, there seems to be a scarcity of results concerning CSM varieties which are not congruence modular. The results that are known include the facts that the variety of sets is CSM, [20]; the variety of semilattices is CSM, [6]; the CSM varieties of regular semigroups have been characterized, [8]; and the CSM varieties of irregular semigroups have almost been characterized, [8]. There are a number of lattice-theoretic results (see, for example, the chapters of [3] concerning Decomposition Theory) which seem to have been proved especially for algebras with semimodular congruence lattices. For the dual lattice-theoretic condition, **lower semimodularity**, it is known that a variety whose 2-generated free algebra is finite consists of algebras with lower semimodular congruence lattices if and only if the variety is congruence modular, [11].

The class of semimodular lattices is closed under a number of class operators. It is not hard to see that semimodular lattices form an elementary class, so this class is closed under the formation of ultraproducts and elementary sublattices. We will not need these facts; the closure properties that we are interested in are those contained in the next definition.

DEFINITION 1.1. A class of lattices is **full** if it is closed under the formation of subdirect products, interval sublattices and bounded homomorphic images.

(A lattice homomorphism is **bounded** if each class of the kernel contains a least and a largest element. It is easily seen that a homomorphism between complete lattices is bounded if and only if it respects the complete lattice operations.) It is known that semimodular lattices form a full class of lattices (see [11] for a proof). In fact, they form a very large full class. If $\mathscr{K}$ is the class of semimodular lattices and $\mathscr{L}$ is the class of all lattices, then Whitman's proof that every lattice embeds into a partition lattice shows that $\boldsymbol{S}(\mathscr{K}) = \mathscr{L}$. The fact that $\mathscr{K}$ is closed under subdirect products also shows that

$$\mathscr{L} = \boldsymbol{S}(\mathscr{K}) \subseteq \boldsymbol{V}(\mathscr{K}) = \boldsymbol{HP}_s(\mathscr{K}) = \boldsymbol{H}(\mathscr{K}),$$

so every lattice is a quotient of a semimodular lattice.

Semimodularity was first defined only for finite dimensional lattices, the appropriate condition on the lattice $\mathbf{L}$ being

$$\text{(WSM)} \quad x, y \in L, \qquad x \wedge y \prec x, y \rightarrow x, y \prec x \vee y.$$

This definition is equivalent to our definition of semimodularity for finite dimensional lattices. We caution the reader that a number of ways have been suggested to extend the definition of semimodularity to infinite dimensional lattices. We have chosen to follow [3] by defining an arbitrary lattice to be semimodular if

$$x, y \in L, \qquad x \wedge y \prec x \rightarrow y \prec x \vee y.$$

A lattice satisfying condition (WSM) will be called **weakly semimodular**. Clearly a semimodular lattice is weakly semimodular. Theorem 3.7 of [3] proves that a compactly generated strongly atomic lattice is weakly semimodular if and only if it is semimodular. There are algebraic lattices which are weakly semimodular and not semimodular, so there are algebras which are CSM but not congruence weakly semimodular (CWSM). However, we do not know of a variety consisting of CWSM algebras which is not CSM. Hence we ask:

PROBLEM 1. Is there a congruence weakly semimodular variety which is not congruence semimodular?

In the sequel to this paper we solve this problem negatively for regular varieties. In Section 7 of this paper we solve the problem negatively for locally finite varieties with the congruence extension property. In almost all of our results the hypotheses of semimodularity or weak semimodularity can be interchanged.

In this paper, we will generally follow the conventions of [19] to which we refer for the notation and results of universal algebra and lattice theory. For tame congruence theory we follow [7] except as explained in Section 2. If $\mathbf{A}$ is an algebra, $\mathrm{Con}(\mathbf{A})$ denotes the congruence lattice of $\mathbf{A}$ with the usual ordering. Join and meet in a lattice, $\mathbf{L}$, will be denoted by $\vee$ and $\wedge$ respectively while 0_L and 1_L denote the smallest and largest elements of $\mathbf{L}$ when they exist. In $\mathrm{Con}(\mathbf{A})$, 0_A and 1_A denote the smallest and largest congruences on $\mathbf{A}$. If $\mathbf{L}$ is a lattice, $\alpha, \beta \in L$ and $\alpha \leq \beta$, then the **interval** $[\alpha, \beta]$ is the set $\{\gamma \in L: \alpha \leq \gamma \leq \beta\}$. We will say that β **covers** α (α **is covered by** β) and we will write $\alpha \prec \beta$ or $\beta \succ \alpha$ if $\alpha < \beta$ and $[\alpha, \beta] = \{\alpha, \beta\}$. If $\alpha \prec \beta$, then $\langle \alpha, \beta \rangle$ is called a **prime quotient** and if $0_L \prec \alpha$ we will say that α is an **atom**. For $X \subseteq A^2$, $\mathrm{Cg}^{\mathbf{A}}(X)$ will denote the smallest congruence on $\mathbf{A}$ that contains X (or just $\mathrm{Cg}(X)$ if $\mathbf{A}$ is understood). If $X = \{(a, b)\}$, then we may just write $\mathrm{Cg}^{\mathbf{A}}(a, b)$ or $\mathrm{Cg}(a, b)$. If $f: B \to A$ is a function and θ is an equivalence relation on A, then $\theta|_B = \{(x, y) \in B \times B \mid (f(x), f(y)) \in \theta\}$.

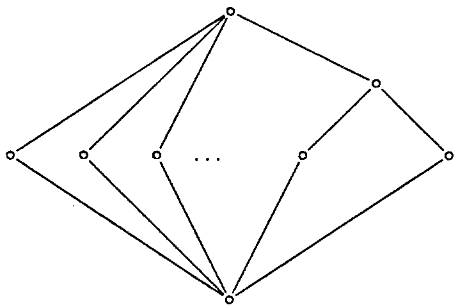
2. Tame congruence theory

There is no room here to include a summary of the results and ideas of tame congruence theory that are required for this paper. A copy of [7] ought to be kept handy when reading many of our proofs. The purpose of this section is to describe how we depart from [7] and to mention results that cannot be found in [7].

We choose to separate the strongly abelian tame quotients into two different types. The following definition is due to E. Kiss (see [16]).

DEFINITION 2.1. If $\mathbf{A}$ is a minimal algebra, then $\mathbf{A}$ is **of type 0** if every nonconstant polynomial is a trivial projection operation ($\mathbf{A}$ is polynomially equivalent to a set). If $\delta < \theta$ in $\mathrm{Con}(\mathbf{B})$ and $\mathbf{B}$ is minimal relative to $\langle \delta, \theta \rangle$, we say that **B has type 0 relative to** $\langle \delta, \theta \rangle$ if the minimal algebra $(\mathbf{B}|_N)/(\delta|_N)$ is of type $\mathbf{0}$ for every $\langle \delta, \theta \rangle$-trace N. If $\langle \alpha, \beta \rangle$ is a tame quotient in a finite algebra $\mathbf{C}$, then $\langle \alpha, \beta \rangle$ **has type 0** if for every $U \in \mathrm{M}_{\mathbf{C}}(\alpha, \beta)$ we have that $\mathbf{C}|_U$ has type $\mathbf{0}$ relative to $\langle \alpha|_U, \beta|_U \rangle$. If a tame quotient $\langle \alpha, \beta \rangle$ is strongly abelian but not of type $\mathbf{0}$ we will say that $\langle \alpha, \beta \rangle$ has type $\mathbf{1}$.

The type-labels $\mathbf{0}$ and $\mathbf{1}$ are more closely related than any other pair of type-labels. For example, it is possible for a prime quotient of type $\mathbf{0}$ to be



THE NUMBER OF ATOMS $= |\mathrm{Aut}(\mathbf{A})| + 2$

Figure 1

perspective with a prime quotient of type **1**. To see an example of this the reader only needs to assign type-labels to the congruence lattice pictured in Figure 1 (the algebra is described in Lemma 2.5). The following exercise describes the limits of such behavior.

EXERCISE 1. Let $\mathbf{A}$ be a finite algebra and assume that for $i = 0, 1$ we have $\alpha_i \prec \beta_i$ in $\mathrm{Con}(\mathbf{A})$ and $\alpha_0 = \beta_0 \wedge \alpha_1$ and $\beta_1 = \beta_0 \vee \alpha_1$. Show that if $\mathrm{typ}(\alpha_0, \beta_0) \neq \mathbf{0}$ or $\mathrm{typ}(\alpha_1, \beta_1) \neq \mathbf{1}$, then $\mathrm{typ}(\alpha_0, \beta_0) = \mathrm{typ}(\alpha_1, \beta_1)$.

Another connection between the type-labels $\mathbf{0}$ and $\mathbf{1}$ is given in Theorem 2.2. There exist varieties whose type-set is $\{\mathbf{i}\}$ for any $\mathbf{i} \in \{\mathbf{0}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ but it is impossible to have $\mathbf{1} \in \mathrm{typ}\{\mathcal{V}\}$ unless we also have $\mathbf{0} \in \mathrm{typ}\{\mathcal{V}\}$. The pathological connections between the type-labels $\mathbf{0}$ and $\mathbf{1}$ will not affect us as we will soon see that type $\mathbf{1}$ prime quotients do not occur in CSM varieties.

THEOREM 2.2. *If* $\mathbf{1} \in \mathrm{typ}\{\mathcal{V}\}$, *then* $\mathbf{0} \in \mathrm{typ}\{\mathcal{V}\}$.

Proof. If $\mathbf{A} \in \mathcal{V}$ is a finite algebra with a minimal congruence β of type $\mathbf{1}$, $U \in \mathrm{M}_{\mathbf{A}}(0_A, \beta)$ and N is a trace of U, then $V(\mathbf{A}|_N)$ contains a two-element algebra $\mathbf{C}$ with only trivial polynomials by Lemma 6.18 of [7]. Theorem 6.17 of [7] implies that there is a finite $\mathbf{A}' \in \mathcal{V}$, an $e \in E(\mathbf{A}')$, $\beta' \in \mathrm{Con}(\mathbf{A}')$, $U' \in \mathrm{M}_{\mathbf{A}}(0_{A'}, \beta')$ and a $\beta'|_{U'}$-equivalence class N' such that $\mathbf{A}'|_{N'}$ is polynomially equivalent to $\mathbf{C}|_C$, a two-element "set". Factoring by a congruence θ maximal for $\theta|_U = 0_U$ if necessary, we may assume that every nonzero congruence on $\mathbf{A}'$ restricts nontrivially to U'. We may further assume that β' is an atom in $\mathrm{Con}(\mathbf{A}')$. N' is connected by $\langle 0_{A'}, \beta' \rangle$-traces so, since $|N'| = 2$, N' is a $\langle 0_{A'}, \beta' \rangle$-trace. Every nonconstant polynomial of $\mathbf{A}'|_{N'}$ is a trivial projection, so $\mathrm{typ}(0_{A'}, \beta') = \mathbf{0}$. □

The following definition is also due to E. Kiss and can be found in [16].

DEFINITION 2.3. An E-**trace** of $\mathbf{A}$ is a nonempty subset $N \subseteq A$ which is the intersection of a congruence class and the range of an idempotent polynomial.

THEOREM 2.4. *If* $\mathbf{A}$ *is an algebra and* $\mathscr{K}$ *is a full class of lattices containing the congruence lattice of every member of* $V(\mathbf{A})$, *then* $\mathscr{K}$ *contains the congruence lattice of every member of* $V(\mathbf{A}I_N)$ *for every* E*-trace* $N \subseteq A$. *Conversely, if* $\mathbf{B}$ *is finite and* $\mathscr{K}$ *contains* $\mathrm{Con}(\mathbf{B}I_U)$ *for every* $U \in \mathrm{M}_{\mathbf{B}}(\alpha, \beta)$ *when* $\alpha \prec \beta$ *in* $\mathrm{Con}(\mathbf{B})$, *then* $\mathscr{K}$ *contains* $\mathrm{Con}(\mathbf{B})$.

Proof. Theorem 6.17 of [7] proves that if $\mathbf{C} \in V(\mathbf{A}I_N)$, then there is an $\mathbf{A}' \in V(\mathbf{A})$ and a congruence $\beta' \in \mathrm{Con}(\mathbf{A}')$ such that $\mathrm{Con}(\mathbf{C})$ is a complete (hence bounded) homomorphic image of the interval $[0_{A'}, \beta']$ in $\mathrm{Con}(\mathbf{A}')$. If $\mathrm{Con}(\mathbf{A}') \in \mathscr{K}$, then $\mathrm{Con}(\mathbf{C}) \in \mathscr{K}$. For the converse, Lemma 6.1 of [7] proves that if $\mathbf{B}$ is finite, then $\mathrm{Con}(\mathbf{B})$ is a subdirect product of lattices of the form $\mathrm{Con}(\mathbf{B}I_U)$ for $U \in \mathrm{M}_{\mathbf{B}}(\alpha, \beta)$ where $\alpha \prec \beta$ in $\mathrm{Con}(\mathbf{B})$. If $\mathscr{K}$ contains all lattices of this form it also contains $\mathrm{Con}(\mathbf{B})$. □

Since semimodular lattices form a full class of lattices Theorem 2.4 suggests that the methods of tame congruence theory might lead to a characterization of locally finite CSM varieties, or at least of the locally finite varieties in which the finite algebras are CSM. Chapter 8 of [7] contains characterizations of other well-known congruence conditions for locally finite varieties in terms of the type-set of the variety and the structure of the minimal sets. For example, a locally finite variety is congruence modular if and only if its type-set is a subset of $\{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$ and all $\langle \alpha, \beta \rangle$-minimal sets have empty tail. We do not know such a characterization of locally finite CSM varieties. If one exists we feel that it must be complicated. There is one type-omitting theorem that holds for CSM varieties, so we proceed with its proof.

LEMMA 2.5. *If* $\mathbf{G}$ *is finite, then the variety of all* $\mathbf{G}$*-sets is CSM if and only if* $\mathbf{G}$ *is trivial.*

Proof. Let $\mathscr{G}$ be the variety of all $\mathbf{G}$-sets for some fixed, nontrivial, finite group $\mathbf{G}$. $\mathscr{G}$ contains a simple, transitive $\mathbf{G}$-set $\mathbf{A}$ and a simple, intransitive $\mathbf{G}$-set $\mathbf{B}$. ($\mathbf{B}$ has two elements and every element of $\mathbf{G}$ acts as the identity permutation on B.) $\mathbf{A} \times \mathbf{B}$ is isomorphic to the disjoint union of two copies of $\mathbf{A}$. It is not hard to verify that $\mathrm{Con}(\mathbf{A} \times \mathbf{B})$ is the lattice pictured in Figure 1 which is not a semimodular lattice. □

THEOREM 2.6. *If $\mathscr{V}$ is CSM, then* $\mathbf{1} \notin \mathrm{typ}\{\mathscr{V}\}$.

Proof. If $\mathbf{1} \in \mathrm{typ}\{\mathscr{V}\}$, then there is a finite algebra $\mathbf{A} \in \mathscr{V}$ which has a minimal nonzero congruence β such that $\mathrm{typ}(0_A, \beta) = \mathbf{1}$. Any $\langle 0_A, \beta \rangle$-trace, N, is an E-trace which is polynomially equivalent to a transitive $\mathbf{G}$-set. Now Theorem 2.4 proves that $V(\mathbf{A}|_N)$ is CSM and this contradicts Lemma 2.5. □

This is the only type-omitting theorem that holds for CSM varieties, since the varieties of sets, vector spaces, boolean algebras, lattices and semilattices are all CSM. However, more can be said about the position of the different type labels in the congruence lattice of a finite algebra whose congruence lattice is semimodular. The following result is used a great deal in this paper.

THEOREM 2.7. [11] *Let α, β and γ be congruences on a finite algebra* $\mathbf{A}$ *that satisfy $\alpha \vee \gamma = \beta \vee \gamma$, $\alpha \wedge \gamma = \beta \wedge \gamma$ and $\alpha < \beta$. If* Con(**A**) *is semimodular, then* $\mathbf{typ}\{\alpha, \beta\} \subseteq \{\mathbf{0}, \mathbf{1}, \mathbf{5}\}$.

It is plausible that the reason that types **2**, **3**, and **4** do not occur in the critical quotient of a pentagon is that the associated minimal sets have empty tail. We have not been able to prove nor to disprove this. We pose the following problems concerning the structure of minimal sets in a CSM variety.

PROBLEM 2. Assume that **A** is a finite member of a CSM variety $\mathscr{V}$ and $\langle \alpha, \beta \rangle$ is a prime quotient in Con(**A**) which has type **2**, **3** or **4**. If $U \in \mathrm{M}_{\mathbf{A}}(\alpha, \beta)$ must U have empty tail?

The answer to Problem 2 is "yes" if $\mathrm{typ}\{\mathscr{V}\} \cap \{\mathbf{0}, \mathbf{5}\} = \varnothing$. The reason for this is that Theorems 2.6 and 2.7 show that any CSM variety satisfying $\mathrm{typ}\{\mathscr{V}\} \cap \{\mathbf{0}, \mathbf{5}\} = \varnothing$ is congruence modular. Theorem 8.5 of [7] proves that all minimal sets have empty tails in a congruence modular variety.

PROBLEM 3. Assume that **A** is a finite member of a CSM variety and $\langle \alpha, \beta \rangle$ is a prime quotient in Con(**A**) which has type **0**. If $U \in \mathrm{M}_{\mathbf{A}}(\alpha, \beta)$ must the body of U consist of a single trace?

THEOREM 2.8. *Let* **A** *be a locally finite algebra belonging to a CSM variety and assume that $\langle \alpha, \beta \rangle$ is a prime quotient of* Con(**A**) *for which $\alpha \overset{ss}{\not\sim} \beta$. Assume that $\gamma \leq \beta$ but $\gamma \nleq \alpha$. If $\alpha \overset{s}{\sim} \beta$ or if* **A** *is finite or if* $\mathbf{5} \notin \mathrm{typ}\{\mathscr{V}\}$, *then there is a least congruence $\delta \in [\alpha \wedge \gamma, \gamma]$ for which $\alpha \vee \delta = \beta$.*

Proof. First we will prove the theorem under the assumption that either $\alpha \overset{s}{\sim} \beta$ or $\mathbf{5} \notin \mathrm{typ}\{\mathscr{V}\}$. Factoring by $\alpha \wedge \gamma$ we may assume that $\alpha \wedge \gamma = 0_A$. Choose δ to be the least element in $[0_A, \gamma]$ which is locally strongly solvably equivalent to γ. Since $\alpha \overset{ss}{\nsim} \beta$ we have $0_A = \alpha \wedge \gamma \overset{ss}{\nsim} \beta \wedge \gamma = \gamma$, so $\delta > 0_A$. We will show that $0_A \prec \delta$. This will prove the theorem; for if $\theta \in [0_A, \gamma]$, $\alpha \vee \theta = \beta$ and θ is not above δ, then the fact that $\delta \overset{ss}{\sim} \gamma$ implies that $0_A = \delta \wedge \theta \overset{ss}{\sim} \gamma \wedge \theta = \theta$ which forces the contradiction that $\alpha = \alpha \vee 0_A \overset{ss}{\sim} \alpha \vee \theta = \beta$. Therefore we only need to prove that $0_A \prec \delta$.

For the purpose of obtaining a contradiction, assume that $0_A < \xi < \delta$. Since $\xi \overset{ss}{\nsim} \delta$ there is a 1-snag $\{a, b\} \subseteq \delta - \xi$. Let $\mathbf{F}$ be a finitely generated subalgebra of $\mathbf{A}$ containing a and b which has enough elements so that $(a, b) \in \alpha|_{\mathbf{F}} \vee \xi|_{\mathbf{F}}$ and $\{a, \beta\}$ is a 1-snag of $\mathbf{F}$. Let $\bar{\alpha} = \alpha|_{\mathbf{F}}$, $\bar{\xi} = \xi|_{\mathbf{F}}$, and $\bar{\delta} = \bar{\xi} \vee \mathrm{Cg}^{\mathbf{F}}(a, b) \subseteq \delta|_{\mathbf{F}}$. Now, $\bar{\xi} < \bar{\delta}$, $\bar{\alpha} \vee \bar{\xi} = \bar{\alpha} \vee \bar{\delta}$ and $\bar{\alpha} \wedge \bar{\xi} = \bar{\alpha} \wedge \bar{\delta}$, so Theorems 2.6 and 2.7 prove that $\mathrm{typ}\{\bar{\xi}, \bar{\delta}\} \subseteq \{\mathbf{0}, \mathbf{5}\}$. If $\alpha \overset{s}{\sim} \beta$, then $0_A = \alpha \wedge \delta \overset{s}{\sim} \beta \wedge \delta = \delta$, so $0_F \overset{s}{\sim} \bar{\delta}$. In this case $\mathbf{5} \notin \mathrm{typ}\{\bar{\xi}, \bar{\delta}\}$. Similarly, if $\mathbf{5} \notin \mathrm{typ}\{\mathscr{V}\}$ we have $\mathbf{5} \notin \mathrm{typ}\{\bar{\xi}, \bar{\delta}\}$. Either case yields $\mathrm{typ}\{\bar{\xi}, \bar{\delta}\} = \{\mathbf{0}\}$. But this is impossible since $(a, b) \in \bar{\delta} - \bar{\xi}$ is a 1-snag. Thus in these two cases the theorem is proved.

The remaining case is the one where $\mathbf{A}$ is finite. The last paragraph covered the situation where $\alpha \overset{s}{\sim} \beta$, so we may assume that $\langle \alpha, \beta \rangle$ is a prime quotient of nonabelian type. Now Lemma 5.15 of [7] proves that in this case there is a least $\delta \in [\alpha \wedge \gamma, 1_A]$ such that $\alpha \vee \delta = \beta$. (The semimodularity hypothesis is unnecessary for this case.) □

We will call the δ of Theorem 2.8 the **pseudo-complement below γ of α under β.** In Lemma 5.15 of [7] it is proved that the pseudo-complement below γ of α under β exists for any finite algebra if $\mathrm{typ}(\alpha, \beta) \in \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$. The "pseudo-complement" described there agrees with our "pseudo-complement below γ," but it has the additional property that it is the least $\delta \in [\alpha \wedge \gamma, 1_A]$ for which $\alpha \vee \delta = \beta$. On the other hand, in a CSM variety the pseudo-complement below γ of Theorem 2.8 exists for finite algebras when $\mathrm{typ}(\alpha, \beta) \in \{\mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ and also in some situations involving infinite algebras.

3. Congruences on Con(A)

DEFINITION 3.1. If $T \subseteq \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ and $\mathbf{A}$ is a finite algebra, then $\underset{T}{\sim}$ is the binary relation on $\mathrm{Con}(\mathbf{A})$ given by

$$\alpha \underset{T}{\sim} \beta \leftrightarrow \mathrm{typ}\{\alpha \wedge \beta, \alpha \vee \beta\} \subseteq T.$$

THEOREM 3.2. *If* $\mathbf{A}$ *is a finite algebra in a CSM variety, then* $\underset{T}{\sim}$ *is a lattice congruence if* $T = \varnothing$, $T = \{\mathbf{0}\}$, $T = \{\mathbf{0}, \mathbf{2}\}$ *or* $\{\mathbf{0}, \mathbf{5}\} \subseteq T$.

Proof. The only part of the theorem that is not covered by the results of Chapter 7 of [7] is that $\underset{T}{\sim}$ is a congruence when $\{\mathbf{0}, \mathbf{5}\} \subseteq T$. This is what we will prove here. Now for any T the relation $\underset{T}{\sim}$ is a reflexive, symmetric relation on $\mathrm{Con}(\mathbf{A})$. We will show that when $\alpha < \beta < \gamma$ and $\alpha \underset{T}{\sim} \beta \underset{T}{\sim} \gamma$, then $\alpha \underset{T}{\sim} \gamma$. If $\alpha \underset{T}{\not\sim} \gamma$, then there exists δ and θ such that $\alpha \leq \delta \prec \theta \leq \gamma$ where $\mathrm{typ}(\delta, \theta) \notin T$. We may assume that the interval $[\alpha, \gamma]$ is minimal under inclusion with these properties. If $\beta \leq \delta$, then $[\delta, \theta] \subseteq [\beta, \gamma]$ which is false since $\mathrm{typ}(\delta, \theta) \notin \mathrm{typ}\{\beta, \gamma\}$. Hence we can find a congruence α' such that $\beta \wedge \delta \prec \alpha' \leq \beta$. The prime quotients $\langle \beta \wedge \delta, \alpha' \rangle$ and $\langle \delta, \delta \vee \alpha' \rangle$ are perspective prime quotients whose type label is a member of T since the former quotient lies in $[\alpha, \beta]$. Since the label on $\langle \delta, \theta \rangle$ is not in T it must be that $\theta \neq \delta \vee \alpha'$. Now, $\delta \vee \alpha' \prec \theta \vee \alpha'$ and $\mathrm{typ}(\delta \vee \alpha', \theta \vee \alpha') = \mathrm{typ}(\delta, \theta) \notin T$. This is a contradiction to the minimality of $[\alpha, \gamma]$ since $[\alpha', \gamma]$ is a proper subinterval of $[\alpha, \gamma]$ with $\alpha' \underset{T}{\sim} \beta \underset{T}{\sim} \gamma$ which contains a prime quotient $\langle \delta \vee \alpha', \theta \vee \alpha' \rangle$ whose type is not in T.

Now we show that $\underset{T}{\sim}$ is compatible with $\vee$. To see this, assume that $\mathrm{Con}(\mathbf{A})$ is semimodular, $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$ and $\alpha \underset{T}{\sim} \beta$. We must show that $\alpha \vee \gamma \underset{T}{\sim} \beta \vee \gamma$. For this it suffices to consider the case when $\alpha < \beta$, since we can replace α and β by $\alpha \wedge \beta$ and $\alpha \vee \beta$. We can find a chain $\alpha = \theta_0 \prec \cdots \prec \theta_n = \beta$ and $\mathrm{typ}(\theta_i, \theta_{i+1}) \in T$ for all $i < n$. Let $\psi_i = \gamma \vee \theta_i$. By semimodularity, either $\psi_i = \psi_{i+1}$ or $\langle \psi_i, \psi_{i+1} \rangle$ and $\langle \theta_i, \theta_{i+1} \rangle$ are perspective prime quotients. In the latter case, $\mathrm{typ}(\psi_i, \psi_{i+1}) = \mathrm{typ}(\theta_i, \theta_{i+1})$. Using the result of the previous paragraph.

$$\mathrm{typ}\{\alpha \vee \gamma, \beta \vee \gamma\} = \mathrm{typ}\{\psi_0, \psi_n\} \subseteq \mathrm{typ}\{\theta_0, \theta_n\} = \mathrm{typ}\{\alpha, \beta\} \subseteq T.$$

If $\{\mathbf{0}, \mathbf{5}\} \subseteq T$, then $\underset{T}{\sim}$ is compatible with $\wedge$. To see this, take $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$ and $\alpha \underset{T}{\sim} \beta$. We must show that $\alpha \wedge \gamma \underset{T}{\sim} \beta \wedge \gamma$. Again it suffices to consider the case where $\alpha < \beta$. Now, if $\alpha \wedge \gamma \underset{T}{\not\sim} \beta \wedge \gamma$, then we can find a chain $\alpha \wedge \gamma = \theta_0 \prec \cdots \prec \theta_n = \beta \wedge \gamma$ where, for some i, $\mathrm{typ}(\theta_i, \theta_{i+1}) = \mathbf{i} \notin T$. In particular, $\mathbf{i} \in \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$. Now, $\alpha \wedge \theta_i = \alpha \wedge \theta_{i+1}$ and $\theta_i \prec \theta_{i+1}$ so, by Theorem 2.7 and semimodularity, we must have $\alpha \leq \delta = \alpha \vee \theta_i \prec \alpha \vee \theta_{i+1} = \psi \leq \beta$. Of course, $\mathbf{i} = \mathrm{typ}(\delta, \psi) \in \mathrm{typ}\{\alpha, \beta\} \subseteq T$, which shows that $\alpha \underset{T}{\not\sim} \beta$.

Now to finish the proof we must establish that $\underset{T}{\sim}$ is transitive if $\{\mathbf{0}, \mathbf{5}\} \subseteq T$. Assume that α, β and γ are arbitrary elements of $\mathrm{Con}(\mathbf{A})$ and that $\alpha \underset{T}{\sim} \beta \underset{T}{\sim} \gamma$. We need to show that $\mathrm{typ}\{\alpha \wedge \gamma, \alpha \vee \gamma\} \subseteq T$. In fact, we will prove the stronger statement that $\mathrm{typ}\{\alpha \wedge \beta \wedge \gamma, \alpha \vee \beta \vee \gamma\} \subseteq T$. Joining both sides of $\alpha \underset{T}{\sim} \beta$ with α yields $\alpha \underset{T}{\sim} \alpha \vee \beta$. Joining both sides of $\beta \underset{T}{\sim} \gamma$ with $\alpha \vee \beta$ yields $\alpha \vee \beta \underset{T}{\sim} \alpha \vee \beta \vee \gamma$. Since $\underset{T}{\sim}$ is transitive on comparable triples, $\alpha \underset{T}{\sim} \alpha \vee \beta \vee \gamma$. Dually, $\alpha \wedge \beta \wedge \gamma \underset{T}{\sim} \alpha$, so $\alpha \wedge \beta \wedge \gamma \underset{T}{\sim} \alpha \vee \beta \vee \gamma$. This finishes the proof. □

If $\underset{T}{\sim}$ and $\underset{T'}{\sim}$ are congruences on $\mathrm{Con}(\mathbf{A})$, then we have $\underset{T}{\sim} \vee \underset{T'}{\sim} = \underset{T \cup T'}{\sim}$ and $\underset{T}{\sim} \wedge \underset{T'}{\sim} = \underset{T \cap T'}{\sim}$ in $\mathrm{Con}(\mathrm{Con}\mathbf{A}))$. Hence, if $\mathrm{Con}(\mathbf{A})$ is semimodular, then $\mathrm{Con}(\mathrm{Con}(\mathbf{A}))$ has a sublattice which is a homomorphic image of the lattice in Figure 2.

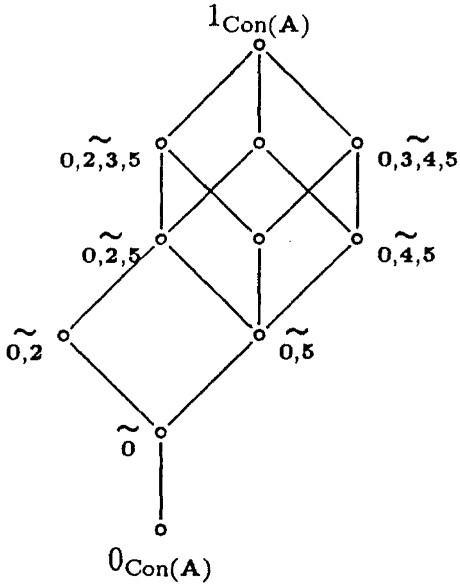


Figure 2

Two congruences $\underset{T}{\sim}$ and $\underset{T'}{\sim}$ on $\mathrm{Con}(\mathbf{A})$ are distinct if and only if $T \cap \mathrm{typ}\{\mathbf{A}\} \neq T' \cap \mathrm{typ}\{\mathbf{A}\}$.

THEOREM 3.3. *If* $\mathbf{A}$ *is a finite algebra which belongs to a CSM variety, then the following hold.*

(1) $\mathrm{Con}(\mathbf{A})/\underset{\mathbf{0,2,5}}{\sim}$ *is a distributive lattice.*
(2) $\mathrm{Con}(\mathbf{A})/\underset{\mathbf{0,5}}{\sim}$ *is a modular lattice.*
(3) $\mathrm{Con}(\mathbf{A})/\underset{\mathbf{0,2}}{\sim}$ *is a meet-semidistributive lattice.*
(4) $\mathrm{Con}(\mathbf{A})/\underset{\mathbf{0}}{\sim}$ *is a subdirect product of a meet-semidistributive lattice and a modular lattice.*

Proof. If we prove (2) and (3), then (1) and (4) follow from the facts that $\underset{\mathbf{0,2}}{\sim} \vee \underset{\mathbf{0,5}}{\sim} = \underset{\mathbf{0,2,5}}{\sim}$ and $\underset{\mathbf{0,2}}{\sim} \wedge \underset{\mathbf{0,5}}{\sim} = \underset{\mathbf{0}}{\sim}$ and also the fact that a modular, meet-semidistributive lattice is distributive. Theorem 7.7 (2) of (7) proves precisely that (3) holds, so we only need to consider (2). To prove it we need to show that if $T = \{\mathbf{0}, \mathbf{5}\}$ and $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$ are congruences such that $\alpha/\underset{T}{\sim} \leq \beta/\underset{T}{\sim}$, $\alpha/\underset{T}{\sim} \wedge \gamma/\underset{T}{\sim} = \beta/\underset{T}{\sim} \wedge \gamma/\underset{T}{\sim}$ and $\alpha/\underset{T}{\sim} \vee \gamma/\underset{T}{\sim} = \beta/\underset{T}{\sim} \vee \gamma/\underset{T}{\sim}$, then $\alpha \underset{T}{\sim} \beta$. To show this, let $\alpha' = (\alpha \wedge \beta) \vee (\beta \wedge \gamma)$ and $\beta' = \beta \wedge (\alpha' \vee \gamma)$. The conditions on α, β and γ imply that $\alpha \underset{T}{\sim} \alpha'$ and $\beta \underset{T}{\sim} \beta'$. Also $\alpha' \wedge \gamma = \beta' \wedge \gamma$, $\alpha' \vee \gamma = \beta' \vee \gamma$ and $\alpha' \leq \beta'$. Now Theorem 2.7 proves that $\alpha' \underset{T}{\sim} \beta'$, so $\alpha \underset{T}{\sim} \alpha' \underset{T}{\sim} \beta' \underset{T}{\sim} \beta$. $\square$

Much of what we will say in the rest of this section does not depend on semimodularity, but only on the fact that $\underset{T}{\sim}$ is a congruence on Con(**A**) for certain T and all finite **A**. Therefore, we will phrase the results in this generality.

If **A** is a finite algebra and $\underset{T}{\sim}$ is a congruence on Con(**A**) for some $T \subseteq \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$, then we get at once that for any $\alpha \in \mathrm{Con}(\mathbf{A})$ there exists a maximal congruence $\rho \in \mathrm{Con}(\mathbf{A})$ such that $\mathrm{typ}\{\alpha, \rho\} \subseteq T$. We call ρ the T-**radical** of α and we may also denote it by ρ_T^α when more precision is needed. If $\alpha = 0_A$ we call ρ the T-radical of **A**. The T-radical of **A** may also be written as ρ_T or $\rho_T^{\mathbf{A}}$ when it is necessary to specify which set T or which algebra **A** is involved. For any $\alpha \in \mathrm{Con}(\mathbf{A})$ there is a minimal congruence $\sigma \in \mathrm{Con}(\mathbf{A})$ such that $\mathrm{typ}\{\sigma, \alpha\} \subseteq T$. We call σ (or σ_T^α) the T-**coradical** of α. If $\alpha = 1_A$ we call σ (or σ_T or $\sigma_T^{\mathbf{A}}$) the T-coradical of **A**. We will call a finite algebra T-**radical-free** if its T-radical is the zero congruence.

Notice that, by Corollary 5.3 of [7], the relation $\underset{T}{\sim}$ is **preserved by homomorphisms**. That is, if $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{A})$ and $\gamma \leq \alpha, \beta$, then $\alpha \underset{T}{\sim} \beta$ if and only if $\alpha/\gamma \underset{T}{\sim} \beta/\gamma$ in $\mathrm{Con}(\mathbf{A}/\gamma)$. If $\mathscr{V}$ is a variety we will say that $\underset{T}{\sim}$ is **hereditary for** $\mathscr{V}$ if whenever $\mathbf{A} \in \mathscr{V}_{fin}$, $\mathbf{B} \leq \mathbf{A}$ and $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ we have

$$\alpha \underset{T}{\sim} \beta \rightarrow \alpha|_B \underset{T}{\sim} \beta|_B.$$

One must wonder whether or not there is some way of extending the definition of the congruences of the form $\underset{T}{\sim}$ so as to be applicable to infinite algebras in locally finite varieties. We will find that in some circumstances this is possible, but first we will need a technical lemma.

LEMMA 3.4. *If* **L** *is an upper continuous lattice and* $h : \mathbf{L} \rightarrow \mathbf{K}$ *is a complete homomorphism onto the lattice* **K**, *then* $h(a)$ *is compact in* **K** *if and only if* $a_\downarrow$ *is compact in* **L**, *where* $a_\downarrow$ *is the least element* $x \in L$ *for which* $h(x) = h(a)$. **K** *is algebraic if and only if for every* $b \in L$ *we have* $(b, \bigvee S) \in \ker h$ *for some* $S \subseteq \{c \leq b \mid c = c_\downarrow$ *and* c *is compact in* $L\}$.

Proof. The map from **K** to **L** which sends $h(a)$ to $a_\downarrow$ is a well-defined, complete $\vee$-homomorphism, so we identify **K** with the $\vee$-subsemilattice of **L** consisting of the elements of the form $x_\downarrow$. Compactness of a lattice element is a notion which only depends on the complete, $\vee$-semilattice structure of the lattice, so if $a_\downarrow$ is compact in **L** it is certainly compact in every complete, $\vee$-semilattice of **L** which contains it. For instance, $a_\downarrow$ is compact in **K** if it is compact in **L**. Conversely, if $a_\downarrow$ is not compact in **L**, then we can find a set $X \subseteq L$ such that $a_\downarrow \leq \bigvee X$ although no finite join of elements from X majorizes $a_\downarrow$. Closing X under finite joins if necessary, we may assume that X is upward-directed. Now, applying upper continuity and the completeness of h we may replace every $x \in X$ by $(x \wedge a_\downarrow)_\downarrow$ and still retain the fact that X is upward-directed and that $a_\downarrow \le \bigvee X$. Now every element of (the upward-directed set) X lies in K, is strictly below $a_\downarrow$ and $a_\downarrow \le \bigvee X$. This shows that $a_\downarrow$ is not compact in **K** either and establishes the first part of the lemma. We leave it to the reader to verify that the second part of the lemma says precisely that every element of the complete lattice **K** is a join of the compact elements that lie below it. □

THEOREM 3.5. *Let $\mathscr{V}$ be a variety for which $\underset{T}{\sim}$ is a congruence on* Con(**A**) *for any $\mathbf{A} \in \mathscr{V}_{fin}$. In order to define a relation $\sim$ on* Con(**B**) *for any locally finite $\mathbf{B} \in \mathscr{V}$ satisfying*:

(1) $\sim\; = \underset{T}{\sim}$ *for all $\mathbf{A} \in \mathscr{V}_{fin}$*

(2) $\sim$ *is a complete congruence on* Con(**B**) *for any locally finite $\mathbf{B} \in \mathscr{V}$*

(3) $\sim$ *is preserved by homomorphisms.*

it is necessary and sufficient that $\underset{T}{\sim}$ be hereditary for $\mathscr{V}$.

Further, when such a congruence can be defined, $\sim$ can be chosen so that for each locally finite $\mathbf{B} \in \mathscr{V}$ the quotient Con(**B**)$/\sim$ *is algebraic.*

Proof. First, assume that (1), (2) and (3) hold and we will prove that $\underset{T}{\sim}$ is hereditary for $\mathscr{V}$. Choose $\mathbf{A} \in \mathscr{V}_{fin}$, $\mathbf{C} \le \mathbf{A}$ and $\alpha, \beta \in \mathrm{Con}(\mathbf{A})$ with $\alpha \underset{T}{\sim} \beta$. We need to prove that $\alpha|_C \underset{T}{\sim} \beta|_C$. Let **D** be the subalgebra of $\mathbf{A}^\omega$ consisting of all tuples which differ in at most finitely many coordinates from a tuple of the form $(c, c, c, \dots)$ where $c \in C$. If η_i is the kernel of the ith coordinate projection, restricted to **D**, then $\mathbf{D}/\eta_i \cong \mathbf{A}$. For $\theta \in \mathrm{Con}(\mathbf{A})$ let θ_i denote the congruence on **D** consisting of those pairs $(\bar{d}, \bar{e})$ where $(d_i, e_i) \in \theta$. For each $i < \omega$ we have $\alpha_i/\eta_i \underset{T}{\sim} \beta_i/\eta_i$ in $\mathrm{Con}(\mathbf{D}/\eta_i)$ so, by (1) and (3), $\alpha_i \sim \beta_i$ in Con(**D**) for all $i < \omega$. Condition (2) implies that $\hat{\alpha} = \bigwedge_{i<\omega} \alpha_i \sim \bigwedge_{i<\omega} \beta_i = \hat{\beta}$. Let $\gamma = \{(\bar{x}, \bar{y}) \in D^2 \mid x_i = y_i$ for all but finitely many $i\}$. $\gamma \in \mathrm{Con}(\mathbf{D})$. Of course, γ is the kernel of the homomorphism $\phi : \mathbf{D} \to \mathbf{C}$ which maps $\bar{x}$ to c if $x_i = c$ for all but finitely many i. Notice that $\hat{\alpha} \vee \gamma \sim \hat{\beta} \vee \gamma$. Also $\hat{\alpha} \vee \gamma = \phi^{-1}(\alpha|_C)$ and $\hat{\beta} \vee \gamma = \phi^{-1}(\beta|_C)$ so, by (3), we get that $\alpha|_C \underset{T}{\sim} \beta|_C$.

Now we prove that if $\underset{T}{\sim}$ is hereditary for $\mathscr{V}$, then we can define $\sim$ satisfying (1), (2) and (3). For the rest of the proof **B** will be a locally finite algebra in $\mathscr{V}$. If $\alpha, \beta \in \mathrm{Con}(\mathbf{B})$ we will say that $\alpha \sim \beta$ if $\alpha|_F \underset{T}{\sim} \beta|_F$ for each finite subalgebra $\mathbf{F} \le \mathbf{B}$. Since $\underset{T}{\sim}$ is hereditary, $\sim$ agrees with $\underset{T}{\sim}$ for all $\mathbf{A} \in \mathscr{V}_{fin}$. This shows that (1) holds. From the definition, $\sim$ is an equivalence relation that respects complete meets. To show that $\sim$ is compatible with joins assume that $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{B})$, $\alpha \sim \beta$ but $\alpha \vee \gamma \not\sim \beta \vee \gamma$. We can find a finite subalgebra $\mathbf{F} \le \mathbf{B}$ such that $(\alpha \vee \gamma)|_F \underset{T}{\not\sim} (\beta \vee \gamma)|_F$ although $\alpha|_F \underset{T}{\sim} \beta|_F$. There must exist a pair (δ, θ) such that

$$(\alpha \vee \gamma)|_F \wedge (\beta \vee \gamma)|_F \le \delta \prec \theta \le (\alpha \vee \gamma)|_F \vee (\beta \vee \gamma)|_F$$

and $\mathrm{typ}(\delta, \theta) \notin T$. On the other hand, since $\alpha|_F \underset{T}{\sim} \beta|_F$ we must have

$$[\delta, \theta] \not\subseteq [(\alpha|_F \vee \gamma|_F) \wedge (\beta|_F \vee \gamma|_F), (\alpha|_F \vee \gamma|_F) \vee (\beta|_F \vee \gamma|_F)].$$

Since θ is $\leq (\alpha \vee \gamma)|_F \vee (\beta \vee \gamma)|_F$ but $\not\leq (\alpha|_F \vee \gamma|_F) \vee (\beta|_F \vee \gamma|_F)$, we can find a finite $\mathbf{F}'$ where $\mathbf{F} \leq \mathbf{F}' \leq \mathbf{B}$ and $\theta \subseteq (\alpha|_{F'} \vee \gamma|_{F'}) \vee (\beta|_{F'} \vee \gamma|_{F'}) = \psi$. For this $\mathbf{F}'$ set $\lambda = (\alpha|_{F'} \vee \gamma|_{F'}) \wedge (\beta|_{F'} \vee \gamma|_{F'})$. The fact that $\alpha \sim \beta$ implies that $\lambda \underset{T}{\sim} \psi$. Further, $\lambda|_F \leq (\alpha \vee \gamma)|_F \wedge (\beta \vee \gamma)|_F \leq \delta \prec \theta \leq \psi|_F$. This contradicts our assumption that $\underset{T}{\sim}$ is hereditary, since $\lambda \underset{T}{\sim} \psi$ but $\lambda|_F \underset{T}{\not\sim} \psi|_F$. Thus, $\sim$ is compatible with $\vee$. To show that it is $\vee$-complete we only need to prove that each $\sim$-congruence class has a largest element (since $\mathrm{Con}(\mathbf{B})$ is algebraic and we already know that each $\sim$-class has a least element). By Zorn's Lemma it suffices to prove that if $\alpha \sim \beta_i$ and $\beta_0 \leq \beta_1 \leq \cdots$, then $\alpha \sim \bigvee \beta_i$. The argument for this is much like the one we just finished. Assume that $\alpha \not\sim \bigvee \beta_i$ and that $a \sim \beta_i$ for each i. We can find a finite $\mathbf{G} \leq \mathbf{B}$ such that $\alpha|_G \wedge (\bigvee \beta_i)|_G \leq \delta \prec \theta \leq \alpha|_G \vee (\bigvee \beta_i)|_G$ and $\mathrm{typ}(\delta, \theta) \notin T$. However, $\alpha|_G \underset{T}{\sim} \bigvee(\beta_i|_G)$. We can find a finite $\mathbf{G}'$ with $\mathbf{G} \leq \mathbf{G}' \leq \mathbf{B}$ such that $\theta \subseteq \alpha|_{G'} \vee \bigvee(\beta_i|_{G'}) = \psi$. For this $\mathbf{G}'$ set $\lambda = \alpha|_{G'} \wedge (\bigvee(\beta_i|_{G'}))$. As before, $\lambda \underset{T}{\sim} \psi$ and $\lambda|_G \leq \delta \prec \theta \leq \psi|_G$. Thus, $\lambda|_G \underset{T}{\not\sim} \psi|_G$, contradicting the fact that $\underset{T}{\sim}$ is hereditary. This establishes (2). (3) is an easy consequence of our definition of $\sim$ and the fact that $\underset{T}{\sim}$ is preserved by homomorphisms for finite algebras.

What is left to prove is that $\mathrm{Con}(\mathbf{B})/\sim$ is algebraic. For this we will need Lemma 3.4 and a new characterization of $\sim$. Suppose that $\mathbf{F} \leq \mathbf{B}$ is finite, $\delta \prec \theta$ in $\mathrm{Con}(\mathbf{F})$ and $\mathrm{typ}(\delta, \theta) = \mathbf{i}$. Choose a $\psi \in \mathrm{Con}(\mathbf{F})$ which is minimal for the properties that $\psi \leq \theta$ and $\psi \not\leq \delta$. ψ is completely join-irreducible with lower cover ψ_* and $\mathrm{typ}(\psi_*, \psi) = \mathbf{i}$. In particular, ψ is a principal congruence. Call any pair $(a, b) \in B^2$ a **snag of type i** if there is some choice of $\mathbf{F}$, δ, θ and ψ as just described such that $\mathrm{Cg}^{\mathbf{F}}(a, b) = \psi$. We allow the possibility that (a, b) is a snag of type $\mathbf{i}$ and also of type $\mathbf{j}$.

CLAIM. *If $\alpha, \beta \in \mathrm{Con}(\mathbf{B})$, then $\alpha \sim \beta$ if and only if $\alpha \wedge \beta$ and $\alpha \vee \beta$ contain the same snags of type* $\mathbf{i}$ *for each* $\mathbf{i} \notin T$.

Proof of Claim. If $(a, b) \in a \vee \beta - a \wedge \beta$ is a snag of type $\mathbf{i} \notin T$, then we can find $\mathbf{F}$, δ, θ and ψ witnessing this fact. Of course, $\psi = \mathrm{Cg}^{\mathbf{F}}(a, b) \leq (\alpha \vee \beta)|_F$ and $\psi \not\leq (\alpha \wedge \beta)|_F$. If $\alpha \sim \beta$, or equivalently $\alpha \wedge \beta \sim a \vee \beta$, then $\psi \wedge (a \vee \beta)|_F \underset{T}{\sim} \psi \wedge (\alpha \wedge \beta)|_F \leq \psi_*$, since ψ is join-irreducible. This would imply $\mathbf{i} = \mathrm{typ}(\psi_*, \psi) \in T$ which is contrary to our assumption. Thus, $\alpha \not\sim \beta$. Now assume that $\alpha \not\sim \beta$ or, equivalently, that $\alpha \wedge \beta \not\sim \alpha \vee \beta$. We can find a finite $\mathbf{F}' \leq \mathbf{B}$ such that $(\alpha \wedge \beta)|_{F'} \underset{T}{\not\sim} (\alpha \vee \beta)|_{F'}$. Say that $(\alpha \wedge \beta)|_{F'} \leq \delta' \prec \theta' \leq (\alpha \vee \beta)|_{F'}$ and $\mathrm{typ}(\delta', \theta') = \mathbf{j} \notin T$. Let ψ' be a congruence below θ' which is minimal for $\psi' \not\leq \delta'$. If ψ'_* is the unique lower cover of ψ' and $(a', b') \in \psi' - \psi'_*$, then (a', b') is a snag of type $\mathbf{j} \notin T$ which is contained in $(\alpha \vee \beta) - (\alpha \wedge \beta)$. This establishes the claim.

Now, if (a, b) is a snag of type $\mathbf{i} \notin T$, then $\gamma = \mathrm{Cg}^{\mathbf{B}}(a, b)$ is the least element of its $\sim$-class. This is because if $\gamma \not\leq \delta$, then $(a, b) \in \gamma \vee \delta - \gamma \wedge \delta$. By Lemma 3.4, for each snag (a, b) of type $\mathbf{i} \notin T$ we have that $\mathrm{Cg}^{\mathbf{B}}(a, b)/\sim$ is compact in $\mathrm{Con}(\mathbf{B})/\sim$. If $\beta \in \mathrm{Con}(\mathbf{B})$ and $\alpha = \mathrm{Cg}^{\mathbf{B}}(\{(a_i, b_i) \in \beta \mid (a_i, b_i)$ is a snag whose type is not in $T\})$, then $\alpha \leq \beta$ and, by the claim, $\alpha \sim \beta$. Lemma 3.4 proves that $\mathrm{Con}(\mathbf{B})/\sim$ is algebraic in these circumstances. □

Of the different possible extensions of $\underset{T}{\sim}$ to congruence lattices of locally finite algebras, the congruence $\sim$ defined in the proof of Theorem 3.5 is the largest. We choose the largest extension to ensure that the quotient $\mathrm{Con}(\mathbf{B})/\sim$ is algebraic. This particular extension of $\underset{T}{\sim}$ will also be called $\underset{T}{\sim}$ throughout the rest of this paper. In particular, we will write $\underset{\mathbf{0,1,2}}{\sim}$ and $\underset{\mathbf{0,1}}{\sim}$ to denote the congruences $\overset{s}{\sim}$ and $\overset{ss}{\sim}$ of [7]. Each $\underset{T}{\sim}$-class of $\mathrm{Con}(\mathbf{B})$ has a least and a largest element. Therefore, we can extend the definitions of the T-radical of a congruence or algebra, the T-coradical of a congruence or algebra and the definition of T-radical-free to any locally finite algebra for which $\underset{T}{\sim}$ makes sense.

THEOREM 3.6. *If $\underset{T}{\sim}$ is a congruence on* $\mathrm{Con}(\mathbf{A})$ *for all $\mathbf{A} \in \mathcal{V}_{fin}$ and $\underset{T}{\sim}$ is not hereditary for $\mathcal{V}$, then there exists a finite, subdirectly irreducible algebra $\mathbf{B} \in \mathcal{V}$ with monolith μ and a subalgebra $\mathbf{C} \leq \mathbf{B}$ such that* $\mathrm{typ}(0_B, \mu) \in T$ *and* $0_C \underset{T}{\not\sim} \mu|_C$. *(In particular, if $\underset{T}{\sim}$ is a congruence on* $\mathrm{Con}(\mathbf{A})$ *for finite $\mathbf{A}$ and every finite subdirectly irreducible algebra with monolith whose type is in T has only trivial subalgebras, then $\underset{T}{\sim}$ is a congruence on* $\mathrm{Con}(\mathbf{A}')$ *for locally finite $\mathbf{A}'$.)*

Proof. If $\underset{T}{\sim}$ is not hereditary, then we can find a finite $\mathbf{B}' \in \mathcal{V}$ with a subalgebra $\mathbf{C}' \leq \mathbf{B}'$ and congruences $\alpha < \beta$ in $\mathrm{Con}(\mathbf{B}')$ such that $\alpha \underset{T}{\sim} \beta$ but $\alpha|_{C'} \underset{T}{\not\sim} \beta|_{C'}$. $\mathbf{C}'$ has a join-irreducible congruence $\psi \leq \beta|_{C'}$ for which $\psi \not\leq \alpha|_{C'}$ and, where ψ_* is the lower cover of ψ, $\mathrm{typ}(\psi_*, \psi) \notin T$. Say that $\psi = \mathrm{Cg}(a, b)$. Let $\theta \in \mathrm{Con}(\mathbf{B}')$ be a congruence containing α which is maximal with respect to not containing (a, b). The congruence θ is completely meet-irreducible with unique upper cover $\theta^* = \theta \vee \mathrm{Cg}(a, b)$. We have $\mathrm{typ}(\theta, \theta^*) \in \mathrm{typ}\{\theta \vee \alpha, \theta \vee \beta\} \subseteq T$, but $\theta|_{C'} \underset{T}{\not\sim} \theta^*|_{C'}$ since $\mathrm{typ}(\psi_*, \psi) \in \mathrm{typ}\{\theta|_{C'}, \theta^*|_{C'}\}$. Hence $\mathbf{B} = \mathbf{B}'/\theta$ is subdirectly irreducible with monolith $\mu = \theta^*/\theta$ and $\mathbf{C} = \mathbf{C}'/\theta|_{C'}$ is a subalgebra for which the conclusions of the theorem hold. □

THEOREM 3.7. *Assume that $\underset{T}{\sim}$ is a congruence on the congruence lattices of members of $\mathcal{V}_{fin}$ and that $\underset{T}{\sim}$ is hereditary for $\mathcal{V}$. If $\mathbf{A}$ is a locally finite T-radical-free subdirectly irreducible algebra contained in $\mathcal{V}$, then $\mathbf{A} \in \boldsymbol{SP}_u(\mathcal{K})$ where $\mathcal{K}$ consists of finite T-radical-free subdirectly irreducibles contained in $\boldsymbol{HS}(\mathbf{A})$. If $\mathbf{B}$ is a locally finite T-radical-free algebra contained in $\mathcal{V}$, then $\mathbf{B} \in \boldsymbol{P}_s(\mathcal{K})$ where $\mathcal{K}$ consists of T-*

radical-free subdirectly irreducible algebras. If $\mathbf{C}$ *is a locally finite algebra in* $\mathscr{V}$, *then*

$$\mathbf{C}/\rho_T^{\mathbf{C}} \in \boldsymbol{SPP}_u(\mathscr{K})$$

where $\mathscr{K}$ *consists of finite, T-radical-free subdirectly irreducible algebras from* $\boldsymbol{HS}(\mathbf{C})$.

Proof. The first two statements of the theorem are equivalent to the third, so we only prove the first two.

The proof of the first statement is a slight modification of Lemma 10.2 of [5]. Let μ denote the monolith of $\mathbf{A}$. Since $\mathbf{A}$ is T-radical-free, we can find a snag $(a, b) \in \mu$ of type $\mathbf{i} \notin T$. There is a finite subalgebra $\mathbf{F} \leq \mathbf{A}$ such that $\mathrm{Cg}^{\mathbf{F}}(a, b) = \psi$ is join-irreducible, ψ has ψ_* as its lower cover and $\mathrm{typ}(\psi_*, \psi) = \mathbf{i}$. Let $\mathscr{S}$ denote the collection of all finite subsets of A which contain the subuniverse F and for $S \in \mathscr{S}$ let $\theta_S \in \mathrm{Con}(\mathrm{Sg}^{\mathbf{A}}(S))$ be a maximal congruence not containing (a, b). The congruence θ_S is meet-irreducible with upper cover θ_S^*. Further, $\theta_S \underset{T}{\not\sim} \theta_S^*$ since this would lead to

$$\psi = \psi \wedge (\theta_S^*)|_F \underset{T}{\sim} \psi \wedge (\theta_S)|_F \leq \psi_*$$

or $\psi \underset{T}{\sim} \psi_*$. Hence, $\mathrm{Sg}(S)/\theta_S$ is a finite, T-radical-free subdirectly irreducible algebra for each $S \in \mathscr{S}$. Let $\mathscr{F}$ be the filter on $\mathscr{S}$ consisting of all $\mathscr{T} \subseteq \mathscr{S}$ for which there is an $S_0 \in \mathscr{S}$ such that $\{S \in \mathscr{S} \mid S_0 \subseteq S\} \subseteq \mathscr{T}$. Let $\mathscr{U}$ be an ultrafilter extending $\mathscr{F}$ and let $\mathbf{D} = \prod_{\mathscr{S}} \mathrm{Sg}(S)/\theta_S$. Define a function $\phi : A \to D$ by $\phi(x)_S = x/\theta_S$ if $x \in S$ and $\phi(x)_S$ is arbitrary if $x \notin S$. We leave it to the reader to check that the composite map $\mathbf{A} \xrightarrow{\phi} \mathbf{D} \to \mathbf{D}/\mathscr{U}$ is a homomorphism. This homomorphism is an embedding because it fails to identify a and b. This establishes our first claim.

For the second statement, $\mathrm{Con}(\mathbf{B})/\underset{T}{\sim}$ is an algebraic lattice, so we can express the bottom element as a meet of completely meet-irreducible elements: say $0/\underset{T}{\sim} = \bigwedge \alpha_i/\underset{T}{\sim}$. We can choose the $\alpha_i \in \mathrm{Con}(\mathbf{B})$ so that they are the largest elements of their respective $\underset{T}{\sim}$-classes. This, and the fact that each $\alpha_i/\underset{T}{\sim}$ is completely meet-irreducible, implies that each α_i is completely meet-irreducible and equal to its T-radical. Now, $\bigwedge \alpha_i \underset{T}{\sim} 0_B$, so $\bigwedge \alpha_i = 0_B$. This shows that we can take $\mathscr{K} = \{\mathbf{B}/\alpha_i\}$. □

We leave it to the reader to prove that Theorem 2.7 extends to locally finite algebras in a CSM variety if $\underset{\mathbf{0,5}}{\sim}$ is hereditary. That is, if $\mathbf{B} \in \mathscr{V}$ is locally finite and in $\mathrm{Con}(\mathbf{B})$ we have $\alpha < \beta$, $\alpha \vee \gamma = \beta \vee \gamma$ and $\alpha \wedge \gamma = \beta \wedge \gamma$, then $\alpha \underset{\mathbf{0,5}}{\sim} \beta$. From this the reader can show that Theorem 3.3 holds for locally finite algebras in a CSM variety if $\underset{\mathbf{0,5}}{\sim}$ is hereditary.

Now we begin exploring applications of Theorem 3.3. R. McKenzie is partly responsible for the following generalization of Jónsson's Theorem.

THEOREM 3.8. *Assume that $V(\mathscr{K})$ is CSM. If* $\mathbf{A}$ *is a finite subdirectly irreducible algebra in $V(\mathscr{K})$ with monolith of type* **3** *or* **4**, *then* $\mathbf{A} \in \boldsymbol{HS}(\mathscr{K})$. *If* $\mathbf{B}$ *is any finite algebra in $V(\mathscr{K})$ and ρ is the* $\{\mathbf{0}, \mathbf{2}, \mathbf{5}\}$*-radical of* $\mathbf{B}$, *then*

$$\mathbf{B}/\rho \in \boldsymbol{P}_s\boldsymbol{HS}(\mathscr{K}).$$

Proof. The first statement of the theorem follows from the second, so we will only prove the second statement. Let $\mathbf{B}' = \mathbf{B}/\rho$; $\mathbf{B}'$ is a finite $\{\mathbf{0}, \mathbf{2}, \mathbf{5}\}$-radical-free algebra in $V(\mathscr{K})$. We can find a finite algebra $\mathbf{C} \in \boldsymbol{SP}(\mathscr{K})$ and a congruence $\alpha \in \mathrm{Con}(\mathbf{C})$ such that $\mathbf{B}' \cong \mathbf{C}/\alpha$. Necessarily, α equals its $\{\mathbf{0}, \mathbf{2}, \mathbf{5}\}$-radical. Let $\eta_0, \ldots, \eta_n \in \mathrm{Con}(\mathbf{C})$ be congruences such that each $\mathbf{C}/\eta_i$ is embeddable in a member of $\mathscr{K}$ and $\bigwedge \eta_i = 0_C$. Since $\mathrm{Con}(\mathbf{C})/\underset{\mathbf{0,2,5}}{\sim}$ is distributive,

$$\alpha = \alpha \vee (\bigwedge \eta_i) \underset{\mathbf{0,2,5}}{\sim} \bigwedge (\alpha \vee \eta_i) = \beta,$$

so $\alpha \leq \beta$ and $\alpha \underset{\mathbf{0,2,5}}{\sim} \beta$. We must have $\alpha = \beta$ since β is contained in the $\{\mathbf{0}, \mathbf{2}, \mathbf{5}\}$-radical of α. Thus, $\alpha = \bigwedge(\alpha \vee \eta_i)$ and so $\mathbf{B}' \cong \mathbf{C}/\alpha$ is a subdirect product of the algebras $\{\mathbf{C}/(\alpha \vee \eta_i)\} \subseteq \boldsymbol{HS}(\mathscr{K})$. □

In Theorem 3.8, if $\mathscr{K}$ is a finite set of finite algebras and $\mathscr{V} = V(\mathscr{K})$ is congruence distributive, then $\mathscr{V}_{fin} \subseteq \boldsymbol{P}_s\boldsymbol{HS}(\mathscr{K})$. For any locally finite variety, this inclusion is equivalent to $\mathscr{V} = \boldsymbol{P}_s\boldsymbol{HS}(\mathscr{K})$. This is just Jónsson's Theorem for finitely generated, congruence distributive varieties (see [9]). We can improve this result for CSM varieties in which $\underset{\mathbf{0,2,5}}{\sim}$ is hereditary (which is always true if $\mathbf{5} \notin \mathrm{typ}\{\mathscr{V}\}$ or even if we just have that the finite subdirectly irreducibles with type **5** monolith have only trivial subalgebras).

THEOREM 3.9. *Assume that $V(\mathscr{K})$ is CSM and that $\underset{\mathbf{0,2,5}}{\sim}$ is hereditary for $V(\mathscr{K})$. If* $\mathbf{A}$ *is a locally finite subdirectly irreducible algebra in $V(\mathscr{K})$ which is* $\{\mathbf{0}, \mathbf{2}, \mathbf{5}\}$*-radical-free, then* $\mathbf{A} \in \boldsymbol{SP}_u\boldsymbol{HS}(\mathscr{K}) \subseteq \boldsymbol{HSP}_u(\mathscr{K})$. *If* $\mathbf{B}$ *is any locally finite algebra in $V = (\mathscr{K})$ and ρ is the* $\{\mathbf{0}, \mathbf{2}, \mathbf{5}\}$*-radical of* $\mathbf{B}$, *then*

$$B/\rho \in \boldsymbol{SPP}_u\boldsymbol{HS}(\mathscr{K}).$$

Proof. This follows from Theorem 3.7 and the fact, proved in Theorem 3.8, that the finite $\{\mathbf{0}, \mathbf{2}, \mathbf{5}\}$-radical-free subdirectly irreducible algebras are contained in $\boldsymbol{HS}(\mathscr{K})$. □

Let $T, T' \subseteq \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$. We say that a finite algebra $\mathbf{A}$ **satisfies the** $\langle T, T'\rangle$-**transfer principle**, if for any $\alpha, \beta, \gamma \in \operatorname{Con}(\mathbf{A})$, with $\alpha \prec \beta \prec \gamma$ such that $\operatorname{typ}(\alpha, \beta) \in T$ and $\operatorname{typ}(\beta, \gamma) \in T'$, we can find a δ with $\alpha \prec \delta \leq \gamma$ with $\operatorname{typ}(\alpha, \delta) = \operatorname{typ}(\beta, \gamma)$. This definition is due to M. Valeriote. (See Chapter 3 of [23].) In all of our references to the $\langle T, T'\rangle$-transfer principle we will assume that $\{\mathbf{0}, \mathbf{1}\} \cap T = \emptyset$ or $\{\mathbf{0}, \mathbf{1}\} \subseteq T$ and similarly that $\{\mathbf{0}, \mathbf{1}\} \cap T' = \emptyset$ or $\{\mathbf{0}, \mathbf{1}\} \subseteq T'$. We do this in order to avoid the difficulties that arise when type $\mathbf{0}$ and type $\mathbf{1}$ prime quotients are perspective. With this convention, $\mathbf{A}$ fails the $\langle T, T'\rangle$-transfer principle if and only if $\operatorname{Con}(\mathbf{A})$ contains a three-element interval $[\alpha, \gamma] = \{a, \beta, \gamma\}$ where $\operatorname{typ}(\alpha, \beta) \in T$ and $\operatorname{typ}(\beta, \gamma) \in T'$. We say that $\mathcal{V}$ satisfies the $\langle T, T'\rangle$-transfer principle if every finite member of $\mathcal{V}$ does.

THEOREM 3.10. *Assume that* $\mathbf{A}$ *is finite,* $T \cup T' = \operatorname{typ}\{\mathbf{A}\}$ *and* $T \cap T' = \emptyset$. *The* $\langle T, T'\rangle$ *and the* $\langle T', T\rangle$*-transfer principles hold for* $\mathbf{A}$ *if and only if* $\underset{T}{\sim}$ *and* $\underset{T'}{\sim}$ *are permuting congruences on* $\operatorname{Con}(\mathbf{A})$.

Proof. If $\underset{T}{\sim}$ and $\underset{T'}{\sim}$ are (complementary) permuting congruences on $\operatorname{Con}(\mathbf{A})$, then $\operatorname{Con}(\mathbf{A})$ decomposes as a direct product of labeled lattices. From this it easily follows that $\mathbf{A}$ satisfies the $\langle T, T'\rangle$ and the $\langle T', T\rangle$-transfer principles. We will prove the converse.

First, each transfer principle implies its "dual". That is, if the $\langle T, T'\rangle$-transfer principle holds, $\alpha \prec \beta \prec \gamma$, $\operatorname{typ}(\alpha, \beta) = \mathbf{i} \in T$ and $\operatorname{typ}(\beta, \gamma) = \mathbf{j} \in T'$, then there is a δ such that $\delta \prec \gamma$ and $\operatorname{typ}(\delta, \gamma) = \mathbf{i}$. To see this, use the $\langle \mathbf{i}, \mathbf{j}\rangle$-transfer principle to obtain θ where $\alpha \prec \theta \leq \gamma$ and $\operatorname{typ}(\alpha, \theta) = \mathbf{j}$. θ complements β in $[\alpha, \gamma]$, so we can choose any $\delta \in [\theta, \gamma]$ such that $\delta \prec \gamma$. Since $\langle \delta, \gamma\rangle$ is perspective with $\langle \alpha, \beta\rangle$ the former quotient must have type $\mathbf{i}$.

Now, as $\underset{T}{\sim}$ is reflexive and symmetric, we only need to show that it is transitive and compatible with $\wedge$. We can then use the dual of the transfer principle to infer that $\underset{T'}{\sim}$ is compatible with $\vee$. Then, as T and T' can be interchanged in these arguments, we may conclude that $\underset{T}{\sim}$ and $\underset{T'}{\sim}$ are lattice congruences.

Compatibility of $\underset{T}{\sim}$ with $\wedge$ can be argued as follows: suppose $\alpha < \beta$, $\alpha \underset{T}{\sim} \beta$ and $\alpha \wedge \gamma \underset{T}{\not\sim} \beta \wedge \gamma$. Using the $\langle T, T'\rangle$-transfer principle in $[\alpha \wedge \gamma, \beta \wedge \gamma]$ we find that there must be a δ such that $\alpha \wedge \gamma \prec \delta$ and $\operatorname{typ}(\alpha \wedge \gamma, \delta) \in T'$. If θ is a congruence in $[\alpha, \beta]$ maximal for not containing δ, then θ has a unique upper cover in $[\alpha, \beta]$, call it θ^*, and $\langle \alpha \wedge \gamma, \delta\rangle$ is perspective with $\langle \theta, \theta^*\rangle$. The former quotient has a type-label belonging to T' while the latter has type label belonging to T. This is a contradiction.

As $\underset{T}{\sim}$ is a lattice tolerance on $\operatorname{Con}(\mathbf{A})$, to prove that it is transitive it suffices to prove transitivity for comparable triples (as we did in the proof of Theorem 3.2). Assume that $\alpha < \beta < \gamma$ and $\alpha \underset{T}{\sim} \beta \underset{T}{\sim} \gamma$. If $\alpha \underset{T}{\not\sim} \gamma$, we can find a prime quotient in $[\alpha, \gamma]$ whose type-label lies in T'. Using the $\langle T, T'\rangle$-transfer principle we can find such a quotient of the form $\langle\alpha, \delta\rangle$. We cannot have $\delta \leq \beta$, so $\alpha = \delta \wedge \beta \underset{T}{\sim} \delta \wedge \gamma = \delta$, which is false. This contradiction proves that $\underset{T}{\sim}$ and $\underset{T'}{\sim}$ are lattice congruences. We will be finished if we show that they permute.

Choose $\alpha \in \mathrm{Con}(\mathbf{A})$ and let ρ^{α}_{T} and $\rho^{\alpha}_{T'}$ denote respectively the T-radical and the T'-radical of α. We claim that

$$\mathrm{typ}\{\rho^{\alpha}_{T}, 1_A\} \cap T = \mathrm{typ}\{\rho^{\alpha}_{T'}, 1_A\} \cap T' = \varnothing.$$

To prove this, assume that $\mathrm{typ}\{\rho^{\alpha}_{T}, 1_A\} \cap T \neq \varnothing$. We can choose $\gamma \prec \delta$ in the interval $[\rho^{\alpha}_{T}, 1_A]$ where γ is minimal for the property that $\mathrm{typ}(\gamma, \delta) \in T$. If $\gamma > \rho^{\alpha}_{T}$, then for some θ satisfying $\rho^{\alpha}_{T} \leq \theta \prec \gamma$ we must have $\mathrm{typ}(\theta, \gamma) \in T'$. But then, by the $\langle T, T'\rangle$-transfer principle, we can find a congruence ψ with

$$\theta \prec \psi < \delta \quad \text{and} \quad \mathrm{typ}(\theta, \psi) \in T.$$

This contradicts the minimality of γ, so that we must have $\rho^{\alpha}_{T} = \gamma$. But then $\rho^{\alpha}_{T} \prec \delta$ and $\mathrm{typ}(\rho^{\alpha}_{T}, \delta) \in T$; this contradicts the fact that ρ^{α}_{T} is the T-radical of α. Thus, $\mathrm{typ}\{\rho^{\alpha}_{T}, 1_A\} \subseteq T'$ and so $\mathrm{typ}\{\rho^{\alpha}_{T}, 1_A\} \cap T = \varnothing$. Similarly, $\mathrm{typ}\{\rho^{\alpha}_{T'}, 1_A\} \cap T' = \varnothing$. This proves that ρ^{α}_{T}, the T-radical of α, contains the T'-coradical of $\mathbf{A}$ (in symbols: $\rho^{\alpha}_{T} \underset{T'}{\sim} 1_A$). Similarly, the T'-radical of α contains the T-coradical of $\mathbf{A}$.

Now choose any $\beta, \gamma \in \mathrm{Con}(\mathbf{A})$. From the previous paragraph we have that $\rho^{\beta}_{T} \underset{T'}{\sim} 1_A$ and $\rho^{\gamma}_{T'} \underset{T}{\sim} 1_A$. It follows that $\rho^{\beta}_{T} \wedge \rho^{\gamma}_{T'} \underset{T}{\sim} \rho^{\beta}_{T} \underset{T}{\sim} \beta$ and similarly that $\rho^{\beta}_{T} \wedge \rho^{\gamma}_{T'} \underset{T'}{\sim} \rho^{\gamma}_{T'} \underset{T'}{\sim} \gamma$. Hence, $\beta \underset{T}{\sim} \rho^{\beta}_{T} \wedge \rho^{\gamma}_{T'} \underset{T'}{\sim} \gamma$ which shows that $(\beta, \gamma) \in \underset{T}{\sim} \circ \underset{T'}{\sim}$. Since β and γ were arbitrary, $\underset{T}{\sim} \circ \underset{T'}{\sim} = 1_{\mathrm{Con}(\mathbf{A})}$ and $\underset{T}{\sim}$ and $\underset{T'}{\sim}$ permute. □

COROLLARY 3.11. *Let* $\mathbf{A}$ *be a finite algebra, and* $T, T' \subseteq \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ *such that* $T \cup T' = \mathrm{typ}\{\mathbf{A}\}$ *and* $T \cap T' = \varnothing$. *If both the* $\langle T, T'\rangle$ *and the* $\langle T', T\rangle$*-transfer principles hold, then* $\mathbf{A} \leq_{sd} \mathbf{A}/\rho^{\mathbf{A}}_{T} \times \mathbf{A}/\rho^{\mathbf{A}}_{T'}$ *and* $\mathrm{typ}\{\mathbf{A}/\rho^{\mathbf{A}}_{T'}\} \subseteq T'$ *and* $\mathrm{typ}\{\mathbf{A}/\rho^{\mathbf{A}}_{T'}\} \subseteq T$. *Further,* $\mathrm{Con}(\mathbf{A}) \cong \mathrm{Con}(\mathbf{A}/\rho^{\mathbf{A}}_{T}) \times \mathrm{Con}(\mathbf{A}/\rho^{\mathbf{A}}_{T'})$.

Proof. Straightforward from the preceding result. □

Corollary 3.11 shows how transfer principles help decompose finite algebras and varieties into algebras and varieties with smaller type-set. The next theorem localizes failures of some of the transfer principles and is a handy tool for establishing whether or not they hold.

THEOREM 3.12. *Assume that* $\underset{T}{\sim}$ *is a congruence on* $\mathrm{Con}(\mathbf{A})$ *for all finite* $\mathbf{A} \in \mathcal{V}$ *and that* $T \cap T' = \varnothing$ *and* $T \cup T' = \mathrm{typ}\{\mathcal{V}\}$. *Suppose that some finite algebra in* $\mathcal{V}$ *fails the* $\langle T, T'\rangle$*-transfer principle. Then there is a finite subdirectly irreducible algebra with congruences* $0 \prec \mu \prec \nu$ *where* $\mathrm{typ}(0, \mu) \in T$ *and* $\mathrm{typ}(\mu, \nu) \in T'$.

Proof. Assume that $\mathbf{B} \in \mathscr{V}$ is the smallest algebra in $\mathscr{V}$ that fails the $\langle T, T' \rangle$-transfer principle. Say that $[\alpha, \gamma]$ is a three-element interval in $\mathrm{Con}(\mathbf{B})$ where $\alpha \prec \beta \prec \gamma$ and $\mathrm{typ}(\alpha, \beta) \in T$ and $\mathrm{typ}(\beta, \gamma) \in T'$. By the minimality of $|B|$ we must have $\alpha = 0_B$. If $\mathbf{B}$ is subdirectly irreducible we are done, so assume otherwise. Let $\delta \in \mathrm{Con}(\mathbf{B})$ be a maximal congruence with the property that $\beta \wedge \delta = 0_B$. δ is completely meet-irreducible with unique upper cover δ^*. The prime quotients $\langle 0_B, \beta \rangle$ and $\langle \delta, \delta^* \rangle$ are perspective and both have type in T. The $\langle T, T' \rangle$-transfer principle and the fact that δ is completely meet-irreducible with $\mathrm{typ}(\delta, \delta^*) \in T$ imply that $\mathrm{typ}\{\delta, 1_B\} \subseteq T$. Thus, $1_B \underset{T}{\sim} \delta$ and $\gamma = \gamma \wedge 1_B \underset{T}{\sim} \gamma \wedge \delta = 0_B$. But $\gamma \underset{T}{\not\sim} 0_B$, so our assumption that $\mathbf{B}$ is not subdirectly irreducible is false. □

If one examines the last proof to see exactly where we used the hypothesis that $T \cup T' = \mathrm{typ}\{\mathscr{V}\}$ one finds that this hypothesis is unnecessary if $\mathscr{V}$ is CSM. We can therefore write down a better version of Theorem 3.12 for CSM varieties. The proof is essentially the same as the proof for Theorem 3.12.

THEOREM 3.13. *Assume that $\mathscr{V}$ is a CSM variety. Suppose that some finite algebra in $\mathscr{V}$ fails the $\langle \mathbf{i}, \mathbf{j} \rangle$-transfer principle for some $\mathbf{i} \neq \mathbf{j}$. Except possibly in the cases $\langle \mathbf{i}, \mathbf{j} \rangle = \langle \mathbf{i}, \mathbf{0} \rangle$, $\langle \mathbf{3}, \mathbf{5} \rangle$ or $\langle \mathbf{4}, \mathbf{5} \rangle$ there must exist a finite subdirectly irreducible algebra $\mathbf{A} \in \mathscr{V}$ with congruences $0_A \prec \mu \prec \nu$ where* $\mathrm{typ}(0_A, \mu) = \mathbf{i}$ *and* $\mathrm{typ}(\mu, \nu) = \mathbf{j}$. □

A semisimple CSM variety satisfies the $\langle \mathbf{i}, \mathbf{j} \rangle$-transfer principle for every $\mathbf{i} \neq \mathbf{j}$ except possibly the six cases listed in the theorem.

THEOREM 3.14. *If $\mathscr{V}$ is locally finite and $\underset{T}{\sim}$ is a congruence on the finite members of $\mathscr{V}$, then the class of algebras $\mathbf{A} \in \mathscr{V}$ for which* $\mathrm{typ}\{\mathbf{S}(\mathbf{A})\} \subseteq T$ *is a subvariety of $\mathscr{V}$.*

Proof. Let $\mathscr{K}$ be the class of all $\mathbf{A} \in \mathscr{V}$ for which $\mathrm{typ}\{\mathbf{S}(\mathbf{A})\} \subseteq T$. Clearly $\mathscr{K}$ is closed under the formation of subalgebras. $\mathscr{K}$ is closed under homomorphisms since $\mathrm{typ}\{\mathbf{SH}(\mathbf{A})\} \subseteq \mathrm{typ}\{\mathbf{S}(\mathbf{A})\}$, therefore if $\mathbf{A} \in \mathscr{K}$ we have $\mathbf{H}(\mathbf{A}) \subseteq \mathscr{K}$. Suppose $\mathbf{A}_0, \mathbf{A}_1 \in \mathscr{K}$ and $\mathbf{B}$ is a subdirect product $\mathbf{B} \leq_{sd} \mathbf{A}_0 \times \mathbf{A}_1$. If $\mathbf{B}' \leq \mathbf{B}$ is finite, then $\mathbf{B}' \leq_{sd} \mathbf{A}_0' \times \mathbf{A}_1'$ where $\mathbf{A}_i' \leq \mathbf{A}_i \in \mathscr{K}$. Let π_i denote the kernel of the projection of $\mathbf{B}'$ onto $\mathbf{A}_i'$. We have $\pi_0 \underset{T}{\sim} 1_{B'} \underset{T}{\sim} \pi_1$, so $1_{\mathbf{B}'} \underset{T}{\sim} \pi_0 \wedge \pi_1 = 0_{B'}$. Hence $\mathrm{typ}\{\mathbf{S}(\mathbf{B})\} \subseteq T$ and so $\mathbf{B} \in \mathscr{K}$. This shows that $\mathscr{K}$ is closed under finite subdirect products. In particular, the finite algebras in $\mathscr{K}$ form a pseudovariety. $\mathscr{V}$ is locally finite which implies that $\mathrm{typ}\{V(\mathscr{K}_{fin})\} = \mathrm{typ}\{\mathscr{K}_{fin}\} \subseteq T$ (since $\mathscr{K}_{fin} = V(\mathscr{K}_{fin})_{fin}$). Therefore $V(\mathscr{K}_{fin}) \subseteq \mathscr{K}$ by the definition of $\mathscr{K}$. On the other hand, if $\mathbf{C} \in \mathscr{K}$, then every finitely generated subalgebra of $\mathbf{C}$ is in $\mathscr{K}_{fin}$, so $\mathbf{C} \in V(\mathscr{K}_{fin})$. Thus, $\mathscr{K} = V(\mathscr{K}_{fin})$ is a variety. □

We will call the subvariety described in Theorem 3.14 the **locally-T subvariety** of $\mathscr{V}$.

THEOREM 3.15. *Let $\mathscr{V}$ be a locally finite, abelian variety and set $T = \{\mathbf{0}, \mathbf{1}\}$, $T' = \{\mathbf{2}\}$. The following conditions are equivalent:*

(1) *$\mathscr{V}$ satisfies the $\langle T, T'\rangle$ and $\langle T', T\rangle$-transfer principles.*

(2) *$\mathscr{V}$ decomposes as a varietal product $\mathscr{V}_T \otimes \mathscr{V}_{T'}$ where $\mathscr{V}_T$ is the strongly abelian subvariety of $\mathscr{V}$ and $\mathscr{V}_{T'}$ is an affine subvariety of $\mathscr{V}$.*

Proof. If (2) holds, then it is easy to see that $\underset{T}{\sim}$ and $\underset{T'}{\sim}$ are factor congruences on $\mathrm{Con}(\mathbf{A})$ for any finite $\mathbf{A} \in \mathscr{V}$. Hence (1) follows from Theorem 3.10.

Now assume that $\mathscr{V}$ satisfies the $\langle T, T'\rangle$ and $\langle T', T\rangle$-transfer principles. Since $\underset{T}{\sim} = \overset{ss}{\sim}$, $\underset{T}{\sim}$ is a hereditary congruence on $\mathrm{Con}(\mathbf{A})$ for all finite $\mathbf{A} \in \mathscr{V}$. $\underset{T'}{\sim}$ is a congruence on $\mathrm{Con}(\mathbf{A})$ for all finite $\mathbf{A} \in \mathscr{V}$ by Theorem 3.10. We claim that $\underset{T'}{\sim} = \underset{\mathbf{2}}{\sim}$ is hereditary, too. This is far from obvious; we need to quote a sequence of difficult theorems to prove it. First, any locally finite abelian variety is Hamiltonian by Corollary 2.2 of [17]. Every Hamiltonian variety has the congruence extension property by the main result of [15]. Finally, if $\mathbf{A}$ is a finite member of a locally finite variety which has the congruence extension property, $\mathbf{B}$ is a subalgebra of $\mathbf{A}$, $\alpha \prec \beta$ in $\mathrm{Con}(\mathbf{B})$ and $\mathrm{typ}(\alpha, \beta) \neq \mathbf{2}$, then there are congruences $\alpha^* \prec \beta^*$ on $\mathbf{A}$ with $\alpha \subseteq \alpha^*$ and $\beta^* = \mathrm{Cg}^{\mathbf{A}}(\beta)$ such that $\alpha^*|_B = \alpha$ and $\beta^*|_B = \beta$ and $\mathrm{typ}(\alpha^*, \beta^*) \neq \mathbf{2}$. This follows from Lemma 2.3 and Theorem 2.13 of [13]. It is a consequence of this result that if $\delta \underset{\mathbf{2}}{\sim} \theta$ in $\mathrm{Con}(\mathbf{A})$, then $\delta|_B \underset{\mathbf{2}}{\sim} \theta|_B$ in $\mathrm{Con}(\mathbf{B})$; i.e., $\underset{\mathbf{2}}{\sim}$ is hereditary.

Let $\mathscr{V}_T$ and $\mathscr{V}_{T'}$ be the locally-T and the locally-T' subvarieties respectively. $\mathscr{V}_T$ contains all the finite algebras in $\mathscr{V}$ which have type-set contained in T and $\mathscr{V}_{T'}$ contains all the finite algebras in $\mathscr{V}$ which have type-set contained in T'. $\mathscr{V}_T$ is abelian and locally strongly solvable; so, by Theorem 2.0 of [23], $\mathscr{V}_T$ is strongly abelian. $\mathscr{V}_{T'}$ is affine since it is abelian and $\mathrm{typ}\{\mathscr{V}_{T'}\} = \{\mathbf{2}\}$. A sketch of the argument for this claim is as follows: $\mathrm{typ}\{\mathscr{V}_{T'}\} = \{\mathbf{2}\}$ so, by Exercise 8.8 (2) and Theorem 8.5 of [7], $\mathscr{V}_{T'}$ is congruence modular. But any abelian, congruence modular variety is affine by Corollary 5.9 of [5].

By Corollary 3.11, if $\mathbf{A}$ is a finite algebra in $\mathscr{V}$, then ρ_T and $\rho_{T'}$ are lattice-theoretic complements and $\mathbf{A}/\rho_T \in \mathscr{V}_{T'}$ and $\mathbf{A}/\rho_{T'} \in \mathscr{V}_T$. It is a consequence of this that $\mathscr{V} = \mathscr{V}_T \vee \mathscr{V}_{T'}$. In order to show that this varietal join decomposition of $\mathscr{V}$ is a varietal product decomposition, we must show that $\mathscr{V}_T$ and $\mathscr{V}_{T'}$ are independent. For this we must produce a binary decomposition term $b(x, y)$ such that $\mathscr{V}_T \models b(x, y) \approx x$ and $\mathscr{V}_{T'} \models b(x, y) \approx y$ (see Definition 0.7 of [22]). Let's examine the free algebra $\mathbf{F} = \mathbf{F}_{\mathscr{V}}(2)$. Let $\sigma_T^{\mathbf{F}}$ and $\sigma_{T'}^{\mathbf{F}}$ denote the T-coradical and the T'-coradical of $\mathbf{F}$ respectively. Let $\mathbf{F}_T = \mathbf{F}/\sigma_T^{\mathbf{F}}$ and let $\mathbf{F}_{T'}^{\mathbf{F}} = \mathbf{F}/\sigma_{T'}^{\mathbf{F}}$. Using Corollary 3.11 and the facts that $\rho_T^{\mathbf{F}} = \sigma_{T'}^{\mathbf{F}}$ and $\rho_{T'}^{\mathbf{F}} = \sigma_T^{\mathbf{F}}$, we get that $\mathrm{typ}\{\mathbf{F}_T\} = T$, $\mathrm{typ}\{\mathbf{F}_{T'}\} = T'$ and

$F \leq_{sd} \mathbf{F}_T \times \mathbf{F}_{T'}$ is a subdirect product decomposition. $\mathbf{F}_T$ is the maximal homomorphic image of $\mathbf{F}$ into $\mathscr{V}_T$, so $\mathbf{F}_T = \mathbf{F}_{\mathscr{V}_T}(2)$. A similar statement is true with T' in place of T. Let $\mathbf{G} = \mathbf{F}_T \times \mathbf{F}_{T'}$. We will use π_T and $\pi_{T'}$ to denote the kernels of the corresponding coordinate projections on $\mathbf{G}$ and $\sigma_T^{\mathbf{G}}$ and $\sigma_{T'}^{\mathbf{G}}$ to denote the T-coradical and the T'-coradical of $\mathbf{G}$. Since $\mathbf{G}/\pi_T \cong \mathbf{F}_T$ we must have $\mathrm{typ}\{\pi_T, 1_G\} = T$. In particular, $\sigma_T^{\mathbf{G}} \leq \pi_T$ and similarly $\sigma_{T'}^{\mathbf{G}} \leq \pi_{T'}$. If $\sigma_T^{\mathbf{G}} < \pi_T$, then since $\mathrm{Con}(\mathbf{G})$ is (naturally) isomorphic to the direct product $[\sigma_T^{\mathbf{G}}, 1_G] \times [\sigma_{T'}^{\mathbf{G}}, 1_G]$ (by Corollary 3.11 again), we get $0_G < \pi_T \wedge \sigma_{T'}^{\mathbf{G}} \leq \pi_T \wedge \pi_{T'} = 0_G$. This is impossible, so $\sigma_T^{\mathbf{G}} = \pi_T$; similarly $\sigma_{T'}^{\mathbf{G}} = \pi_{T'}$. In particular, $[\sigma_T^{\mathbf{G}}, 1_G] = [\pi_T, 1_G]$ is isomorphic to $\mathrm{Con}(\mathbf{F}_T)$ which in turn is isomorphic to $[\sigma_T^{\mathbf{F}}, 1_F]$. Similarly, $[\sigma_{T'}^{\mathbf{G}}, 1_G]$ and $[\sigma_{T'}^{\mathbf{F}}, 1_F]$ are isomorphic. Corollary 3.11 yields that $\mathrm{Con}(\mathbf{F}) \cong [\sigma_T^{\mathbf{F}}, 1_F] \times [\sigma_{T'}^{\mathbf{F}}, 1_F]$ is isomorphic to $[\sigma_T^{\mathbf{G}}, 1_G] \times [\sigma_{T'}^{\mathbf{G}}, 1_G] \cong \mathrm{Con}(\mathbf{G})$. Since $\mathscr{V}$ has the CEP the restriction map from $\mathrm{Con}(\mathbf{G})$ to $\mathrm{Con}(\mathbf{F})$ is onto; this map must also be 1-1 because the lattices are finite. We will argue that this forces the natural embedding of $\mathbf{F}$ into $\mathbf{G} = \mathbf{F}_T \times \mathbf{F}_{T'}$ to be an isomorphism.

As we mentioned above, any locally finite abelian variety is Hamiltonian. The image of the embedding of $\mathbf{F}$ into $\mathbf{G}$ is a subuniverse of G, so this set is a congruence block. Say that it is a γ-block for some congruence $\gamma \in \mathrm{Con}(\mathbf{G})$. As $\gamma|_F = 1_F = 1_G|_F$ we can only have $\gamma = 1_G$. This implies that the natural embedding of $\mathbf{F}$ into $\mathbf{G}$ is onto and therefore an isomorphism. Thus, $\mathbf{F} \cong \mathbf{G} = \mathbf{F}_T \times \mathbf{F}_{T'}$. Identifying $\mathbf{F}$ with $\mathbf{G}$, there is an element $f \in F$ such that $x \, \pi_T f \, \pi_{T'} \, y$. Let $b(x, y)$ be a binary term representing f. If $\bar{x}$, $\bar{y}$ and $\bar{f}$ denote the elements x/π_T, y/π_T and f/π_T in $\mathbf{F}_T = \mathbf{F}_{\mathscr{V}_T}(2)$, then $b(\bar{x}, \bar{y}) = \bar{f} = \bar{x}$. From this it follows that $\mathscr{V}_T \models b(x, y) \approx x$. A similar argument proves that $\mathscr{V}_{T'} \models b(x, y) \approx y$. Hence $\mathscr{V} = \mathscr{V}_T \otimes \mathscr{V}_{T'}$. □

Earlier versions of this manuscript were circulated before E. Kiss and M. Valeriote had proved that every locally finite, abelian variety is Hamiltonian. Those early versions contained a weaker result than the one just proved. Although not formulated in this way, the early versions contained arguments showing essentially that a locally finite, abelian variety which is Hamiltonian satisfies Theorem 3.15 (1) if and only if it satisfies (2). After Kiss and Valeriote announced their theorem it was pointed out to us that our result could be improved to the version given here. This suggestion was made to us on different occasions by R. McKenzie, M. Valeriote and one of the referees of this paper.

Call an algebra $\mathbf{A}$ **commutator neutral** or just **neutral** if the commutator in $\mathrm{Con}(\mathbf{A})$ is equal to the meet operation. Thus, $\mathbf{A}$ is neutral if $C(\alpha, \beta; \delta) \to \alpha \wedge \beta \leq \delta$ is true for $\mathrm{Con}(\mathbf{A})$, $\mathbf{A}$ is **herditarily neutral** if every subalgebra of $\mathbf{A}$ is neutral. It is a result of modular commutator theory that a finite set $\mathscr{K}$ of finite algebras contained in a congruence modular variety generates a congruence distributive subvariety if and only if the algebras are hereditarily neutral. The corresponding result for CSM varieties is the next corollary.

COROLLARY 3.16. *Let $\mathscr{U}$ be a CSM variety which omits* $\mathbf{0}$ *and contains the finite set $\mathscr{K}$ of finite algebras. $\mathscr{V} = V(\mathscr{K})$ is a congruence meet-semidistributive subvariety if and only if the algebras in $\mathscr{K}$ are hereditarily neutral.*

Proof. $\mathscr{V}$ is congruence meet-semidistributive if and only if $\mathrm{typ}\{\mathscr{V}\} \subseteq \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$ which is equivalent to the condition that every member of $\mathscr{V}$ is hereditarily neutral (see Theorem 9.10 of [7] for a proof of this). Thus, the forward direction of the corollary is clear. For the other direction we can apply Theorem 3.14. The relation $\underset{\mathbf{3,4,5}}{\sim}$ is a congruence on the congruence lattice of any finite member of $\mathscr{V}$ and the locally-$\{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$ subvariety contains $\mathscr{K}$, so it is all of $\mathscr{V}$. Hence $\mathrm{typ}\{\mathscr{V}\} \subseteq \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$ and $\mathscr{V}$ is congruence meet-semidistributive. □

Example 5.3 shows that the hypothesis that $\mathbf{0} \notin \mathrm{typ}(\mathscr{U}\}$ is a necessary one, for that example describes a finite hereditarily neutral algebra which generates a variety which is CSM although not congruence meet-semidistributive.

The following theorem is a decomposition result for certain locally finite CSM varieties that omit type $\mathbf{0}$.

THEOREM 3.17. *Assume that $\mathscr{V}$ is a locally finite CSM variety that omits type* $\mathbf{0}$. *Assume also that any non-solvable, finite, subdirectly irreducible algebra in $\mathscr{V}$ is hereditarily neutral. Then $\mathscr{V} = \mathscr{V}_2 \vee \mathscr{V}_{3,4,5} = \boldsymbol{P}_s(\mathscr{V}_2 \cup \mathscr{V}_{3,4,5})$, there are binary terms $b_0(x, y)$ and $b_1(x, y)$ and the following conclusions hold:*

(1) *$\mathscr{V}_2$ is locally solvable, congruence permutable and*

$$\mathscr{V}_2 \models b_0(x, y) \approx x, \qquad b_1(x, y) \approx y.$$

(2) *$\mathscr{V}_{3,4,5}$ is congruence meet-semidistributive and*

$$\mathscr{V}_{3,4,5} \models b_0(x, y) \approx b_1(x, y), \qquad b_i(x, y) \approx x.$$

Proof. Notice that $\underset{\mathbf{2}}{\sim}$ and $\underset{\mathbf{3,4,5}}{\sim}$ are congruences on the congruence lattice of any finite algebra in $\mathscr{V}$. Define $\mathscr{V}_2$ to be locally solvable subvariety and $\mathscr{V}_{3,4,5}$ to be the locally-$\{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$ subvariety. $\mathscr{V}_2$ is congruence permutable since $\mathrm{typ}\{\mathscr{V}_2\} \subseteq \{\mathbf{2}\}$ and $\mathscr{V}_{3,4,5}$ is congruence meet-semidistributive since $\mathrm{typ}\{\mathscr{V}_{3,4,5}\} \subseteq \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$. $\mathscr{V}_2 \cap \mathscr{V}_{3,4,5}$ contains only trivial algebras and $\boldsymbol{SP}_u(\mathscr{V}_2 \cup \mathscr{V}_{3,4,5}) = \mathscr{V}_2 \cup \mathscr{V}_{3,4,5}$ contains all of the subdirectly irreducibles in $\mathscr{V}$, so $\mathscr{V} = \mathscr{V}_2 \vee \mathscr{V}_{3,4,5} = \boldsymbol{P}_s(\mathscr{V}_2 \cup \mathscr{V}_{3,4,5})$. Let $T = \{\mathbf{2}\}$ and let $T' = \{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$. Theorem 3.13 and our hypothesis about non-solvable subdirectly irreducibles guarantees that the $\langle T, T'\rangle$ and the $\langle T', T\rangle$-principles hold. Let $\mathbf{F} = \mathbf{F}_{\mathscr{V}}(x, y)$. The argument in the proof of Theorem 3.10 can be used to show that if σ_2 is the $\{\mathbf{2}\}$-coradical of $\mathrm{Cg}^{\mathbf{F}}(x, y)$ and $\sigma_{3,4,5}$ is the $\{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$-coradical of $\mathrm{Cg}^{\mathbf{F}}(x, y)$, then $\sigma_2 \wedge \sigma_{3,4,5} = 0_F$ and $\sigma_2 \vee \sigma_{3,4,5} = \mathrm{Cg}^{\mathbf{F}}(x, y)$. This means that $\sigma_{3,4,5}$ is solvable. Corollary 7.13 of [7] implies that $\mathrm{Cg}^{\mathbf{F}}(x, y) = \sigma_2 \vee \sigma_{3,4,5} = \sigma_2 \circ \sigma_{3,4,5} \circ \sigma_2$; so there exist $b_0, b_1 \in F$ such that

$$x \,\sigma_2\, b_0 \,\sigma_{3,4,5}\, b_1 \,\sigma_2\, y.$$

We also use the symbols $b_0(x, y)$ and $b_1(x, y)$ to denote terms representing b_0 and b_1. Since $(x, b_0), (b_1, y) \in \mathrm{Cg}^{\mathbf{F}}(x, y)$ it follows that $\mathscr{V} \models x \approx b_i(x, x)$. The kernel of the maximal quotient of $\mathbf{F}$ into $\mathscr{V}_2$ is precisely the $\{\mathbf{3}, \mathbf{4}, \mathbf{5}\}$-radical of $\mathbf{F}$ and this (fully invariant) congruence contains (x, b_0) and (b_1, y). A consequence is that $\mathscr{V}_2 \models b_0(x, y) \approx x, b_1(x, y) \approx y$. Similarly, the $\{\mathbf{2}\}$-radical of $\mathbf{F}$ is the kernel of the maximal quotient of $\mathbf{F}$ into $\mathscr{V}_{3,4,5}$. This fully invariant congruence contains (b_0, b_1), so $\mathscr{V}_{3,4,5} \models b_0(x, y) \approx b_1(x, y)$. This finishes the proof. □

We remark that the assumption that $\mathscr{V}$ is semisimple and has the CEP is enough to guarantee our hypothesis on finite non-solvable subdirectly irreducible algebras. (Semisimplicity insures that the non-solvable subdirectly irreducibles are neutral; the CEP insures that this is inherited by subalgebras.) We remark that if $\mathbf{5} \notin \mathrm{typ}\{\mathscr{V}\}$ holds in the above theorem, then $\mathscr{V} = \mathscr{V}_2 \otimes \mathscr{V}_{3,4}$ where $\mathscr{V}_2$ is congruence permutable and $\mathscr{V}_{3,4}$ is congruence distributive. We do not know if one always obtains a varietal product decomposition of $\mathscr{V}$ without the assumption that $\mathbf{5} \notin \mathrm{typ}\{\mathscr{V}\}$. The problem lies in showing that $\mathscr{V}_2$ and $\mathscr{V}_{3,4,5}$ are independent. Proving this is equivalent to showing that one can choose the binary terms in Theorem 3.17 so that $\mathscr{V}_{3,4,5} \models b_0(x, y) \approx y$.

Added in proof: Theorem 2.7 of [24] implies $\mathscr{V}_2$ and $\mathscr{V}_{3,4,5}$ are independent, so $\mathscr{V} = \mathscr{V}_2 \otimes \mathscr{V}_{3,4,5}$ in Theorem 3.17.

4. Congruences on $\mathbf{L}_{\mathscr{V}}$

As we have already mentioned, the two relations $\underset{\mathbf{0,1}}{\sim}$ and $\underset{\mathbf{0,1,2}}{\sim}$ are lattice congruences on the congruence lattice of any finite algebra. In fact, these congruences are heridatary for any locally finite variety, so they can be defined on the congruence lattices of locally finite algebras. In this way we obtain two complete congruences on the congruence lattice of any locally finite algebra in $\mathscr{V}$; e.g., on $\mathrm{Con}(\mathbf{F}_{\mathscr{V}}(\omega))$ and therefore on $\mathbf{L}_{\mathscr{V}}$: the lattice of subvarieties of $\mathscr{V}$. Hence $\underset{\mathbf{0,1}}{\sim}$ and $\underset{\mathbf{0,1,2}}{\sim}$ induce congruences on $\mathbf{L}_{\mathscr{V}}$ for any locally finite $\mathscr{V}$. It turns out that, for *any* set of type labels T, if $\underset{T}{\sim}$ is a congruence on the finite algebras of some locally finite variety $\mathscr{V}$, then $\underset{T}{\sim}$ induces a congruence on the lattice of subvarieties of $\mathscr{V}$.

For the rest of this section we will use the notation $\theta^{\mathbf{A}}_{\mathscr{U}}$ to denote the least congruence $\theta \in \mathrm{Con}(\mathbf{A})$ for which $\mathbf{A}/\theta \in \mathscr{U}$.

DEFINITION 4.1. Assume that $\mathscr{V}$ is a locally finite variety and that $\underset{T}{\sim}$ is a congruence on the congruence lattices of the finite members of $\mathscr{V}$ for some $T \subseteq \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$. We will write $\mathscr{U} \underset{T}{\simeq} \mathscr{W}$ for subvarieties $\mathscr{U}, \mathscr{W} \subseteq \mathscr{V}$ to mean that for each n we have $\theta^{\mathbf{F}}_{\mathscr{U}} \underset{T}{\sim} \theta^{\mathbf{F}}_{\mathscr{W}}$ where $\mathbf{F} = \mathbf{F}_{\mathscr{V}}(n)$.

LEMMA 4.2. *Assume that $\mathscr{V}$ is locally finite, $\mathscr{U}, \mathscr{W} \subseteq \mathscr{V}$ are subvarieties and $\underset{T}{\sim}$ is a congruence on the congruence lattices of the finite members of $\mathscr{V}$. The following conditions are equivalent.*

(1) $\mathscr{U} \underset{T}{\simeq} \mathscr{W}$.

(2) *$\mathscr{U}$ and $\mathscr{W}$ contain the same finite, T-radical-free algebras.*

Proof. Assume that $\mathscr{U} \underset{T}{\simeq} \mathscr{W}$ and that $\mathbf{A} \in \mathscr{U}$ is a finite, T-radical-free algebra. Choose $n < \omega$ large enough so that $\mathbf{F} = \mathbf{F}_{\mathscr{V}}(n)$ has a homomorphism onto $\mathbf{A}$ and let α be the kernel of such a homomorphism. Since $\mathbf{A} \in \mathscr{U}$ we have $\theta^{\mathbf{F}}_{\mathscr{U}} \le \alpha$. Hence $\alpha = \alpha \vee \theta^{\mathbf{F}}_{\mathscr{U}} \underset{T}{\sim} \alpha \vee \theta^{\mathbf{F}}_{\mathscr{W}}$. But the fact that $\mathbf{A}$ is T-radical-free implies that α is its own T-radical, so $\alpha \vee \theta^{\mathbf{F}}_{\mathscr{W}} = \alpha$. This shows that $\mathbf{A} \cong \mathbf{F}/\alpha$ is a homomorphic image of $\mathbf{F}/\theta^{\mathbf{F}}_{\mathscr{W}} \cong \mathbf{F}_{\mathscr{W}}(n)$, so $\mathbf{A} \in \mathscr{W}$. This is enough to show that (1) implies (2).

For the other direction, assume that $\mathscr{U} \underset{T}{\not\simeq} \mathscr{W}$. There is a finite algebra $\mathbf{G}$ which is free relative to $\mathscr{V}$ for which $\theta^{\mathbf{G}}_{\mathscr{U}} \underset{T}{\not\sim} \theta^{\mathbf{G}}_{\mathscr{W}}$. Since $\underset{T}{\sim}$ is a congruence on $\mathrm{Con}(\mathbf{G})$, we may assume that $\theta^{\mathbf{G}}_{\mathscr{W}}$ is not below the T-radical, ρ, of the congruence $\theta^{\mathbf{G}}_{\mathscr{U}}$. But this means that $\mathbf{G}/\rho$ is a finite, T-radical-free algebra which is in $\mathscr{U} - \mathscr{W}$ since $\theta^{\mathbf{G}}_{\mathscr{U}} \le \rho$ and $\theta^{\mathbf{G}}_{\mathscr{W}} \not\le \rho$. Thus, (2) implies (1). □

THEOREM 4.3. *If $\mathscr{V}$ is locally finite and $\underset{T}{\sim}$ is a congruence on the congruence lattices of the finite members of $\mathscr{V}$, then $\underset{T}{\simeq}$ is a congruence on $\mathbf{L}_{\mathscr{V}}$. Furthermore, the quotient lattice $\mathbf{L}_{\mathscr{V}}/\underset{T}{\simeq}$ is both algebraic and dually algebraic and the natural map of $\mathbf{L}_{\mathscr{V}}$ onto $\mathbf{L}_{\mathscr{V}}/\underset{T}{\simeq}$ is complete.*

Proof. Lemma 4.2 makes it clear that $\underset{T}{\simeq}$ is an equivalence relation on $\mathbf{L}_{\mathscr{V}}$ which is compatible with arbitrary meets. Suppose that $\mathscr{U}, \mathscr{W}, \mathscr{S} \subseteq \mathscr{V}$ and $\mathscr{U} \underset{T}{\simeq} \mathscr{W}$. To show that $\mathscr{S} \vee \mathscr{U} \underset{T}{\simeq} \mathscr{S} \vee \mathscr{W}$ take a finite T-radical-free algebra $\mathbf{A} \in \mathscr{S} \vee \mathscr{U}$. We can find algebras $\mathbf{B}_{\mathscr{S}} \in \mathscr{S}, \mathbf{B}_{\mathscr{U}} \in \mathscr{U}$ such that some subdirect product $\mathbf{B} \le_{sd} \mathbf{B}_{\mathscr{S}} \times \mathbf{B}_{\mathscr{U}}$ has a homomorphism onto $\mathbf{A}$; we choose $\mathbf{B}_{\mathscr{S}}$ and $\mathbf{B}_{\mathscr{U}}$ of the smallest possible cardinality. Let $\pi_{\mathscr{S}}$ and $\pi_{\mathscr{U}}$ denote the kernels of the maps of $\mathbf{B}$ onto $\mathbf{B}_{\mathscr{S}}$ and $\mathbf{B}_{\mathscr{U}}$ respectively. Let α denote the kernel of the homomorphism of $\mathbf{B}$ onto $\mathbf{A}$. Let $\rho_{\mathscr{S}}$ denote the T-radical of $\pi_{\mathscr{S}}$ and $\rho_{\mathscr{U}}$ denote the T-radical of $\pi_{\mathscr{U}}$. If $\pi_{\mathscr{U}} < \rho_{\mathscr{U}}$, then the minimality of $\mathbf{B}_{\mathscr{U}}$ forces $\rho_{\mathscr{U}} \wedge \pi_{\mathscr{S}} \nleq \alpha$. But in this case

$$\pi_{\mathscr{U}} \underset{T}{\sim} \rho_{\mathscr{U}} \to 0_B \underset{T}{\sim} \rho_{\mathscr{U}} \wedge \pi_{\mathscr{S}} \to \alpha \underset{T}{\sim} \alpha \vee (\rho_{\mathscr{U}} \wedge \pi_{\mathscr{S}})$$

shows that α is strictly smaller than its T-radical. This is impossible since $\mathbf{B}/\alpha$ is T-radical-free. Thus, $\pi_{\mathscr{U}} = \rho_{\mathscr{U}}$ and $\mathbf{B}_{\mathscr{U}}$ is T-radical-free. Both $\mathscr{U}$ and $\mathscr{W}$ contain the same T-radical-free algebras, so $\mathbf{B}_{\mathscr{U}} \in \mathscr{W}$. This gives us that $\mathbf{B}$ and therefore $\mathbf{A}$ are in $\mathscr{S} \vee \mathscr{W}$. Using symmetry and Lemma 4.2 we conclude that $\mathscr{S} \vee \mathscr{U} \underset{T}{\simeq} \mathscr{S} \vee \mathscr{W}$.

In proving that the natural map of $\mathbf{L}_{\mathscr{V}}$ onto $\mathbf{L}_{\mathscr{V}}/\underset{T}{\simeq}$ is complete it is enough to show that every $\underset{T}{\simeq}$-class has a least and a largest element. It is clear that the class $\mathscr{U}/\underset{T}{\simeq}$ has a least element: it is the subvariety of $\mathscr{U}$ that is generated by the T-radical-free algebras in $\mathscr{U}$. To prove that $\mathscr{U}/\underset{T}{\simeq}$ has a largest element we first prove that this congruence class is closed under unions of chains. If $\mathscr{U}_0 \subseteq \mathscr{U}_1 \subseteq \cdots$ is a chain of subvarieties of $\mathscr{V}$ where each $\mathscr{U}_i$ contains exactly the same finite T-radical-free algebras as $\mathscr{U}$, then $(\bigvee \mathscr{U}_i)_{fin} = \bigcup (\mathscr{U}_i)_{fin}$ since all of the varieties are contained in the locally finite variety $\mathscr{V}$. Therefore the join of this chain contains the same finite T-radical-free algebras as $\mathscr{U}$, i.e. the join of a chain in $\mathscr{U}/\underset{T}{\simeq}$ again lies in $\mathscr{U}/\underset{T}{\simeq}$. Zorn's Lemma shows that every variety in $\mathscr{U}/\underset{T}{\simeq}$ can be extended to a maximal variety in $\mathscr{U}/\underset{T}{\simeq}$. If $\mathscr{S}$ and $\mathscr{W}$ are maximal varieties in $\mathscr{U}/\underset{T}{\simeq}$, then the fact that $\underset{T}{\simeq}$ is a congruence shows that $\mathscr{S} \underset{T}{\simeq} \mathscr{S} \vee \mathscr{W} \underset{T}{\simeq} \mathscr{W}$, so $\mathscr{S} = \mathscr{W}$. This gives us that the class $\mathscr{U}/\underset{T}{\simeq}$ has a largest element.

Now we will show that $\mathbf{L}_{\mathscr{V}}/\underset{T}{\simeq}$ is algebraic. We will apply Lemma 3.4 to establish this. Notice that $\mathbf{L}_{\mathscr{V}}$ is complete and its compact members are precisely the finite generated subvarieties of $\mathscr{V}$. Every locally finite variety is the join of the finitely generated subvarieties that it contains, so $\mathbf{L}_{\mathscr{V}}$ is algebraic. In particular, it is upper continuous. We've shown that the natural map from $\mathbf{L}_{\mathscr{V}}$ to $\mathbf{L}_{\mathscr{V}}/\underset{T}{\simeq}$ is complete. Now we can use Lemma 3.4. If $\mathbf{A}$ is a finite, T-radical-free algebra, then $V(\mathbf{A})$ is a compact member of $\mathbf{L}_{\mathscr{V}}$ since it is a finitely generated variety. Further, a variety of this form must be the least element of its $\underset{T}{\simeq}$-class, since no proper subvariety contains the T-radical-free algebra $\mathbf{A}$. By Lemma 3.4, $V(\mathbf{A})/\underset{T}{\simeq}$ is a compact member of $\mathbf{L}_{\mathscr{V}}/\underset{T}{\simeq}$. Now suppose that $\mathscr{U}$ is an arbitrary member of $\mathbf{L}_{\mathscr{V}}$. Let

$$\mathscr{W} = \bigvee \mathscr{S} \qquad \text{where } \mathscr{S} = \{V(\mathbf{A}) \mid \mathbf{A} \text{ is a finite, } T\text{-radical-free algebra in } \mathscr{U}\}.$$

$\mathscr{W}$ is a subvariety of $\mathscr{U}$ which contains all the finite, T-radical-free algebras in $\mathscr{U}$, hence $\mathscr{U} \underset{T}{\simeq} \mathscr{W}$. Lemma 3.4 shows that $\mathbf{L}_{\mathscr{V}}/\underset{T}{\simeq}$ is algebraic.

To prove that $\mathbf{L}_{\mathscr{V}}/\underset{T}{\simeq}$ is dually algebraic we will use the dual of Lemma 3.4. First observe that $\mathbf{L}_{\mathscr{V}}$ is dually algebraic since it is dually isomorphic to $\mathrm{Con}(\mathbf{F})$ where $\mathbf{F}$ is $\mathbf{F}_{\mathscr{V}}(\omega)$ with all endomorphisms adjoined as unary operations. This implies that $\mathbf{L}_{\mathcal{V}}$ is lower continuous and that the dual of Lemma 3.4 applies. Now suppose that $\mathcal{U} \subseteq \mathcal{V}$ and that $\mathbf{A} \in \mathcal{V} - \mathcal{U}$ is a finite, T-radical-free algebra. The collection of varieties $\mathcal{W} \in [\mathcal{U}, \mathcal{V}]$ which do not contain $\mathbf{A}$ is closed under unions of chains (again we are using the fact that for a chain of subvarieties of $\mathcal{V}$ we have $(\bigvee \mathcal{U}_i)_{fin} = \bigcup (\mathcal{U}_i)_{fin}$). Hence by Zorn's Lemma $\mathcal{U}$ can be extended to a variety $\mathcal{U}^{\mathbf{A}}$ which is maximal in $\mathbf{L}_{\mathcal{V}}$ with respect to not containing $\mathbf{A}$. Every larger subvariety of $\mathcal{V}$ contains $\mathbf{A}$, so $\mathcal{U}^{\mathbf{A}}$ is the largest element of its $\underset{T}{\simeq}$-class. It is also dually compact in $\mathbf{L}_{\mathcal{V}}$. To show this, we must verify that if $X \subseteq L_{\mathcal{V}}$ is a downward-directed set of varieties containing $\mathcal{U}^{\mathbf{A}}$ and $\mathcal{U}^{\mathbf{A}} \notin X$, then $\bigwedge X \nsubseteq \mathcal{U}^{\mathbf{A}}$. Suppose that X is a downward-directed set of varieties, each properly containing $\mathcal{U}^{\mathbf{A}}$. Each member of X contains $\mathbf{A}$, so $\mathbf{A} \in \bigwedge X$ and $\bigwedge X \nsubseteq \mathcal{U}^{\mathbf{A}}$. By the dual of Lemma 3.4, $\mathcal{U}^{\mathbf{a}}/\underset{T}{\simeq}$ is dually compact in $\mathbf{L}_{\mathcal{V}}/\underset{T}{\simeq}$. Further, if $\mathcal{U} \in \mathbf{L}_{\mathcal{V}}$ and $\mathcal{W}$ is the intersection of all varieties of the form $\mathcal{U}^{\mathbf{A}}$ where $\mathbf{A}$ is a finite, T-radical-free algebra not in $\mathcal{U}$, then $\mathcal{W}$ is a subvariety of $\mathcal{V}$ extending $\mathcal{U}$ which has no T-radical-free algebras not already in $\mathcal{U}$. Thus, $\mathcal{W} \underset{T}{\simeq} \mathcal{U}$ and so $\mathbf{L}_{\mathcal{V}}/\underset{T}{\simeq}$ dually algebraic. □

THEOREM 4.4. *If $\mathcal{V}$ is a locally finite CSM variety, then the following hold.*

(1) $\mathbf{L}_{\mathcal{V}}/\underset{\mathbf{0},\mathbf{2},\mathbf{5}}{\simeq}$ *is a distributive lattice.*
(2) $\mathbf{L}_{\mathcal{V}}/\underset{\mathbf{0},\mathbf{5}}{\simeq}$ *is a modular lattice.*
(3) $\mathbf{L}_{\mathcal{V}}/\underset{\mathbf{0},\mathbf{2}}{\simeq}$ *is a join-semidistributive lattice.*
(4) $\mathbf{L}_{\mathcal{V}}/\underset{\mathbf{0}}{\simeq}$ *is a subdirect product of a join-semidistributive lattice and a modular lattice.*

Proof. In $\mathrm{Con}(\mathbf{L}_{\mathcal{V}})$ we have $\underset{T}{\simeq} \cap \underset{T'}{\simeq} = \underset{T \cap T'}{\simeq}$ and $\underset{T}{\simeq} \vee \underset{T'}{\simeq} \subseteq \underset{T \cup T'}{\simeq}$ for any T and T' as long as $\underset{T}{\simeq}$ and $\underset{T'}{\simeq}$ are congruences. This fact applied when $T = \{\mathbf{0}, \mathbf{5}\}$ and $T' = \{\mathbf{0}, \mathbf{2}\}$ shows that we only need to verify (2) and (3). To prove (2) assume that $\mathbf{L}_{\mathcal{V}}/\underset{\mathbf{0},\mathbf{5}}{\simeq}$ is nonmodular. That is, assume that there are $\mathcal{U}$, $\mathcal{W}$ and $\mathcal{S}$ such that $\mathcal{U}/\underset{\mathbf{0},\mathbf{5}}{\simeq} \leq \mathcal{W}/\underset{\mathbf{0},\mathbf{5}}{\simeq}$, $(\mathcal{S} \vee \mathcal{U})/\underset{\mathbf{0},\mathbf{5}}{\simeq} = (\mathcal{S} \vee \mathcal{W})/\underset{\mathbf{0},\mathbf{5}}{\simeq}$ and $(\mathcal{S} \cap \mathcal{U})/\underset{\mathbf{0},\mathbf{5}}{\simeq} = (\mathcal{S} \cap \mathcal{W}/\underset{\mathbf{0},\mathbf{5}}{\simeq}$ but $\mathcal{U} \underset{\mathbf{0},\mathbf{5}}{\not\simeq} \mathcal{W}$. Using the projectivity of $\mathbf{N}_5$, we may assume that $\mathcal{U} \subseteq \mathcal{W}$, $\mathcal{S} \vee \mathcal{U} = \mathcal{S} \vee \mathcal{W}$ and $\mathcal{S} \wedge \mathcal{U} = \mathcal{S} \wedge \mathcal{W}$. Now $\mathcal{U} \underset{\mathbf{0},\mathbf{5}}{\not\simeq} \mathcal{W}$, so there is a finite $\{\mathbf{0}, \mathbf{5}\}$-radical-free algebra $\mathbf{A} \in \mathcal{W} - \mathcal{U}$. Since $\mathcal{W} \subseteq \mathcal{S} \vee \mathcal{U}$ we can find finite algebras $\mathbf{B} \in \mathcal{S}$ and $\mathbf{C} \in \mathcal{U}$ such that some subdirect product $\mathbf{D} \leq_{sd} \mathbf{B} \times \mathbf{C}$ has a homomorphism onto $\mathbf{A}$. Let $n = |D|$ and let $\mathbf{F}$ denote $\mathbf{F}_{\mathcal{V}}(v)$. $\mathbf{F}$ has a homomorphism onto $\mathbf{D}$ and therefore onto $\mathbf{A}$. Let $\delta \leq \alpha \in \mathrm{Con}(\mathbf{F})$ denote kernels of a homomorphism of $\mathbf{F}$ onto $\mathbf{D}$ and $\mathbf{A}$ respectively. Consider the congruences $\theta^{\mathbf{F}}_{\mathcal{U}}$, $\theta^{\mathbf{F}}_{\mathcal{W}}$ and $\theta^{\mathbf{F}}_{\mathcal{S}}$. Of course we have $\theta^{\mathbf{F}}_{\mathcal{W}} < \theta^{\mathbf{F}}_{\mathcal{U}}$ and $\theta^{\mathbf{F}}_{\mathcal{W}} \leq \alpha$. Also, we have

$$\theta^{\mathbf{F}}_{\mathcal{S}} \vee \theta^{\mathbf{F}}_{\mathcal{W}} = \theta^{\mathbf{F}}_{\mathcal{S} \cap \mathcal{W}} = \theta^{\mathbf{F}}_{\mathcal{S} \cap \mathcal{U}} = \theta^{\mathbf{F}}_{\mathcal{S}} \vee \theta^{\mathbf{F}}_{\mathcal{U}}.$$

Let $\beta = \theta^{\mathbf{F}}_{\mathcal{S}}$ and $\gamma = \theta^{\mathbf{F}}_{\mathcal{U}}$. Since $\mathbf{B} \in \mathcal{S}$ and $\mathbf{C} \in \mathcal{U}$ we get that $\beta \wedge \gamma \leq \delta \leq \alpha$. Now let

$\psi = (\beta \wedge \gamma) \vee \theta^{\mathbf{F}}_{\mathscr{W}}$. Certainly $\psi \leq \alpha$ and $\psi \leq \gamma$ although $\gamma \nleq \alpha$. This shows that $\psi < \gamma$. The previous displayed equation implies that $\beta \vee \psi = \beta \vee \gamma$. Since $\psi < \gamma$, $\beta \wedge \psi = \beta \wedge \gamma \wedge ((\beta \wedge \gamma) \vee \theta^{\mathbf{F}}_{\mathscr{W}}) = \beta \wedge \gamma$. Thus, $\psi \underset{\mathbf{0,5}}{\sim} \gamma$ by Theorem 2.7. From this we get $\alpha \vee \gamma \underset{\mathbf{0,5}}{\sim} \alpha \vee \psi = \alpha$. But then $(\alpha \vee \gamma)/\alpha$ is a nonzero congruence on $\mathbf{F}/\alpha \cong \mathbf{A}$ which is contained in the $\{\mathbf{0}, \mathbf{5}\}$-radical of $\mathbf{A}$. Since $\mathbf{A}$ is $\{\mathbf{0}, \mathbf{5}\}$-radical-free this is impossible. This contradiction proves (2).

If $\mathbf{F} = \mathbf{F}_{\mathscr{V}}(\omega)$, the complete congruence $\overset{s}{\sim}$ of [7] on Con(**F**) induces a complete congruence $\overset{s}{\simeq}$ on $\mathbf{L}_{\mathscr{V}}$:

$$\mathscr{U} \overset{s}{\simeq} \mathscr{W} \quad \text{if and only if } \theta^{\mathbf{F}}_{\mathscr{U}} \overset{s}{\sim} \theta^{\mathbf{F}}_{\mathscr{W}}.$$

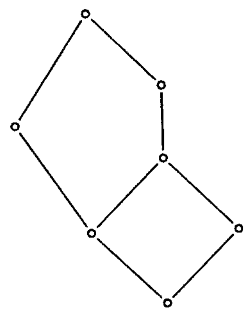
For an equivalent way of defining this congruence notice that the restriction of $\overset{s}{\sim}$ on Con(**F**) to the sublattice of fully invariant congruences on **F** is a complete congruence. The proof of this requires only that the inclusion map of the lattice of fully invariant congruences into Con(**F**) is a complete embedding. The natural dual isomorphism between $\mathbf{L}_{\mathscr{V}}$ and the lattice of fully invariant congruences on **F** allows us to transfer this complete congruence to a complete congruence on $\mathbf{L}_{\mathscr{V}}$; this congruence on $\mathbf{L}_{\mathscr{V}}$ is just $\overset{s}{\simeq}$ as we have defined it. An argument similar to the proof of Lemma 4.2 shows that $\mathscr{U} \overset{s}{\simeq} \mathscr{W}$ if and only if $\mathscr{U}$ and $\mathscr{W}$ contain the same countably generated algebras which have trivial solvable radical. In particular, $\mathscr{U}$ and $\mathscr{W}$ contain the same finite, $\{\mathbf{0}, \mathbf{2}\}$-radical-free algebras. This shows that $\overset{s}{\simeq} \subseteq \underset{\mathbf{0,2}}{\simeq}$. Now $\mathbf{L}_{\mathscr{V}}/\overset{s}{\simeq}$ is dually embedded into $\mathrm{Con}(\mathbf{F})/\overset{s}{\sim}$ which, by Theorem 7.7 of [7], is meet-semidistributive. Hence $\mathbf{L}_{\mathscr{V}}/\overset{s}{\simeq}$ is join-semidistributive. By (an infinitary version of) the second isomorphism theorem we have that $\mathbf{L}_{\mathscr{V}}/\underset{\mathbf{0,2}}{\simeq}$ is a complete homomorphic image of $\mathbf{L}_{\mathscr{V}}/\overset{s}{\simeq}$. But the class of complete, join-semidistributive lattices is closed under complete homomorphisms, so $\mathbf{L}_{\mathscr{V}}/\underset{\mathbf{0,2}}{\simeq}$ is join-semidistributive. □

THEOREM 4.5. *Assume that $\mathscr{V}$ is a locally finite CSM variety and that $\mathscr{U} \subseteq \mathscr{V}$. If $\mathbf{A} \in \mathscr{V}$ is a finite $\{\mathbf{0}, \mathbf{5}\}$-radical-free algebra, then in $\mathbf{L}_{\mathscr{V}}$*

$$\mathscr{U} \prec \mathscr{U} \vee V(\mathbf{A}) \rightarrow \mathscr{U} \cap V(\mathbf{A}) \prec V(\mathbf{A}).$$

Proof. We certainly have $\mathscr{U} \cap V(\mathbf{A}) < V(\mathbf{A})$ since $\mathbf{A} \notin \mathscr{U}$. Choose any $\mathscr{W}$ such that $\mathscr{U} \cap V(\mathbf{A}) < \mathscr{W} \leq V(\mathbf{A})$. Theorem 4.4 (2) proves that $\mathscr{W} \underset{\mathbf{0,5}}{\simeq} V(\mathbf{A})$. Since $\mathbf{A}$ is $\{\mathbf{0}, \mathbf{5}\}$-radical-free, Lemma 4.2 proves that $\mathbf{A} \in \mathscr{W}$, so $\mathscr{W} = V(\mathbf{A})$. This shows that $\mathscr{U} \cap V(\mathbf{A}) \prec V(\mathbf{A})$. □

EXAMPLE 4.6. Theorem 4.5 shows that a certain amount of dual semimodularity must hold in $\mathbf{L}_{\mathscr{V}}$ when $\mathscr{V}$ is a locally finite CSM variety. There *is* a natural



$\mathbf{L}_{\mathcal{U}}$

Figure 3

dual embedding of $\mathbf{L}_{\mathscr{V}}$ into the semimodular lattice $\mathrm{Con}(\mathbf{F}_{\mathscr{V}}(\omega))$ but, since semimodularity is not usually inherited by sublattices, we should not expect $\mathbf{L}_{\mathscr{V}}$ to be dually semimodular. To construct an example of a CSM variety whose subvariety lattice is not dually semimodular, let $\mathbf{R}$ denote the 2-element semigroup satisfying the equation $xy \approx y$. Let $\mathbf{S}$ denote the 2-element semilattice. $\mathrm{Con}(\mathbf{R} \times \mathbf{S})$ is isomorphic to the 5-element non-(semi)modular lattice, $\mathbf{N}_5$. Let M be the monoid obtained by adjoining a unit element to $\mathbf{R} \times \mathbf{S}$. M satisfies the equations $x^2 \approx x$ and $xy \approx yxy$ so, by Theorem 3.18 of [2], the unary variety, $\mathscr{U}$, of all left M-sets is CSM. However, $\mathbf{L}_{\mathscr{U}}$ is dually isomorphic to $\mathrm{Con}(M)$ which contains an interval isomorphic to $\mathrm{Con}(\mathbf{R} \times \mathbf{S})$. Thus $\mathbf{L}_{\mathscr{U}}$ is not dually semimodular. See Figure 3.

5. Subdirect products

In this section we compare the type-set of a subdirect product of algebras in a CSM variety to the type-sets of the factors. We also investigate the existence of irredundant subdirect product decompositions in CSM varieties.

THEOREM 5.1. *If* $\mathbf{A}$ *is a finite algebra in a CSM variety and* $\mathbf{A} \leq_{sd} \prod_{i \in I} \mathbf{A}_i$ *is a subdirect representation of* $\mathbf{A}$, *then*

$$\mathrm{typ}\{\mathbf{A}\} \subseteq \left(\bigcup_{i \in I} \mathrm{typ}\{\mathbf{A}_i\} \right) \cup \{\mathbf{0}, \mathbf{5}\}.$$

Proof. Let $\pi_i \in \mathrm{Con}(\mathbf{A})$ denote the restriction of the kernel of the ith projection on $\prod_{i \in I} \mathbf{A}_i$ to $\mathbf{A}$. Let $T = (\bigcup_{i \in I} \mathrm{typ}\{\mathbf{A}_i\}) \cup \{\mathbf{0}, \mathbf{5}\}$. By Theorem 3.2, $\underset{T}{\sim}$ is a congruence on $\mathrm{Con}(\mathbf{A})$. Clearly $\pi_i \underset{T}{\sim} 1_A$ for all i, so $1_A \underset{T}{\sim} (\bigwedge_{i \in I} \pi_i) = 0_A$. Hence $\mathrm{typ}\{\mathbf{A}\} \subseteq T$. □

COROLLARY 5.2. *If $\mathscr{K}$ is a finite set of finite, similar algebras which generate the CSM variety $\mathscr{V}$, then*

$$\mathrm{typ}\{\mathscr{V}\} \subseteq \mathrm{typ}\{\mathbf{S}(\mathscr{K})\} \cup \mathrm{typ}\{\mathbf{F}_{\mathscr{V}}(2)\} \cup \{\mathbf{5}\}. \qquad (*)$$

Proof. The type-set of $\mathscr{V}$ equals $\bigcup_{n<\omega} \mathrm{typ}\{\mathbf{F}_{\mathscr{V}}(n)\}$ and each free algebra, $\mathbf{F}_{\mathscr{V}}(n)$, is a finite subdirect product of members of $\mathbf{S}(\mathscr{K})$. Theorem 5.1 implies that

$$\mathrm{typ}\{\mathscr{V}\} \subseteq \mathrm{typ}\{\mathbf{S}(\mathscr{K})\} \cup \{\mathbf{0}, \mathbf{5}\}.$$

However, if there is a finite algebra $\mathbf{A} \in \mathscr{V}$ with a minimal nonzero congruence, α, where $\mathrm{typ}(0_A, \alpha) = \mathbf{0}$, then the subalgebra $\mathbf{B} \leq \mathbf{A}$ generated by a pair of elements $a, b \in A$ where $(a, b) \in \alpha - 0_A$ is 2-generated and has a nonzero strongly abelian congruence, $\mathrm{Cg}^{\mathbf{B}}(a, b)$, which must be of type $\mathbf{0}$. Hence, if $\mathbf{0} \in \mathrm{typ}\{\mathscr{V}\}$, then $\mathbf{0} \in \mathrm{typ}\{\mathbf{B}\} \subseteq \mathrm{typ}\{\mathbf{F}_{\mathscr{V}}(2)\}$. Thus, $\mathrm{typ}\{\mathscr{V}\} \subseteq \mathrm{typ}\{\mathbf{S}(\mathscr{K})\} \cup \mathrm{typ}\{\mathbf{F}_{\mathscr{V}}(2)\} \cup \{\mathbf{5}\}$ holds. □

If $\mathbf{0} \notin \mathrm{typ}\{\mathscr{V}\}$ in Corollary 5.2, then $\mathbf{5} \in \mathrm{typ}\{\mathscr{V}\}$ if and only if $\mathbf{5} \in \mathrm{typ}\{\mathbf{F}_{\mathscr{V}}(4)\}$. This follows from Theorem 2.7 and A. Day's characterization of congruence modular varieties given in [4]. We conjecture that Corollary 5.2 can be improved so that (*) reads $\mathrm{typ}\{\mathscr{V}\} \subseteq \mathrm{typ}\{\mathbf{S}(\mathscr{K})\} \cup \mathrm{typ}\{\mathbf{F}_{\mathscr{V}}(4)\}$. See Corollary 7.6 for a result of this kind.

The proof of Corollary 5.2 shows that if $\mathbf{0}, \mathbf{5} \notin \mathrm{typ}\{\mathscr{V}\}$, i.e., $\mathscr{V}$ is congruence modular, then $\mathrm{typ}\{\mathscr{V}\} \subseteq \mathrm{typ}\{\mathbf{S}(\mathscr{K})\}$. This stronger result does not hold for other CSM varieties as the next example shows.

EXAMPLE 5.3. Let $\mathbf{A}$ be the algebra on $\{a, b, c\}$ with two binary operations, f, g, defined by $f^{\mathbf{A}}(a, a) = a$ and $f^{\mathbf{A}}(x, y) = c$ otherwise, $g^{\mathbf{A}}(b, b) = b$ and $g^{\mathbf{A}}(x, y) = c$ otherwise. We leave it to the reader to check that $\mathrm{typ}\{\mathbf{S}(\mathbf{A})\} = \{\mathbf{5}\}$. Every term operation of $\mathscr{V} = \mathbf{V}(\mathbf{A})$ which is composed from both f and g depends on none of its variables in $\mathscr{V}$. Any two such terms are equivalent; they are constant and they name the same element. Now, if $X = A^2 - \{(a, b)\}$, then X is a block of a congruence α on $\mathbf{A}^2$ since (a, b) is not in the range of $f^{\mathbf{A}^2}$ or $g^{\mathbf{A}^2}$. The quotient $\mathbf{N} = \mathbf{A}^2/\alpha$ consists of two elements $0 = X/\alpha$ and $1 = (a, b)/\alpha$ and $f^{\mathbf{N}}(x, y) = 0 = g^{\mathbf{N}}(x, y)$ for all $x, y \in N$. This shows that $\mathrm{typ}\{\mathbf{N}\} = \{\mathbf{0}\}$, so $\mathbf{0} \in \mathrm{typ}\{\mathscr{V}\} \nsubseteq \mathrm{typ}\{\mathbf{S}(\mathbf{A})\} = \{\mathbf{5}\}$. What remains to show is that $\mathscr{V}$ is CSM. We leave it to the reader to apply Theorem 3.13 of [2] to $\mathscr{K} = \{\mathbf{A}, \mathbf{S}_{\langle 2,2\rangle}\}$ and show that reg $\mathscr{V}$ is a regular, polynomially orderable variety. It follows from this and Theorem 3.11 of [2] that $\mathscr{V}$ is CSM.

Corollary 3.5 of [2] implies that the regularization of any congruence modular variety fails to be CSM, since congruence modular varieties are strongly irregular.

Thus the join of two CSM varieties need not be CSM (reg $\mathscr{V} = \mathscr{V} \vee V(\mathbf{S}_\tau)$). The variety described in Example 5.3 is CSM and equal to a join of two subvarieties which are congruence meet-semidistributive. ($\mathbf{A}$ of that example is a subdirect product of subalgebras $\mathbf{P}_a = \mathrm{Sg}^{\mathbf{A}}(\{a, c\})$ and $\mathbf{P}_b = \mathrm{Sg}^{\mathbf{A}}(\{b, c\})$, so $V(\mathbf{A}) = V(\mathbf{P}_a) \vee V(\mathbf{P}_b)$. Each of $V(\mathbf{P}_a)$ and $V(\mathbf{P}_b)$ is equivalent to the variety of semilattices with zero, so they are congruence meet-semidistributive.) However, $V(\mathbf{A})$ is not congruence meet-semidistributive since $\mathbf{0} \in \mathrm{typ}\{V(\mathbf{A})\}$. Thus, the join of two congruence meet-semidistributive subvarieties of a CSM variety need not be congruence meet-semidistributive. We do not know if the join of two congruence modular (or congruence distributive) subvarieties of a CSM variety must be congruence modular (congruence distributive), so we pose this as a problem.

PROBLEM 4. Is it true that the join of two congruence modular (congruence distributive) subvarieties of a CSM variety is congruence modular (congruence distributive)?

DEFINITION 5.4. If a is an element of the complete lattice $\mathbf{L}$, then a **decomposition** of a is a representation, $a = \bigwedge S$, of a as the meet of a set S of completely meet-irreducible elements of $\mathbf{L}$. A decomposition $a = \bigwedge S$ is **irredundant** if for all $s \in S$ we have $a \neq \bigwedge S - \{s\}$. A decomposition $a = \bigwedge S$ is **strongly irredundant** if whenever $s \in S$ and s^* is the unique upper cover of s we have $a \neq s^* \wedge \bigwedge(S - \{s\})$. A complete lattice has **replaceable (strongly) irredundant decompositions** if every element has a (strongly) irredundant decomposition and whenever $a = \bigwedge S = \bigwedge S'$ are (strongly) irredundant decompositions of a and $s \in S$ there is an $s' \in S'$ such that $a = s' \wedge \bigwedge(S - \{s\})$ is a (strongly) irredundant decomposition of a.

Morgan Ward was the first to prove that if every element of a lattice has a unique irredundant decomposition and every decomposition contains an irredundant decomposition, then the lattice is semimodular. Conversely, one finds that semimodularity seems to be a natural hypothesis for many existence and uniqueness results concerning lattice decompositions, see chapters 6 and 7 of [3]. Decompositions of elements in a congruence lattice correspond to representations of algebras as a subdirect product of subdirectly irreducible factors, so it is especially germane to our investigation of algebras with semimodular congruence lattices to examine the results of Decomposition Theory and to see how they apply to congruence lattices. We make the following definitions.

DEFINITION 5.5. Assume that $\mathbf{A} \leq_{sd} \prod_{i \in I} \mathbf{A}_i$ is a representation of $\mathbf{A}$ as a subdirect product of subdirectly irreducibles and that $\pi_i \in \mathrm{Con}(\mathbf{A})$ is the kernel of the projection onto the ith factor. Let $\Pi = \{\pi_i : i \in I\}$. We say that the representation $\mathbf{A} \leq_{sd} \prod_{i \in I} \mathbf{A}_i$ is **irredundant (strongly irredundant)** if Π is an irredundant (strongly irredundant) decomposition of 0_A in $\mathrm{Con}(\mathbf{A})$. We say that a class $\mathscr{K}$ of algebras has **replaceable (strongly) irredundant subdirect representations** if the members of $\mathscr{K}$ have congruence lattices which have replaceable (strongly) irredundant decompositions.

THEOREM 5.6 (from Theorem 6.9 of [3]). *If* $\mathbf{L}$ *is a semimodular, atomic, algebraic lattice, then* 0_L *has an irredundant decomposition.* □

It is easy to show that any irredundant decomposition of 0_L in a semimodular, atomic lattice $\mathbf{L}$ must in fact be strongly irredundant, so the statement of this theorem could be strengthened slightly.

COROLLARY 5.7. *If* $\mathrm{Con}(\mathbf{A})$ *is semimodular and atomic, then* $\mathbf{A}$ *has a strongly irredundant subdirect representation.* □

On the other hand, the existence of strongly irredundant subdirect representations for all algebras in a variety is very restrictive.

THEOREM 5.8. *If* $\mathscr{V}$ *is a locally finite variety in which every algebra has a strongly irredundant subdirect representation, then* $\mathscr{V}$ *must be locally solvable. Further, if* $\mathscr{V}$ *is CSM, then any minimal congruence on any* $\mathbf{A} \in \mathscr{V}$ *must be central or locally strongly solvable.*

Proof. Assume that every algebra in $\mathscr{V}$ has a strongly irredundant subdirect representation. If $\mathscr{V}$ is not locally solvable, then we can find a finite subdirectly irreducible algebra $\mathbf{A} \in \mathscr{V}$ within a nonabelian monolith. Let T be a compact, Hausdorff, totally disconnected topological space which has no isolated points. T must be infinite. Giving A the discrete topology, let $\mathbf{B}$ be the subdirect power of $\mathbf{A}$ consisting of the continuous functions from T to A where the operations act coordinatewise on $\mathbf{B} < \mathbf{A}^T$. For a subset $X \subseteq T$ let π_X be the congruence on $\mathbf{B}$ consisting of pairs (f, g) for which $f(x) = g(x)$ for all $x \in X$. If $X = \{x\}$ we will just write π_x. Now let Λ be a set of completely meet-irreducible congruences which is a strongly irredundant decomposition of θ_B. We will use the notation λ^* to denote the unique upper cover of $\lambda \in \Lambda$. If δ is a locally solvable congruence on $\mathbf{B}$, then $(\delta \vee \pi_t)/\pi_t$ is a solvable congruence on $\mathbf{B}/\pi_t \cong \mathbf{A}$ for any $t \in T$. But $\mathbf{A}$ has no solvable congruence, so $\delta \leq \pi_t$ for all t; hence $\delta = 0_B$. From this we can conclude that

$$\mathrm{Con}(\mathbf{B}) \models x \wedge y = 0 = x \wedge z \to x \wedge (y \vee z) = 0.$$

This follows from the fact that **B** has no locally solvable congruence and $\mathrm{Con}(\mathbf{B})/\overset{s}{\sim}$ is meet-semidistributive. For some $\lambda \in \Lambda$ let $\alpha = \lambda^* \wedge \bigwedge(\Lambda - \{\lambda\})$. Since Λ is a strongly irredundant decomposition of 0_B we have $\alpha > 0_B$. Now if $S \subseteq T$ is a clopen subset let $\alpha_S = \alpha \wedge \pi_S$ and let $\alpha_{T-S} = \alpha \wedge \pi_{T-S}$. We have

$$\alpha_S \wedge \alpha_{T-S} \subseteq \pi_S \wedge \pi_{T-S} = 0_B \quad \text{and} \quad \alpha_S \wedge \lambda \subseteq \alpha \wedge \lambda = 0_B,$$

so we must have $\alpha_S \wedge (\alpha_{T-S} \vee \lambda) = 0_B$. Since $0_B \leq \alpha_S \leq \lambda^*$ we must have that either $\alpha_S = 0_B$ or else $\alpha_{T-S} \leq \lambda$. The latter possibility leads to $\alpha_{T-S} \leq \alpha \wedge \lambda = 0_B$, so either $\alpha_S = 0_B$ or $\alpha_{T-S} = 0_B$. Assume that $0_B = \alpha_S = \alpha \wedge \pi_S$. Since we also have $\alpha \wedge \lambda = 0_B$ we even get $\alpha \wedge (\lambda \vee \pi_S) = 0_B$. We know that $0_B < \alpha \leq \lambda^*$ so this can only mean that $\pi_S \leq \lambda$. Similarly, if we assume that $\alpha_{T-S} = 0_B$ we get $\pi_{T-S} \leq \lambda$. Therefore, given $\lambda \in \Lambda$ and a clopen $S \subseteq T$, we have either $\pi_S \leq \lambda$ or $\pi_{T-S} \leq \lambda$.

If S and S' are clopen and $\pi_S \leq \lambda$ and $\pi_{S'} \leq \lambda$, then $\pi_{S \cap S'} = \pi_S \vee \pi_{S'} \leq \lambda < 1_B$, so $S \cap S'$ is nonempty. This shows that the collection of clopen sets $\{S \mid \pi_S \leq \lambda\}$ has the finite intersection property. T is compact, so the intersection, $L = \bigcap_{\pi_S \leq \lambda} S$, of these clopen sets is nonempty. If $x, y \in T$ are distinct we can find a clopen set S containing, say, x and not y and exactly one of π_S and π_{T-S} is $\leq \lambda$. Thus, at most one of the elements x, y can be in L. Hence L consists of exactly one element, l. We claim that $\pi_l = \bigvee_{\pi_S \leq \lambda} \pi_S$. Since $l \in S$ whenever S is a clopen set for which $\pi_S \leq \lambda$, the inclusion $\pi_l \supseteq \bigvee_{\pi_S \leq \lambda} \pi_S$ is clear. Now suppose that $(f, g) \in \pi_l$, or equivalently that $f(l) = g(l)$. Since the set $E \subseteq T$ on which any pair of functions agree is a clopen set, there is a clopen set S containing l such that $(f, g) \in \pi_S$. Since $l \notin T - S$ we cannot have $\pi_{T-S} \leq \lambda$, so $\pi_S \leq \lambda$. This shows that $\pi_l \subseteq \bigvee_{\pi_S \leq \lambda} \pi_S$.

To summarize what we have shown up to this point, if $\lambda \in \Lambda$ there is an $l \in T$ such that $\pi_l \leq \lambda$. Since $\mathbf{B}/\pi_l \cong \mathbf{A}$ each π_l is completely meet-irreducible and has a unique upper cover, π_l^*. For every $\lambda \in \Lambda$ there is an $l \in T$ such that $\lambda = \pi_l$ or $\pi_l^* \leq \lambda$. Let $D = \{l \in T \mid \pi_l = \lambda \text{ for some } \lambda \in \Lambda\}$ and let G be a clopen set disjoint from D. Let $\Lambda' = \{\lambda \in \Lambda \mid \pi_1 < \lambda \text{ and } l \in G\}$ and let $\Lambda'' = \Lambda - \Lambda'$. We have

$$0_B = \bigwedge \Lambda = \left(\bigwedge \Lambda'\right) \wedge \left(\bigwedge \Lambda''\right) \geq \left(\bigwedge_{l \in G} \pi_l^*\right) \wedge \left(\bigwedge_{l \notin G} \pi_l\right) = \theta.$$

But θ is *not* zero if G is any *nonempty* clopen set. If a and b are distinct elements of A which are related by the monolith of **A**, then $(h, k) \in \theta - 0_B$ where $h(x) = a$ for all x and $k(x) = b$ if $x \in G$ and a otherwise. This argument shows that any clopen set disjoint from D is empty and, since T has a basis of clopen sets, this implies that D is dense in T. Now, $\{\pi_l \mid l \in D\} \subseteq \Lambda$ and, since D is dense in T, $\bigwedge_{l \in D} \pi_l = 0_B$. Since $\bigwedge \Lambda$ is irredundant, $\Lambda = \{\pi_l \mid l \in D\}$. If $d \in D$, then $D - \{d\}$ is also dense in T since d is not an isolated point of T (T has no isolated points), so

$\bigwedge(\Lambda - \{\pi_d\}) = 0_B$. This contradicts the irredundance of Λ. Our conclusion is that $\mathscr{V}$ contains no finite, subdirectly irreducible $\mathbf{A}$ with nonabelian monolith. This proves that $\mathscr{V}$ is locally solvable.

To finish the proof we must show that if $\mathscr{V}$ is also CSM, then every minimal congruence on any $\mathbf{A} \in \mathscr{V}$ is either central or locally strongly solvable. Assume that $\alpha \in \mathrm{Con}(\mathbf{A})$ is a minimal congruence on $\mathbf{A} \in \mathscr{V}$ and that α is neither strongly solvable nor central. Let $\beta \in \mathrm{Con}(\mathbf{A})$ be a congruence maximal for the property that $\alpha \wedge \beta = 0_A$. β is completely meet-irreducible with unique upper cover $\beta^* = \alpha \vee \beta$. Now $\beta \overset{ss}{\not\sim} \beta^*$, for otherwise $0_A = \alpha \wedge \beta \overset{ss}{\sim} \alpha \wedge \beta^* = \alpha$ which is false. Since the implication

$$C(\gamma, \delta; \theta) \to C(\gamma \wedge \lambda, \delta; \theta \wedge \lambda)$$

is valid for any congruences γ, δ, θ and λ, we cannot have $\mathrm{C}(\beta^*, 1_A; \beta)$. This would mean that $\mathrm{C}(\beta^* \wedge \alpha, 1_A; \beta \wedge \alpha) = \mathrm{C}(\alpha, 1_A; 0_A)$ holds which is contrary to our assumption. Thus, $\mathbf{A}/\beta$ is a subdirectly irreducible algebra in $\mathscr{V}$ whose monolith is neither locally strongly solvable nor central. Replacing $\mathbf{A}$ by $\mathbf{A}/\beta$ and α by β^*/β and changing notation we may assume that $\mathbf{A}$ is subdirectly irreducible with monolith α.

Construct an algebra $\mathbf{B}$ as in the earlier part of the proof: let T be a compact, Hausdorff, totally disconnected topological space which has no isolated points and set $\mathbf{B}$ equal to the subdirect power of $\mathbf{A}$ consisting of the continuous functions from T to the discrete space A where the operations act coordinatewise on $\mathbf{B} \leq \mathbf{A}^T$. Let π_i denote the projection congruence onto the ith factor and π_i^* its upper cover. If $\beta \in \mathrm{Con}(\mathbf{B})$ is a nonzero congruence, then for some i we must have $\beta \nleq \pi_i$. For this i we have $\pi_i = \pi_i \vee 0_B \overset{ss}{\not\sim} \pi_i^* \leq \pi_i \vee \beta$. Therefore β cannot be locally strongly solvable and so $\mathbf{B}$ has no nonzero locally strongly solvable congruences. Now suppose that Λ is a strongly irredundant decomposition of 0_A. If $\lambda \in \Lambda$ has unique upper cover λ^*, then $\lambda \overset{ss}{\not\sim} \lambda^*$. Otherwise $\gamma = \lambda^* \wedge (\bigwedge\Lambda - \{\lambda\})$ is a nonzero, locally strongly solvable congruence of $\mathbf{B}$. However, $\lambda \overset{s}{\sim} \lambda^*$ since $\mathscr{V}$ is locally solvable. The hypotheses of Theorem 2.8 are satisfied, so the pseudo-complement below γ of λ under λ^* exists; call it δ. If δ' is a congruence strictly smaller than δ, then δ' must be smaller than λ or else δ is not the pseudo-complement below γ of λ under λ^*. But then $\delta' \leq \lambda \wedge \delta \leq \lambda \wedge \gamma = 0_A$. This shows that $0_B \prec \delta \leq \gamma$. By semimodularity we have $\delta \leq \pi_i \vee \delta \leq \pi_i^*$ for all i. Thus, if $\delta = \mathrm{Cg}^{\mathbf{B}}(f, g)$, then for each i we must have $(f(i), g(i)) \in \alpha$. Let $X_0 \cup X_1 \cup \cdots \cup X_n$ be a partition of T into disjoint clopen subsets where f and g are constant on each X_j and $a = f(i) \neq g(i) = b$ when $i \in X_0$. We can decompose X_0 as $Y \cup Z$ where Y and Z are nonempty clopen subsets.

Now, since $\mathrm{C}(\alpha, 1_A; 0_A)$ fails and $\alpha = \mathrm{Cg}^{\mathbf{A}}(a, b) \succ 0_A$ we can find an n, an $(n+1)$-ary term p and n-tuples $\bar{u} \in A^n$ and $\bar{v} \in A^n$ such that

$$p^{\mathbf{A}}(a, \bar{u}) = p^{\mathbf{A}}(a, \bar{v}) \quad \text{but} \quad p^{\mathbf{A}}(b, \bar{u}) \neq p^{\mathbf{A}}(b, \bar{v}),$$

or else the same condition with a and b interchanged. Let $\bar{y}, \bar{z} \in \mathbf{B}$ be the tuples of continuous functions

$$\bar{y}(i) = \bar{u} \quad \text{and} \quad \bar{z}(i) = \begin{cases} \bar{u} & \text{if } i \notin Y \\ \bar{v} & \text{if } i \in Y. \end{cases}$$

Evaluating functions at each $i \in T$, we find that $c = p^{\mathbf{B}}(g, \bar{y}) \, \delta \, p^{\mathbf{B}}(f, \bar{y}) = p^{\mathbf{B}}(f, \bar{z}) \, \delta \, p^{\mathbf{B}}(g, \bar{z}) = d$, so $(c, d) \in \delta$. Now, $c(i) = d(i)$ if and only if $i \notin Y$. One consequence of this is that $\mathrm{Cg}^{\mathbf{B}}(c, d) > 0$. Further, every pair $(r, s) \in \mathrm{Cg}^{\mathbf{B}}(c, d)$ satisfies $r(i) = s(i)$ for $i \in Z$, so $(f, g) \notin \mathrm{Cg}^{\mathbf{B}}(c, d)$. This contradicts our proof that δ is a minimal congruence on $\mathbf{B}$. Our assumption that $\mathscr{V}$ contains an algebra with a minimal congruence which is neither locally strongly solvable nor central is false; this proves the theorem. □

It seems likely that the existence of a strongly irredundant subdirect representation for every algebra in a locally finite CSM variety forces each algebra to have an atomic congruence lattice. (This does hold whenever $\mathbf{0} \notin \mathrm{typ}\{\mathscr{V}\}$.) If this is true, then Theorem 2.3 of [10] proves that such a variety must be locally nilpotent. Theorem 5.8 falls just short of establishing this.

Next we next consider the uniqueness of representations.

THEOREM 5.9. *Let $\mathscr{V}$ be a variety for which $\mathbf{F}_{\mathscr{V}}(2)$ is finite. The following are equivalent*:

(1) *$\mathscr{V}$ has replaceable, strongly irredundant subdirect representations.*

(2) *$\mathscr{V}$ is congruence modular and every $\mathbf{A} \in \mathscr{V}$ has an atomic congruence lattice. If $\mathscr{V}$ is finitely generated or is of finite similarity type, then these conditions are equivalent to*:

(3) *$\mathscr{V}$ is a nilpotent, congruence permutable variety.*

Proof. The equivalence of (2) and (3) under the assumptions that $\mathbf{F}_{\mathscr{V}}(2)$ is finite and $\mathscr{V}$ is of finite type is contained in Corollary 3.12 of [10]. The hypothesis that $\mathscr{V}$ is of finite type is only used to prove that (2) implies (3) and is only necessary to show that $\mathscr{V}$ has a uniform bound on the nilpotency degree of any congruence. This fact is proved for any finitely generated congruence modular variety in [21]. Hence (2) is equivalent to (3) when $\mathscr{V}$ is finitely generated. If (2) holds, then the congruence lattice of any $\mathbf{A} \in \mathscr{V}$ is modular and strongly atomic, since $\mathscr{V}$ is closed under homomorphisms, and Theorem 7.6 of [3] proves that such lattices have replaceable, (strongly) irredundant decompositions. Therefore we only need to show that (1) implies (2).

If we assume (1), then the type-set of $\mathscr{V}$ is a subset of $\{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$. If we show that $\mathbf{0} \notin \mathrm{type}\{\mathscr{V}\}$, then, by Theorem 2.2, the type-set of the locally finite variety

$\mathcal{V}_2 = V(\mathbf{F}_{\mathcal{V}}(2))$ is contained in $\{\mathbf{2}\}$. Every finite algebra with type-set $\{\mathbf{2}\}$ has permuting congruences, so $\mathbf{F}_{\mathcal{V}_2}(4)$ has permuting congruences. This is enough to conclude that $\mathcal{V}_2$ is congruence modular and therefore, by Theorem 2.3 of [11], that $\mathcal{V}$ is congruence modular. At this point we can apply Theorem 6.3 of [3] which states that if every element of a compactly generated, modular lattice has an irredundant decomposition, then the lattice is strongly atomic. Hence, showing that $\mathbf{0} \notin \mathrm{typ}\{\mathcal{V}\}$ will finish the proof.

If $\mathbf{0} \in \mathrm{typ}\{\mathcal{V}\}$, then we can find a finite subdirectly irreducible algebra $\mathbf{A} \in \mathcal{V}$ with monolith, μ, of type $\mathbf{0}$. Let N be a $\langle 0_A, \mu\rangle$-trace and let $\mathbf{B} = \mathrm{Sg}^{\mathbf{A}^2}(N^2 \cup 0_A)/\psi$ where ψ is a congruence on $\mathrm{Sg}^{\mathbf{A}^2}(N^2 \cup 0_A)$ which is maximal with respect to $\psi \cap (N^2)^2 = 0_N$. In $\mathbf{B}$, $M = N^2/\psi$ is a 4-element E-trace and every nonzero congruence of $\mathbf{B}$ restricts nontrivially to M. Further, $\mathbf{B}|_M$ is polynomially equivalent to the 4-element set. Lemma 2.4 of [7] guarantees that if θ is any equivalence relation on $M = \{a, b, c, d\}$, then $\mathrm{Cg}^{\mathbf{B}}(\theta)|_M = \theta$. Now choose $\alpha \in \mathrm{Con}(\mathbf{B})$ containing (a, c) and (b, d) which is maximal for the property that $(a, b) \notin \alpha$; α is completely meet-irreducible. Choose $\beta \in \mathrm{Con}(\mathbf{B})$ containing (a, b) and (c, d) which is maximal for the property that $\alpha \wedge \beta = 0_B$. Such a β exists since $(\alpha \wedge \mathrm{Cg}^{\mathbf{B}}((a, b), (c, d)))|_M = 0_B|_M$, so $\alpha \wedge \mathrm{Cg}^{\mathbf{B}}((a, b), (c, d)) = 0_B$. Let Λ be a strongly irredundant decomposition of β (which exists since $\mathbf{B}/\beta$ has a strongly irredundant subdirect decomposition). $\Lambda \cup \{\alpha\}$ is a strongly irredundant decomposition of 0_B. Now let $\gamma \in \mathrm{Con}(\mathbf{B})$ be a congruence containing (c, d) which is maximal with respect to not containing (a, b) and let $\delta \in \mathrm{Con}(\mathbf{B})$ be a congruence containing (a, b) which is maximal for the property that $\gamma \wedge \delta = 0_B$. Let Ω be a strongly irredundant decomposition of δ; $\Omega \cup \{\gamma\}$ is a strongly irredundant decomposition of 0_B. Since $\mathcal{V}$ has replaceable strongly irredundant subdirect decompositions and $\Lambda \cup \{\alpha\}$ and $\Omega \cup \{\gamma\}$ are strongly irredundant decompositions of 0_B we must be able to find a $\tau \in \Omega \cup \{\gamma\}$ such that $\Lambda \cup \{\tau\}$ is a strongly irredundant decomposition of 0_B. But in fact there is no such τ since every member of $\Omega \cup \{\gamma\}$ contains either (a, b) or (c, d) and every congruence in Λ contains both (a, b) and (c, d). Hence for any $\tau \in \Omega \cup \{\gamma\}$ we have that

$$\bigwedge(\Lambda \cup \{\tau\}) \supseteq \mathrm{Cg}^{\mathbf{B}}(a, b) \quad \text{or} \quad \mathrm{Cg}^{\mathbf{B}}(c, d) > 0_B.$$

This shows that $\mathbf{0} \notin \mathrm{typ}\{\mathcal{V}\}$ and finishes the proof. □

DEFINITION 5.10. If $\mathbf{A} \leq_{sd} \prod_{i \in I} \mathbf{A}_i$ is a representation of the finite algebra $\mathbf{A}$ as a subdirect product of subdirectly irreducible algebras, then the **type-set** of this representation is the set $\bigcup_{i \in I} \mathrm{typ}(0_{A_i}, \mu_{A_i})$ where μ_{A_i} is the monolith of $\mathbf{A}_i$.

THEOREM 5.11. *If* $\mathbf{A}$ *is finite and belongs to a CSM variety, then the type-set of any two strongly irredundant subdirect representations of* $\mathbf{A}$ *is the same.*

Proof. We will prove this by showing that the type-set of any strongly irredundant decomposition of **A** is equal to

$$S = \bigcup_{0_A \prec \alpha} \{\mathrm{typ}(0_A, \alpha)\}.$$

Let Λ be a strongly irredundant decomposition of 0_A and let L be the type-set of this decomposition. If α is an atom of Con(**A**), then there is a $\lambda \in \Lambda$ which has unique upper cover λ^* where $\alpha \nleq \lambda$. By semimodularity the prime quotients $\langle 0_A, \alpha \rangle$ and $\langle \lambda, \lambda^* \rangle$ are perspective, so $\mathrm{typ}(0_A, \alpha) = \mathrm{typ}(\lambda, \lambda^*)$. The atom α was arbitrary, so $S \subseteq L$. To show that $L \subseteq S$ choose $\nu \in \Lambda$ which has unique upper cover ν^*. Let $\nu' = \bigwedge(\Lambda - \{\nu\}) > 0_A$. If β is an atom below ν', then $\langle \nu, \nu^* \rangle$ and $\langle 0_A, \beta \rangle$ are perspective prime quotients and therefore these quotients have the same type; thus, $L \subseteq S$. □

Although the type-set of a decomposition is not defined for infinite algebras, Theorem 2.8 can be used to prove an analogous result for some infinite algebras: If **A** belongs to a locally finite CSM variety which omits type **5** and Λ and Ω are strongly irredundant decompositions of 0_B, then for each $\lambda \in \Lambda$ there is a $\gamma \in \Omega$ such that both $[\lambda, \lambda^*]$ and $[\gamma, \gamma^*]$ are locally strongly solvable intervals or $[\lambda, \lambda^*]$ is projective in two steps to $[\gamma, \gamma^*]$. We don't know if the hypothesis that $\mathbf{5} \notin \mathrm{typ}\{\mathscr{V}\}$ is necessary.

6. Relatively modular and relatively distributive subquasivarieties

If $\mathscr{K}$ is a quasivariety, $\mathbf{A} \in \mathscr{K}$ and $\theta \in \mathrm{Con}(\mathbf{A})$ is a congruence satisfying $\mathbf{A}/\theta \in \mathscr{K}$, then we call θ a $\mathscr{K}$-congruence of **A**. For $X \subseteq A \times A$ we will use the notation X' to denote the least $\mathscr{K}$-congruence containing X. The collection of all $\mathscr{K}$-congruences on **A** forms a complete lattice, $\mathrm{Con}_{\mathscr{K}}(\mathbf{A})$, which we call the $\mathscr{K}$-congruence lattice of **A**. If every $\mathbf{A} \in \mathscr{K}$ has a modular (distributive) $\mathscr{K}$-congruence lattice, then we say that $\mathscr{K}$ is **relatively modular (relatively distributive)**. $\mathbf{A} \in \mathscr{K}$ is **relatively subdirectly irreducible** if the bottom element of $\mathrm{Con}_{\mathscr{K}}(\mathbf{A})$ is completely meet-irreducible in this lattice.

The study of relatively distributive quasivarieties was motivated by the desire to extend Baker's finite basis theorem to quasivarieties. This was accomplished by D. Pigozzi. Now relatively modular quasivarieties are being studied with the hope of extending McKenzie's finite basis theorem to quasivarieties. In [14], Kearnes and McKenzie extend much of modular commutator theory to relatively modular quasivarieties as a step toward achieving this goal. For information and references concerning relatively modular and relatively distributive quasivarieties see [14].

In [12], the relatively distributive subquasivarieties of a congruence modular variety were characterized using the techniques of modular commutator theory. For locally finite quasivarieties this characterization can be stated as follows: if $\mathscr{V}$ is a congruence modular variety and $\mathscr{K} \subseteq \mathscr{V}$ is locally finite, then $\mathscr{K}$ is relatively distributive if and only if (1) every finite, relatively subdirectly algebra in $\mathscr{K}$ is subdirectly irreducible, and (2) every finite algebra in $\mathscr{K}$ is $\underset{2}{\sim}$-radical-free. Later, in [14], a complete characterization of all relatively modular and relatively distributive quasivarieties was given. This characterization involved "quasi-Mal'cev" conditions similar to the familiar Mal'cev conditions characterizing congruence modular and congruence distributive varieties.

In this section we are going to characterize the locally finite relatively modular and relatively distributive subquasivarieties of a CSM variety in the spirit of [12]. This approach avoids mentioning "quasi-Mal'cev" conditions.

THEOREM 6.1. *If $\mathscr{V}$ is a CSM variety and $\mathscr{K} \subseteq \mathscr{V}$ is a locally finite subquasivariety of $\mathscr{V}$, then $\mathscr{K}$ is relatively distributive if and only if*

(1) *every finite relatively subdirectly algebra in $\mathscr{K}$ is subdirectly irreducible, and*

(2) *every finite algebra in $\mathscr{K}$ is $\{\mathbf{0}, \mathbf{2}, \mathbf{5}\}$-radical-free.*

$\mathscr{K}$ is relatively modular if and only if

(3) *$\mathscr{K}_{fin} \models_{con} \alpha \wedge \beta = 0 \rightarrow a' \wedge \beta' = 0$, and*

(4) *every finite algebra in $\mathscr{K}$ is $\{\mathbf{0}, \mathbf{5}\}$-radical-free.*

Proof. Some of this proof already appears in [14]. For example, it is a consequence of Theorem 1.1 of [14] that if $\mathscr{K}$ is relatively modular, then $\mathscr{K} \models_{con} \alpha \wedge \beta = 0 \rightarrow \alpha' \wedge \beta' = 0$. This implies that every finite relatively subdirectly irreducible algebra is subdirectly irreducible. Lemma 4.4 of [14] proves that if $\mathscr{K}$ is relatively distributive, then the $\{\mathbf{0}, \mathbf{2}, \mathbf{5}\}$-radical of any finite algebra in $\mathscr{K}$ is trivial; if $\mathscr{K}$ is relatively modular, then the $\{\mathbf{0}, \mathbf{5}\}$-radical is trivial. Further, the remarks after Theorem 4.1 of [14] explain why, when $\mathscr{K}$ is locally finite, $\mathscr{K}$ is relatively modular if and only if $\mathscr{K}_{fin}$ is. Therefore, if $\mathscr{K}$ is relatively distributive, then (1) and (2) hold. If $\mathscr{K}$ is relatively modular, then (3) and (4) hold. We must show that if (1) and (2) hold (or (3) and (4) hold), then $\mathscr{K}_{fin}$ is relatively distributive (relatively modular).

Assume that $\mathscr{K}_{fin}$ is not relatively distributive and that (1) holds. We will show that (2) fails. We can find an algebra $\mathbf{A} \in \mathscr{K}_{fin}$ with $\mathscr{K}$-congruences α, β and γ witnessing a failure of relative distributivity. That is, we can find $\mathscr{K}$-congruences α, β and γ satisfying $\theta = \alpha \wedge (\beta \vee \gamma)' > ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))' = \delta$. The congruences δ and θ are $\mathscr{K}$-congruences, so we can find a complete meet-irreducible $\mathscr{K}$-congruence ψ above δ but not above θ. By (1), ψ is completely meet-irreducible as an ordinary congruence. Let ψ^* denote the upper cover of ψ. Now, $\alpha \not\leq \psi$ and either

$\beta \nleq \psi$ or $\gamma \nleq \psi$. Assume that $\beta \nleq \psi$. Then $\alpha \vee \psi \geq \psi^*$, $\beta \vee \psi \geq \psi^*$ and $\alpha \wedge \beta \leq \psi$. This means that

$$\psi = \psi \vee (\alpha \wedge \beta) \underset{\mathbf{0,2,5}}{\sim} (\psi \vee \alpha) \wedge (\psi \vee \beta) \geq \psi^*.$$

This shows that $\mathbf{A}/\psi$ is a member of $\mathscr{K}_{fin}$ with nontrivial $\{\mathbf{0}, \mathbf{2}, \mathbf{5}\}$-radical, so (2) fails.

Now assume that $\mathscr{K}_{fin}$ fails to be relatively modular and that (3) holds. We will show that (4) fails. Since $\mathscr{K}_{fin}$ fails to be relatively modular, we can find an algebra $\mathbf{A} \in \mathscr{K}_{fin}$ which has $\mathscr{K}$-congruences α, β and γ satisfying $\theta = \alpha \wedge (\beta \vee (\alpha \wedge \gamma))' > ((\alpha \wedge \beta) \vee (\alpha \wedge \gamma))' = \delta$. Now, $\mathbf{A}/\delta \in \mathscr{K}_{fin}$, $(\theta/\delta)' = \theta/\delta$ and $((\beta \vee \delta)/\delta)' \supseteq \theta/\delta$. Hence, $(\theta \wedge (\beta \vee \delta)')/\delta = \theta/\delta > 0_{A/\delta}$. Using (3) we get that $(\theta \wedge (\beta \vee \delta))/\delta > 0_{A/\delta}$ or, in Con(**A**), $\theta \wedge (\beta \vee \delta) > \delta$. Let $\lambda = \theta \wedge (\beta \vee \delta)$. Now, $\delta < \lambda$, $\delta \vee \beta = \lambda \vee \beta$ and $\delta \wedge \beta = \lambda \wedge \beta$ ($=\alpha \wedge \beta$). Hence, $\delta \underset{\mathbf{0,5}}{\sim} \lambda$ which means that λ/δ is a nonzero congruence contained in the $\{\mathbf{0}, \mathbf{5}\}$-radical of $\mathbf{A}/\delta \in \mathscr{K}$. Thus, (4) fails and the proof is finished. □

7. CSM Varieties with the CEP

A semimodular lattice has the property that every interval sublattice is semimodular, so a variety is CSM if and only if free algebras have semimodular congruence lattices. This is a property shared by many other well-studied congruence conditions, e.g. congruence modularity, congruence distributivity and congruence permutability. For the three congruence conditions just listed it is even true that $\mathscr{V}$ has the congruence condition if and only if $\mathbf{F}_{\mathscr{V}}(4)$ does if and only if the subvariety generated by $\mathbf{F}_{\mathscr{V}}(2)$ does (see [4], [9], [11], [18]). It turns out that there do not exist finite m and n such that $\mathscr{V}$ is CSM if and only if $\mathbf{F}_{\mathscr{V}}(m)$ is, or that $\mathscr{V}$ is CSM if and only if the subvariety generated by $\mathbf{F}_{\mathscr{V}}(n)$ is (see Example 7.1). We do not even know if there is an infinite cardinal λ, independent of $\mathscr{V}$, with the property that $\mathscr{V}$ is CSM if and only if $\mathbf{F}_{\mathscr{V}}(\lambda)$ is. (One of the referees points out that λ = cardinality of the language of $\mathscr{V}$ always works.) However, in this section we will show that a variety with the congruence extension property is CSM if and only if its finitely generated free algebras are. In fact, we show that if $\mathscr{V}$ has the CEP, then $\mathscr{V}$ is congruence semimodular if and only if $\mathbf{F}_{\mathscr{V}}(5)$ is and that $\mathscr{V}$ is congruence weakly semimodular if and only if $\mathbf{F}_{\mathscr{V}}(4)$ is.

EXAMPLE 7.1. Let $A = \{0, \ldots, n\}, n > 1$, and for any permutation of A, $\sigma \in \mathrm{Sym}(A)$, define an operation on A:

$$f_\sigma^{\mathbf{A}}(x, y_0, \ldots, y_n) = \begin{cases} \sigma(x) & \text{if } y_i = i \text{ for all } i \\ x & \text{otherwise.} \end{cases}$$

If $F = \{f_\sigma \mid \sigma \in \mathrm{Sym}(A)\}$ let $\mathbf{A} = \langle A; F\rangle$. $\mathbf{A}$ has unary polynomials (of the form $f_\sigma^{\mathbf{A}}(x, 0, 1, \ldots, n)$) which act completely transitively on $\mathbf{A}$, so $\mathbf{A}$ is simple. Every proper subset $B \subset A$ is a subuniverse and, for all $\sigma, f_\sigma^{\mathbf{B}}(x, \bar{y}) = x$. This means that every proper subalgebra of $\mathbf{A}$ is equivalent to a set. In $\mathcal{V} = V(\mathbf{A})$ the k-generated free algebra is a subdirect product of the k-generated subalgebras of $\mathbf{A}$ so, if $k \le n$, this free algebra is just a set. Thus $V(\mathbf{F}_{\mathcal{V}}(n))$ is CSM since it is equivalent to the variety of sets. However, we will show that $\mathcal{V}$ is not CSM.

$\mathcal{V}$ contains a two-element set $\mathbf{S}$. We leave it to the reader to check that the algebra $\mathbf{A} \times \mathbf{S} \in \mathcal{V}$ has a congruence lattice isomorphic to the (non-semimodular) lattice pictured in Fig. 4. Hence, $\mathcal{V}$ is not CSM.

In a variety with the congruence extension property the kind of bad behavior exhibited by Example 7.1 cannot occur. The reason for this is that prime quotients in congruence lattices restrict well to subalgebras, allowing us to push failures of congruence semimodularity down into "small" algebras.

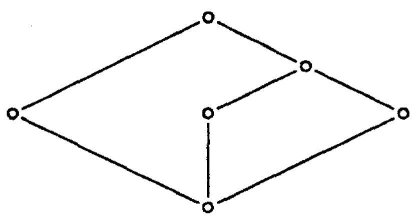
LEMMA 7.2 [13]. *Assume that* $\mathbf{B}$ *is a subalgebra of* $\mathbf{A} \in \mathcal{V}$ *and that* $H(\mathbf{A})$ *has the CEP. If* $\alpha \prec \beta$ *in* $\mathrm{Con}(\mathbf{A})$, *then* $\alpha|_B = \beta|_B$ *or* $\alpha|_B \prec \beta|_B$. □

Therefore in a *variety* with the CEP, a covering pair of congruences on $\mathbf{A}$ either restrict to the same congruence on a subalgebra or they restrict to a covering pair.

THEOREM 7.3. *If* $\mathcal{V}$ *has the CEP, then* $\mathcal{V}$ *is CSM if and only if* $\mathbf{F}_{\mathcal{V}}(5)$ *is CSM.* $\mathcal{V}$ *is CWSM if and only if* $\mathbf{F}_{\mathcal{V}}(4)$ *is CWSM.*

Proof. It is enough to show that if $\mathcal{V}$ has the CEP and fails to be CSM, then $\mathcal{V}$ contains a 5-generated algebra $\mathbf{A}$ that fails to be CSM. $\mathbf{A}$ is a homomorphic image of $\mathbf{F} = \mathbf{F}_{\mathcal{V}}(5)$, so $\mathbf{F}$ also fails to be CSM. Now, suppose that $\mathbf{B} \in \mathcal{V}$ fails to be CSM. We may assume that there are $\alpha, \beta, \gamma \in \mathrm{Con}(\mathbf{B})$ such that $0_B \prec \alpha$ and $\alpha \vee \beta > \gamma > \beta$. Observe that since $\mathrm{Con}(\mathbf{B})$ is upper continuous the set

$$D = \{\delta \in [0_B, \beta] \mid \delta \prec \alpha \vee \delta\}$$



$\mathrm{Con}(\mathbf{A} \times \mathbf{S})$

Figure 4

is closed under unions of chains. The argument for this is as follows. Suppose that $C \subseteq D$ is a chain and that $\bigvee_{\delta \in C} \delta = \theta \leq \beta$. Suppose also that $\theta \leq \psi < \alpha \vee \theta = \bigvee_{\delta \in C} \alpha \vee \delta$. Since $\alpha \nleq \psi$, for every $\delta \in C$ we must have $\delta \leq (\alpha \vee \delta) \wedge \psi < \alpha \vee \delta$, so $\delta = (\alpha \vee \delta) \wedge \psi$. Thus,

$$\theta = \bigvee_{\delta \in C} \delta = \bigvee_{\delta \in C} ((\alpha \vee \delta) \wedge \psi) = \left(\bigvee_{\delta \in C} (\alpha \vee \delta) \right) \wedge \psi = (\alpha \vee \theta) \wedge \psi = \psi,$$

so $\theta \prec \alpha \vee \theta$. By Zorn's Lemma we can find an element $\nu \in [0_B, \beta]$ maximal for the property that $\nu \prec \alpha \vee \nu$. Now replacing $\mathbf{B}$ by $\mathbf{B}/\nu$, α by $(\alpha \vee \nu)/\nu$ and β by β/ν and relabeling we may assume that $\alpha \wedge \beta = 0_B$ and for every nonzero $\delta \in [0_B, \beta]$ we have $\delta \nprec \alpha \vee \delta$.

If the nontrivial α-classes are all disjoint from nontrivial β-classes, then $\alpha \vee \beta = \alpha \cup \beta \succ \beta$ which is false. Hence we can find $a \neq b \neq c$ such that $(a, b) \in \alpha$ and $(b, c) \in \beta$. Let $\beta' = \mathrm{Cg}^{\mathbf{B}}(b, c) \leq \beta$. Of course, $\alpha \wedge \beta' = 0_B \prec \alpha$ and $\beta' \nprec \alpha \vee \beta'$; say that $\beta' < \gamma' < \alpha \vee \beta'$. Choose $(d, e) \in \gamma' - \beta'$. Let $\mathbf{A} = \mathrm{Sg}^{\mathbf{B}}(a, b, c, d, e)$. Since $\mathcal{V}$ has the CEP, $0_A \prec \mathrm{Cg}^{\mathbf{A}}(a, b) = \lambda$. This uses Lemma 7.2 and the fact that $\lambda = \mathrm{Cg}^{\mathbf{B}}(\lambda)|_A = \alpha|_A$. We also have $0_A < \beta'|_A = \psi < \mathrm{Cg}^{\mathbf{A}}((b, c), (d, e)) = \chi$. Now $\lambda \wedge \psi = 0_A \prec \lambda$ and $\psi < \lambda \vee \psi$. We claim that $\psi \nprec \lambda \vee \psi$ which is a failure of semimodularity in $\mathrm{Con}(\mathbf{A})$. To show this we can use the CEP to conclude that

$$\lambda \vee \psi = \mathrm{Cg}^{\mathbf{B}}(\lambda \vee \psi)|_A \supseteq \mathrm{Cg}^{\mathbf{B}}((a, b), (b, c))|_A = (\alpha \vee \beta')|_A \supseteq \gamma'|_A \supseteq \chi.$$

On the other hand, $\psi < \chi$ since $(d, e) \in \chi - \psi$ and $\lambda \nleq \chi$ since $\lambda \wedge \chi \subseteq (\alpha \wedge \gamma')|_A = 0_A$. Thus, $\psi < \chi < \lambda \vee \psi$. This shows that a 5-generated algebra in $\mathcal{V}$ fails to be CSM, so $\mathbf{F}_{\mathcal{V}}(5)$ also fails to be CSM.

Now suppose that $\mathcal{V}$ fails to be CWSM. As in the last paragraph, we can find a finitely generated algebra $\mathbf{A}$ which has congruences α and β such that $0_A \prec \alpha, \beta$ but, say, $\beta < \gamma < \alpha \vee \beta$. The reason for this is that if one begins with a failure of weak semimodularity in $\mathrm{Con}(\mathbf{B})$ in the last paragraph, then one obtains a failure of weak semimodularity in $\mathrm{Con}(\mathbf{A})$. If $(a, b) \in \gamma - \beta$ we can find a chain of elements $a = x_0, \ldots, x_n = b$ where the pairs (x_i, x_{i+1}) belonging alternately to $\alpha - \beta$ and $\beta - \alpha$. Let $\mathbf{C} = \mathrm{Sg}^{\mathbf{A}}(x_0, x_1, x_2, x_n)$. $\mathbf{C}$ is 4-generated and we claim that $\mathbf{C}$ is not CWSM. Certainly $0_C \prec \alpha|_C = \lambda$ since $0_A \prec \alpha$, $\mathcal{V}$ has the CEP and $\alpha|_C$ contains either (x_0, x_1) or (x_1, x_2). Similarly, $0_C \prec \beta|_C = \psi$. Let $\chi = \mathrm{Cg}^{\mathbf{C}}(x_0, x_n) \vee \psi$. To show that $\mathbf{C}$ is not CWSM it is necessary to prove that $\lambda \wedge \psi = 0_C \prec \lambda$ but $\psi < \chi < \lambda \vee \psi$. The only part left to show is that $\chi < \lambda \vee \psi$. Since $\lambda \wedge \chi = 0_C$ it is enough to prove that $\chi \leq \lambda \vee \psi$. The argument for this is exactly the same as in the last paragraph:

$$\lambda \vee \psi = \mathrm{Cg}^{\mathbf{A}}(\lambda \vee \psi)|_C = (\mathrm{Cg}^{\mathbf{A}}(x_0, x_1) \vee \mathrm{Cg}^{\mathbf{A}}(x_1, x_2))|_C = (\alpha \vee \beta)|_C \supseteq \gamma|_C \supseteq \chi.$$

We've shown that if $\mathscr{V}$ is not CWSM, then $\mathscr{V}$ contains a 4-generated algebra that fails to be CWSM. Hence if $\mathbf{F}_{\mathscr{V}}(4)$ is CWSM, $\mathscr{V}$ will also be CWSM. $\square$

THEOREM 7.4. *If $\mathscr{V}$ is a unary variety, then $\mathscr{V}$ is CSM if and only if $\mathbf{F}_{\mathscr{V}}(3)$ is CSM.*

Proof. As in the proof of the first part of Theorem 7.3, if $\mathscr{V}$ fails to be CSM, then we can find an algebra $\mathbf{B} \in \mathscr{V}$ which has elements a, b and c such that $\alpha = \mathrm{Cg}(a, b) \succ 0_B$ and for $\beta' = \mathrm{Cg}(b, c)$ we have $\alpha \wedge \beta' = 0_B$, but $\beta' \not\prec \alpha \vee \beta'$. Now let $\mathbf{A} = \mathrm{Sg}^{\mathbf{B}}(a, b, c)$. Since $\mathscr{V}$ is unary, $\mathrm{Con}(\mathbf{A})$ is isomorphic to the interval $[0_B, \mathrm{Cg}^{\mathbf{B}}(A \times A)]$ in $\mathrm{Con}(\mathbf{B})$. This interval contains $[0_B, \alpha \vee \beta']$, so it contains a failure of semimodularity. It follows that $\mathbf{A}$ and therefore $\mathbf{F}_{\mathscr{V}}(3)$ is not CSM. $\square$

It turns out that the type-set of a finitely generated CSM variety with CEP can be easily determined.

THEOREM 7.5 [13]. *If $\mathscr{V}$ is a locally finite variety with the CEP, then* $\mathrm{typ}\{\mathscr{V}\} \subseteq \mathrm{typ}\{\mathbf{F}_{\mathscr{V}}(2)\} \cup \{\mathbf{3}\}$. $\square$

COROLLARY 7.6. *Assume that $\mathbf{A}$ is finite and that $\mathscr{V} = \mathscr{V}(\mathbf{A})$ is a CSM variety with the CEP. Then* $\mathrm{typ}\{\mathscr{V}\} = \mathrm{typ}\{\mathbf{S}(\mathbf{A})\} \cup \mathrm{typ}\{\mathbf{F}_{\mathscr{V}}(2)\}$.

Proof. Combining the results of Corollary 5.2 and Theorem 7.5 we get that

$$\mathrm{typ}\{\mathscr{V}\} - (\mathrm{typ}\{\mathbf{S}(\mathbf{A})\} \cup \mathrm{typ}\{\mathbf{F}_{\mathscr{V}}(2)\}) \subseteq \{\mathbf{5}\} \cap \{\mathbf{3}\} = \varnothing.$$

Hence $\mathrm{typ}\{\mathscr{V}\} = \mathrm{typ}\{\mathbf{S}(\mathbf{A})\} \cup \mathrm{typ}\{\mathbf{F}_{\mathscr{V}}(2)\}$. $\square$

We leave it to the reader to verify that the algebra $\mathbf{A}$ of Example 5.3 generates a variety that has the CEP. Now for a given n it is not hard to produce an example of a finite algebra $\mathbf{B}$ for which $\mathscr{V} = V(\mathbf{B})$ is a congruence modular variety with the CEP such that $\mathrm{typ}\{\mathscr{V}\} \nsubseteq \mathrm{typ}\{\mathbf{F}_{\mathscr{V}}(n)\}$. Two such examples appear in [13]. Taking the varietal product of such an example with the variety of Example 5.3 produces a finitely generated CSM variety which has the CEP and the properties that $\mathscr{V} = V(\mathbf{A} \times \mathbf{B})$, $\mathrm{typ}\{\mathscr{V}\} \nsubseteq \mathrm{typ}\{\mathbf{F}_{\mathscr{V}}(n)\}$ and $\mathrm{typ}\{\mathscr{V}\} \nsubseteq \mathrm{typ}\{\mathbf{S}(\mathbf{A} \times \mathbf{B})\}$, although of course $\mathrm{typ}\{\mathscr{V}\} \subseteq \mathrm{typ}\{\mathbf{S}(\mathbf{A} \times \mathbf{B})\} \cup \mathrm{typ}\{\mathbf{F}_{\mathscr{V}}(2)\}$. This shows that the result of Corollary 7.6 cannot be improved.

8. Geometric varieties

A complete lattice is **atomistic** if every element is a join of atoms. A lattice is **geometric** if it is complete, semimodular and atomistic. A variety is **geometric** if the congruence lattice of any member is a geometric lattice. S. MacLane showed that a lattice is geometric if and only if it is isomorphic to the lattice of closed sets of some set under a closure operator which satisfies the exchange principle. Therefore our comments from the Introduction show that the variety of sets or any variety of vector spaces are examples of geometric varieties. Are there any others? The answer to this question is "yes" although all of the examples that we know are built up in a straightforward way from varieties of vector spaces and the variety of sets.

EXAMPLE 8.1. Varieties of sets and of vector spaces are geometric. More generally, varieties in which every operation is essentially nullary or varieties which are affine over a division ring are geometric. The kth-power construction defined in [22] provides a way of obtaining new varieties, $\mathcal{V}^{[k]}$, categorically isomorphic to a given variety, $\mathcal{V}$. A categorical isomorphism between varieties preserves congruence lattices, so a kth-power of a variety of nullary algebras is geometric. The kth-power of an affine variety over a division ring is geometric. (These varieties are precisely the affine varieties over a simple Artinian ring.)

The congruence lattices of algebras in a varietal product $\mathcal{U} \otimes \mathcal{V}$ are just the direct products of congruence lattices from algebras in $\mathcal{U}$ with congruence lattices of algebras in $\mathcal{V}$. The class of geometric lattices is closed under products, so the class of geometric varieties is closed under (finite) varietal products. We don't have a good description of finite varietal products of kth-powers of nullary varieties, but finite varietal products of affine varieties over simple Artinian rings are just affine varieties over semisimple Artinian rings. Thus, many geometric varieties may be described as a varietal product of an affine variety over a semisimple Artinian ring with a varietal product of k-th powers of nullary varieties. We suspect that there are few other locally finite geometric varieties. We will find, as a consequence of Theorem 8.5, that at least there are no locally finite geometric varieties omitting type $\mathbf{0}$ other than the affine varieties over a finite semisimple ring.

LEMMA 8.2. *An algebraic semimodular lattice is atomistic if and only if it is atomic and relatively complemented.*

Proof. Certainly an atomistic lattice is atomic. Further, it is not too hard to see that every interval in an atomistic semimodular lattice is atomistic and semimodular. Thus, for the forward direction, it suffices to prove that an algebraic, semimodular atomistic lattice, $\mathbf{L}$, is complemented. The top and the bottom elements of $\mathbf{L}$ are complements. If $x \in L$ is not the top element, then there is an atom $a \in L$ such that $x \wedge a = 0_L$. By upper continuity, we can extend a to an element y maximal for the property that $x \wedge y = 0_L$. If we show that $x \vee y = 1_L$ we will be done; y is a complement for x. If $x \vee y < 1_L$ we can find an atom $b \in L$ such that $b \nleq x \vee y$. By semimodularity, $y \prec y \vee b$ and $x \vee y \prec x \vee y \vee b$. The maximality of y implies that $x \wedge (y \vee b) > 0_L$ and, since $\mathbf{L}$ is atomic, we can find an atom $c \leq x \wedge (y \vee b)$. Of course, $c \nleq y$, so $y \prec y \vee c \leq y \vee b$. This forces $y \vee c = y \vee b$. From this we derive the contradiction that $x \vee y = x \vee c \vee y = x \vee y \vee b > x \vee y$. We conclude that x and y are complements.

Now suppose that $\mathbf{L}$ is algebraic, semimodular, atomic and relatively complemented. We must show that any $z \in L$ is the join of the atoms that lie below it. Let u denote the join of the atoms that lie below z. If $u \neq z$, then there is a nonzero element $v \in [0_L, z]$ which is a complement of u relative to this interval. Let w be an atom below v. Since $u \wedge v = 0_L$ we cannot have $w \leq u$. But w is an atom below z and every such atom lies below u. This contradiction means that $u = z$ and we are done. □

Every interval in a geometric lattice is complemented and therefore cannot contain three elements $0 \prec \alpha < \beta$ where α is completely meet-irreducible in $[0, \beta]$; α needs a complement. It follows that any subdirectly irreducible algebra in a geometric variety is simple. Further, $\mathrm{Con}(\mathbf{A})$ contains no 3-element intervals if $\mathrm{Con}(\mathbf{A})$ is geometric. Hence every possible transfer principle holds in a geometric variety.

THEOREM 8.3 [10]. *Assume that every $\mathbf{A} \in \mathscr{V}$ has an atomic congruence lattice. Then if $\alpha \in \mathrm{Con}(\mathbf{A})$ is an atom, α is a central congruence.* □

COROLLARY 8.4. *A geometric variety is semisimple, abelian and its type-set is contained in $\{\mathbf{0}, \mathbf{2}\}$. Both the $\langle \mathbf{0}, \mathbf{2} \rangle$-transfer principle and the $\langle \mathbf{2}, \mathbf{0} \rangle$transfer principle hold.*

Proof. This corollary follows directly from Theorem 8.3 and the remark that precedes it. Nevertheless, we give a proof of this result that only depends on the fact that the members of a geometric variety have relatively complemented congruence lattices.

We have already pointed out that all transfer principles and semisimplicity follows from the fact that all algebras in our variety have relatively complemented congruence lattices. If $\mathscr{V}$ is our variety, we need to prove that $\mathscr{V}$ is abelian and has type-set contained in $\{\mathbf{0}, \mathbf{2}\}$. If $\mathscr{V}$ is not abelian, we can find a nonabelian simple algebra $\mathbf{A} \in \mathscr{V}$. As in the proof of Theorem 5.8, we construct a certain subdirect power of $\mathbf{A}$: $\mathbf{B}$ is the algebra of all continuous functions from a compact, Hausdorff, totally disconnected space without isolated points, T, to A considered as a discrete space. As in that proof we use the notation π_X to denote the congruence $\{(f, g) \in B^2 \mid f(x) = g(x) \text{ for all } x \in X\}$. If $t \in T$, we claim that the proper congruence π_t has no complement in $\mathrm{Con}(\mathbf{B})$. Indeed, there is no nonzero congruence

$\theta \in \mathrm{Con}(\mathbf{B})$ such that $\theta \wedge \pi_t = 0_B$. To prove this, assume that θ is a nonzero congruence on $\mathbf{B}$. We can find $(f, g) \in \theta - 0_B$. If $f(t) = g(t)$, then $(f, g) \in \theta \wedge \pi_t - 0_B$ and we are done. Therefore we may as well assume that (f, g) is not in π_t and more specifically that $f(t) = a \neq b = g(t)$. Now f and g are functions from a compact Hausdorff space to a discrete space, so they can only assume finitely many distinct values. This means that we can partition T into a finite disjoint union of clopen sets, $T = X_0 \cup \cdots \cup X_m$, where both f and g are constant on each X_i. We may assume that $t \in X_0$. Each X_i is infinite since T contains no isolated points. This means that we can partition X_0 into the disjoint union of two infinite clopen sets, $X_0 = Y \cup Z$. We may assume that $t \in Y$.

Since $\mathbf{A}$ is simple, $1_A = \mathrm{Cg}^{\mathbf{A}}(a, b)$. We have assumed that $\mathbf{A}$ is nonabelian, so for some n there exists an $(n+1)$-ary term p and n-tuples $\bar{u}, \bar{v} \in A^n$ such that $p^{\mathbf{A}}(a, \bar{u}) = p^{\mathbf{A}}(a, \bar{v})$ but $p^{\mathbf{A}}(b, \bar{u}) \neq p^{\mathbf{A}}(b, \bar{v})$. Now let $\bar{c} \in B^n$ be the n-tuple of constant functions where for each $x \in T$ we have $\bar{c}(x) = \bar{u}$. Let $\bar{d} \in B^n$ be the n-tuple of functions where $\bar{d}(x) = \bar{u}$ if $x \notin Z$ and $\bar{d}(x) = \bar{v}$ if $x \in Z$. Checking coordinatewise we learn that

$$r = p^{\mathbf{B}}(g, \bar{c})\ \theta\ p^{\mathbf{B}}(f, \bar{c}) = p^{\mathbf{B}}(f, \bar{d})\ \theta\ p^{\mathbf{B}}(g, \bar{d}) = s,$$

but that $r \neq s$ since these functions disagree at all points of Z. On the other hand, r and s agree at all points not in Z, so $r(t) = s(t)$. This shows that $(r, s) \in \theta \wedge \pi_t - 0_B$ and finishes this part of the proof.

Since $\mathcal{V}$ is abelian we already have $\mathrm{typ}\{\mathcal{V}\} \subseteq \{\mathbf{0}, \mathbf{1}, \mathbf{2}\}$; we must show that $\mathbf{1} \notin \mathrm{typ}\{\mathcal{V}\}$. Assume otherwise. Say that the finite algebra $\mathbf{C} \in \mathcal{V}$ has a minimal congruence γ where $\mathrm{typ}(0_C, \gamma) = \mathbf{1}$. Let N be a $\langle 0_C, \gamma \rangle$-trace. Since the class of relatively complemented lattices is closed under the formation of interval sublattices and bounded homomorphic images, every $\mathbf{D} \in V(\mathbf{CI}_N)$ has a relatively complemented congruence lattice. But $\mathbf{CI}_N$ is polynomially equivalent to a transitive $\mathbf{G}$-set, so some member of $V(\mathbf{CI}_N)$ has a congruence lattice isomorphic to the lattice in Figure 1 which is not a relatively complemented lattice. This contradiction shows that $\mathbf{1} \notin \mathrm{typ}\{\mathcal{V}\}$, so $\mathrm{typ}\{\mathcal{V}\} \subseteq \{\mathbf{0}, \mathbf{2}\}$. □

THEOREM 8.5. *A locally finite variety $\mathcal{V}$ is geometric if and only if $\mathcal{V} = \mathcal{V}_0 \otimes \mathcal{V}_2$ where $\mathcal{V}_0$ is a strongly abelian, geometric subvariety and $\mathcal{V}_2$ is an affine variety whose corresponding ring is finite and semisimple.*

Proof. The justification for the claim that any variety of the form $\mathcal{V}_0 \otimes \mathcal{V}_2$ where $\mathcal{V}_0$ and $\mathcal{V}_2$ are as described is a geometric variety is contained in the discussion of Example 8.1. For the other direction, $\mathrm{typ}\{\mathcal{V}\} \subseteq \{\mathbf{0}, \mathbf{2}\}$, $\mathcal{V}$ is locally finite and abelian and $\mathcal{V}$ satisfies the $\langle \mathbf{0}, \mathbf{2} \rangle$ and $\langle \mathbf{2}, \mathbf{0} \rangle$-transfer principles. Theorem 3.15 proves that $\mathcal{V}$ decomposes as $\mathcal{V}_0 \otimes \mathcal{V}_2$ where $\mathrm{typ}\{\mathcal{V}_i\} \subseteq \{\mathbf{i}\}$. Further, from

Theorem 3.15 we have that $\mathscr{V}_0$ is strongly abelian and $\mathscr{V}_2$ is affine. $\mathscr{V}_0$ is geometric, since it is a subvariety of a geometric variety. $\mathscr{V}_2$ is locally finite and semisimple, so the ring of $\mathscr{V}_2$ is a finite semisimple ring. $\square$

Theorem 8.5 determines the structure of locally finite geometric varieties up to the determination of their strongly abelian subvariety. We pose the following problem.

PROBLEM 5. Describe the locally finite, strongly abelian geometric varieties.

We were careful to show that all of our conclusions about geometric varieties already follow from the hypothesis that all the congruence lattices in the variety are relatively complemented. Is this hypothesis equivalent to the property of being geometric? We don't know, but for locally finite varieties Theorem 8.5 reduces this to a question about strongly abelian varieties.

PROBLEM 6. Describe the (locally finite) varieties whose members have relatively complemented congruence lattices.

It would be interesting if these two kinds of varieties coincided, even if just in the locally finite case. Before we leave this topic we include an exercise concerning locally finite varieties in which every finite algebra has a geometric congruence lattice.

EXERCISE 2. Assume that $\mathscr{V}$ is locally finite and that every finite algebra in $\mathscr{V}$ has a geometric congruence lattice. Prove that the locally solvable subvariety of $\mathscr{V}$ is abelian and decomposes as $\mathscr{V}_0 \otimes \mathscr{V}_2$ where $\mathrm{typ}\{\mathscr{V}_i\} = \{\mathbf{i}\}$. Show that if $\mathscr{V}$ has the CEP, then $\mathscr{V}$ decomposes as $\mathscr{V}_0 \otimes \mathscr{V}_2 \otimes \mathscr{V}_{3,4}$ where $\mathscr{V}_0$ is strongly abelian, $\mathscr{V}_2$ is affine over a finite, semisimple ring and $\mathscr{V}_{3,4}$ is a semisimple, congruence distributive variety.

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