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Congruence lower semimodularity and 2-finiteness imply congruence modularity

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Abstract. We show that any congruence lower semimodular variety whose 2-generated free algebra is finite must be congruence modular.

1. Introduction

What led us to discover the result in the title was our investigation of *upper* semimodularity as a congruence condition. Upper semimodularity seems to arise naturally as a congruence condition and examples of congruence upper semimodular varieties abound. However, it is very difficult to understand what makes a variety congruence upper semimodular. Perhaps it was a moment of exasperation or of perversity that made us wonder why we knew no examples of non-modular varieties that were congruence *lower* semimodular. Lower semimodularity does play a role in algebra, it is known that the subgroup lattice of a finite nilpotent group is lower semimodular (but not generally upper semimodular). However, the role of lower semimodularity does not seem to be that of an interesting congruence condition, as we shall see.

We rely a great deal on the techniques and results of tame congruence theory. The reader can find all he needs in [6]. Our notation for universal algebra is fairly standard and follows [1].

2. Congruence modularity

Probably the best-known characterization of congruence modularity for a variety is the one in Alan Day's Master's thesis. He shows that a variety is congruence modular if and only if it satisfies a certain Mal'cev condition in four variables. The following theorem is essentially a statement of that result in a slightly expanded form.

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THEOREM 2.1. *The following conditions are equivalent for a variety* $\mathcal{V}$:

(a) $\mathcal{V}$ *is congruence modular.*

(b) *There is an* n *and* 4*-ary terms* $m_0(w, x, y, z), \ldots, m_n(w, x, y, z)$ *such that* $\mathcal{V}$ *satisfies*:

  (i) $m_0(w, x, y, z) \approx w$, $m_n(w, x, y, z) \approx z$

  (ii) $m_i(w, y, y, w) \approx w$, $i \leqslant n$

  (iii) $m_i(w, w, y, y) \approx m_{i+1}(w, w, y, y)$, *for even* $i < n$

  (iv) $m_i(w, y, y, z) \approx m_{i+1}(w, y, y, z)$, *for odd* $i < n$.

(c) $\mathcal{V}_4 = \mathbf{HSP}(\mathbf{F}_{\mathcal{V}}(4))$ *is congruence modular.*

(d) $\operatorname{Con} \mathbf{F}_{\mathcal{V}}(4)$ *is modular.* □

The proof that (a) is equivalent to (b) can be found in [3]. Conditions (c) and (d) follow easily from the proof.

It was a long time before a significant improvement of Day's result was found. It was by an application of modular commutator theory that H.-P. Gumm was able to show that it is not necessary to work with $\mathbf{F}_{\mathcal{V}}(4)$ or with 4-variable terms to verify that a variety is congruence modular. A slightly expanded statement of Gumm's result (for varieties) follows. The proof that (a) is equivalent to (b) can be found in [5].

THEOREM 2.2. *The following conditions are equivalent for a variety* $\mathcal{V}$:

(a) $\mathcal{V}$ *is congruence modular.*

(b) *There is an* n *and* 3*-ary terms* $p(x, y, z)$ *and* $q_0(x, y, z), \ldots, q_n(x, y, z)$ *such that* $\mathcal{V}$ *satisfies*:

  (i) $q_0(x, y, z) \approx x$

  (ii) $q_i(x, y, z) \approx x$, $i \leqslant n$

  (iii) $q_i(x, y, y) \approx q_{i+1}(x, y, y)$, *for even* $i < n$

  (iv) $q_i(x, x, y) \approx q_{i+1}(x, x, y)$, *for odd* $i < n$

  (v) $q_n(x, y, y) \approx p(x, y, y)$

  (vi) $p(x, x, y) \approx y$.

(c) $\mathcal{V}_3 = \mathbf{HSP}(\mathbf{F}_{\mathcal{V}}(3))$ *is congruence modular.* □

It is not true that the conditions of the last theorem are equivalent to the condition that $\operatorname{Con} \mathbf{F}_{\mathcal{V}}(3)$ is modular. For example, in the variety of sets $\mathbf{F}_{\mathcal{V}}(3)$ has three elements and $\operatorname{Con} \mathbf{F}_{\mathcal{V}}(3)$ is just the lattice of equivalence relations on this 3-element set. Thus $\operatorname{Con} \mathbf{F}_{\mathcal{V}}(3)$ is the 5-element simple (modular) lattice. However, the variety of sets is not congruence modular; every set of more than three elements has a non-modular congruence lattice.

After these two results it is natural to ask if the number "3" as it appears in condition (b) and (c) of Gumm's theorem are optimal. The answer is "yes" for (b) and "no" for (c). We provide an example to show that if congruence modularity implies a nontrivial Mal'cev condition, then some of the terms in the Mal'cev condition must have at least three variables.

EXAMPLE. The purpose of this example is to show that if a fairly general sort of congruence condition implies a nontrivial Mal'cev condition for varieties or quasivarieties, then some of the terms in the Mal'cev condition require three variables. The reader who is not familiar with Mal'cev conditions is referred to [7].

We will consider a certain collection of reflexive, compatible binary relations on an algebra $\mathbf{A}$ in a quasivariety $\mathscr{K}$ to be an algebra which we will call $\mathbf{R}(\mathbf{A})$. For this example, the carrier of $\mathbf{R}(\mathbf{A})$ will be the collection of reflexive, compatible binary relations on $\mathbf{A}$ that can be obtained from the congruences on $\mathbf{A}$ by relational composition and intersection. The nullary basic operations of this algebra are the operations 0 and 1 which denote the smallest and largest reflexive, compatible binary relations on $\mathbf{A}$. The unary operations are Θ, which takes a relation to the least congruence relation that contains it, and $\Theta_{\mathscr{K}}$, the operation which takes a relation α to the least congruence relation θ containing α and satisfying $\mathbf{A}/\theta \in \mathscr{K}$. The binary basic operations on $\mathbf{R}(\mathbf{A})$ are $\vee$, where $\alpha \vee \beta$ is defined to be $\Theta(\alpha \cup \beta)$, also $\wedge$, which is our symbol for the intersection of relations, and finally the operation of relational composition $\circ$.

Now, we will write $\mathscr{K} \vDash_{\text{con}} \varphi$ to mean that for every $\mathbf{A} \in \mathscr{K}$ we have $\operatorname{Con} \mathbf{A} \vDash \varphi$. Here φ is a formula whose only nonlogical symbols are among the operation symbols 0, 1, Θ, $\Theta_{\mathscr{K}}$, $\vee$, $\wedge$, and $\circ$. Our observation is that if φ is a universally quantified formula such that $\mathscr{K} \vDash_{\text{con}} \varphi$ for some nontrivial quasivariety $\mathscr{K}$, then $\mathscr{V} \vDash_{\text{con}} \varphi$ for *every* arithmetical variety. (A variety is *arithmetical* if it is both congruence distributive and congruence permutable.) To show this it is enough to choose an arbitrary arithmetical variety $\mathscr{V}$, an arbitrary algebra $\mathbf{X}$ in $\mathscr{V}$ and show that $\operatorname{Con} \mathbf{X} \vDash_{\text{con}} \varphi$. Notice that $\operatorname{Con} \mathbf{X}$ is distributive and satisfies the universal formulae $\Theta_{\mathscr{V}}(\alpha) \approx \Theta(\alpha) \approx \alpha$ and $\alpha \vee \beta \approx \alpha \circ \beta$. Since $\operatorname{Con} \mathbf{X}$ is distributive we can find a 0,1-preserving embedding of $\operatorname{Con} \mathbf{X}$ into a boolean lattice $\mathbf{B}$. $\mathbf{B}$ has a unique expansion to a boolean algebra which can also denote $\mathbf{B}$. Let $\mathbf{B}^*$ be boolean space which is the dual of this boolean algebra. Since $\mathscr{K}$ is nontrivial, it contains an algebra $\mathbf{A}$ of more than one element and $\operatorname{Con} \mathbf{A} \vDash \varphi$. Let $\mathbf{A}[\mathbf{B}^*]$ be the boolean power of $\mathbf{A}$ that is determined by $\mathbf{B}^*$. That is, $\mathbf{A}[\mathbf{B}^*]$ is the algebra of continuous functions from $\mathbf{B}^*$ to the discrete algebra $\mathbf{A}$. $\mathbf{A}[\mathbf{B}^*]$ is a subdirect power of $\mathbf{A}$ so it is a member of $\mathscr{K}$. For each clopen subject $J \subseteq \mathbf{B}^*$ we write η_J for the kernel of the projection onto the coordinates in J. The collection of all such η_J's form a boolean sublattice of $\operatorname{Con} \mathbf{A}[\mathbf{B}^*]$ isomorphic to $\mathbf{B}$ which contains the least and largest congruences on $\mathbf{A}[\mathbf{B}^*]$. The embedding of $\operatorname{Con} \mathbf{X}$ into $\mathbf{B}$ followed by the isomorphism from $\mathbf{B}$ to the sublattice of $\operatorname{Con} \mathbf{A}[\mathbf{B}^*]$ consisting of the η_J's allows us to identify Con $\mathbf{X}$ with a 0,1-sublattice of Con $\mathbf{A}[\mathbf{B}^*]$. Further, $\Theta_{\mathscr{X}}(\eta_J) = \Theta(\eta_J) = \eta_J$ since all the η_J are projection congruences. Also, $\eta_J \vee \eta_{J'} = \eta_J \circ \eta_{J'}$. It follows that the collection of η_J's that correspond to elements of Con $\mathbf{X}$ form a subalgebra of Con $\mathbf{A}[\mathbf{B}^*]$ (in the sense of the last paragraph) that is isomorphic to Con $\mathbf{X}$ (considered as an algebra). Since Con $\mathbf{A}[\mathbf{B}^*] \models \varphi$ and φ is universal, Con $\mathbf{X}$ also satifies φ. $\mathbf{X}$ and $\mathscr{V}$ were arbitrary, so every arithmetical variety satisfies φ.

Suppose that a certain congruence condition is expressible by a universally quantified formula φ whose only nonlogical symbols are among 0, 1, Θ, $\Theta_{\mathscr{X}}$, $\vee$, $\wedge$, and $\circ$ and that this congruence condition is satisfied by a nontrivial quasivariety. Suppose also that this congruence condition implies a nontrivial Mal'cev condition. (A Mal'cev condition is trivial if it holds in every variety.) We wish to show that some of the terms in this Mal'cev condition must have at least three variables. For this we will use Pixley's characterization of arithmetical varieties: a variety $\mathscr{V}$ is arithmetical if and only if $\mathscr{V}$ has a term $m(x, y, z)$ satisfying $m(x, x, y) = m(y, x, y) = m(y, x, x) = y$. Now let $\mathscr{W}$ denote the variety of all algebras whose only basic operation is a ternary operation m which satisfies the equations just listed. $\mathscr{W}$ is an arithmetical variety. Since φ is universally quantified and satisfied by a nontrivial quasivariety it is satisfied by $\mathscr{W}$. This congruence condition implies a nontrivial Mal'cev condition for $\mathscr{W}$. Since the Mal'cev condition is nontrivial it must involve a term different from a trivial projection operation. Now, it is an easy matter to check that in the variety $\mathscr{W}$ that we have defined the only terms that are not equal in $\mathscr{W}$ to a projection operation have at least three variables. This verifies our claim that any universally quantified congruence condition that is satisfied by some nontrivial quasivariety and contains only operations among 0, 1, Θ, $\Theta_{\mathscr{X}}$, $\vee$, $\wedge$, and $\circ$ cannot imply a nontrivial Mal'cev condition in which all terms depend on only two variables.

It is possible to improve condition (c) of Theorem 2.2. This is essentially the content of the next result.

THEOREM 2.3. *The following conditions are equivalent for a variety* $\mathscr{V}$:

(a) $\mathscr{V}$ *is congruence modular.*

(b) $\mathscr{V}_2 = \mathbf{HSP}(\mathbf{F}_{\mathscr{V}}(2))$ *is congruence modular.*

(c) *The subalgebra* $\mathbf{S}$ *of* $\mathbf{F}_{\mathscr{V}}(u, v) \times \mathbf{F}_{\mathscr{V}}(u, v)$ *that is generated by* $\{(u, u), (u, v), (v, u), (v, v)\}$ *is congruence modular.*

Proof. Clearly (a) implies (b) which in turn implies (c). We only need to show that (c) implies (a). What we will actually do is show that (c) implies Theorem 2.1(b).

Consider the homomorphism $\lambda_0 : \mathbf{F}_{\mathscr{V}}(w, x, y, z) \to \mathbf{F}_{\mathscr{V}}(u, v) : w, x \mapsto u$, and $y, z \mapsto v$ and the homomorphism $\lambda_1 : \mathbf{F}_{\mathscr{V}}(w, x, y, z) \to \mathbf{F}_{\mathscr{V}}(u, v) : w, z \mapsto u$, and $x, y \mapsto v$. These yield a homomorphism $\lambda_0 \times \lambda_1 : \mathbf{F}_{\mathscr{V}}(w, x, y, z) \to \mathbf{F}_{\mathscr{V}}(u, v) \times$

$\mathbf{F}_{\mathcal{V}}(u, v) : w \mapsto (u, u), x \mapsto (u, v), y \mapsto (v, v), z \mapsto (v, u)$. The image of $\lambda_0 \times \lambda_1$ is precisely $\mathbf{S}$. The kernel of $\lambda_0 \times \lambda_1$ is the intersection of the kernels of the λ_i which is $\Theta((w, x), (y, z)) \wedge \Theta((w, z), (x, y)) = \delta$. Let $\gamma = \Theta((w, x), (y, z))$, $\beta = \Theta((w, z), (x, y))$ and let $\alpha = \delta \wedge \Theta(x, y)$. Since $\mathbf{S} \cong \mathbf{F}_{\mathcal{V}}(w, x, y, z)/\delta$ is congruence modular, the interval in $\operatorname{Con} \mathbf{F}_{\mathcal{V}}(w, x, y, z)$ above δ is modular. This interval includes α, β and γ, so $\beta = \beta \wedge (\alpha \vee \gamma) = \alpha \vee (\beta \wedge \gamma) = \alpha$. This shows that $(w, z) \in \alpha = \delta \vee \Theta(x, y)$. Hence for some n there are elements $p_0 = w, p_1, \ldots, p_n = z$ of $\mathbf{F}_{\mathcal{V}}(w, x, y, z)$ such that $(p_i, p_{i+1}) \in \delta$ for i even and $(p_i, p_{i+1}) \in \Theta(x, y)$ for i odd. Let $m_0(w, x, y, z), \ldots, m_n(w, x, y, z)$ be terms representing $p_0(w, x, y, z), \ldots, p_n(w, x, y, z)$. Since all the p_i are congruent modulo $\theta = \Theta((w, z), (x, y))$, we get that $w = m_0(w, x, y, z) \theta m_i(w, x, y, z) \theta m_i(w, y, y, w)$. Since θ is trivial when restricted to the subalgebra of $\mathbf{F}_{\mathcal{V}}(w, x, y, z)$ generated by w and y it follows that $m_i(w, y, y, w) = w$ in $\mathbf{F}_{\mathcal{V}}(w, x, y, z)$. It follows that $m_i(w, y, y, w) \approx w$ is an equation of $\mathcal{V}$. The other equations of 2.1(b) can be verified with similar arguments. $\square$

It is not a corollary to this theorem, but it is not too hard to show that $\mathcal{V}$ is congruence distributive if and only if $\mathcal{V}_2$ of part (b) is congruence distributive if and only if $\mathbf{S}$ of part (c) is congruence distributive. Also, $\mathcal{V}$ is congruence permutable if and only if $\mathcal{V}_2$ is if and only if the subalgebra of $\mathbf{S}$ generated by (u, u), (u, v) and (v, v) is congruence permutable.

Of course it is impossible to replace $\mathbf{F}_{\mathcal{V}}(2)$ by $\mathbf{F}_{\mathcal{V}}(1)$ in condition (b) of the last theorem, since there are non-modular varieties in which every basic operation is idempotent (for example, the variety of sets or the variety of semilattices). In such a variety $\mathcal{V}_1 = \mathbf{HSP}(\mathbf{F}_{\mathcal{V}}(1))$ is a trivial (hence congruence modular) variety even though $\mathcal{V}$ is not.

The equivalence of 2.3(a) and 2.3(b) is what interests us. It implies that if the free algebra on two generators in $\mathcal{V}$ is finite, then $\mathcal{V}$ is congruence modular if and only if a certain finitely generated subvariety is. Now that we can focus on a finitely generated variety we may bring the techniques of tame congruence theory to bear.

3. Congruence semimodularity

DEFINITION 3.1. A lattice $\mathbf{L}$ is *upper semimodular* if whenever x, y and z are elements of L and $x \prec y$ and $x \vee z \neq y \vee z$ we have $x \vee z \prec y \vee z$. $\mathbf{L}$ is *lower semimodular* if the dual condition holds. That is, $\mathbf{L}$ is lower semimodular if whenever x, y and z are elements of L and $y \prec x$ and $x \wedge z \neq y \wedge z$ we have $y \wedge z \prec x \wedge z$.

Both upper and lower semimodularity are implied by modularity. Conversely, in the presence of certain finiteness conditions, the conjunction of upper and lower semimodularity implies modularity. See Theorem 3.6 of [2] for a proof of this under fairly weak finiteness hypotheses. The next theorem is well known and much weaker than the result proved in [2].

THEOREM 3.2. *If a lattice of finite height is both upper semimodular and lower semimodular, then it is modular.* □

From this we immediately get:

COROLLARY 3.3. *If* $\mathbf{F}_{\mathscr{V}}(4)$ *is finite and* $\mathscr{V}$ *is congruence upper semimodular and congruence lower semimodular, then* $\mathscr{V}$ *is congruence modular.*

Proof. This follows from the last theorem and Theorem 2.1(d). □

What is somewhat surprising is that the last corollary remains true if we delete the hypothesis that $\mathscr{V}$ is congruence upper semimodular, but is false if we instead delete the hypothesis that $\mathscr{V}$ is congruence lower semimodular.

THEOREM 3.4. *Let* α, β *and* γ *be congruences on a finite algebra* $\mathbf{A}$ *that satisfy* $\alpha \vee \gamma = \beta \vee \gamma$, $\alpha \wedge \gamma = \beta \wedge \gamma$ *and* $\alpha < \beta$.

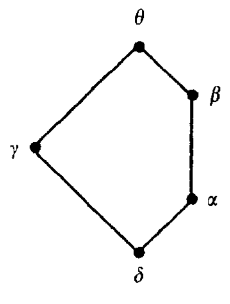
(a) *If* $\mathbf{A}$ *is congruence upper semimodular, then* $\operatorname{typ}\{\alpha, \beta\} \subseteq \{\mathbf{1}, \mathbf{5}\}$.

(b) *If* $\mathbf{A}$ *is congruence lower semimodular, then* $\operatorname{typ}\{\alpha, \beta\} = \{\mathbf{1}\}$.

Proof. First, let us assume that $\mathbf{A}$ is congruence upper semimodular and that $\operatorname{typ}\{\alpha, \beta\} \nsubseteq \{\mathbf{1}, \mathbf{5}\}$. Let $\theta = \alpha \vee \gamma = \beta \vee \gamma$ and let $\delta = \alpha \wedge \gamma = \beta \wedge \gamma$. We may assume that the interval $I[\delta, \theta]$ is a minimal interval (with respect to inclusion) in Con $\mathbf{A}$ that supports our assumptions and the hypotheses of this lemma.

By our assumption that $\operatorname{type}\{\alpha, \beta\} \nsubseteq \{\mathbf{1}, \mathbf{5}\}$, we may find α' and β' in Con $\mathbf{A}$ such that $\alpha < \alpha' \prec \beta' < \beta$ and $\operatorname{typ}(\alpha', \beta') \notin \{\mathbf{1}, \mathbf{5}\}$. Changing notation so that $\alpha = \alpha'$ and $\beta = \beta'$ we may assume that $\alpha \prec \beta$.

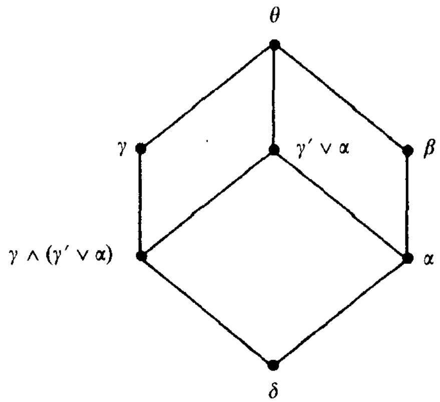
We have the following sublattice in Con $\mathbf{A}$:



It follows from our assumption of upper semimodularity that since $\alpha = \delta \vee \alpha \neq \gamma \vee \alpha = \theta$ and $\alpha \not\prec \theta$ we must have $\delta \not\prec \gamma$. Hence, there is a congruence γ' such that $\delta \prec \gamma' < \gamma$. By semimodularity, $\beta \prec \gamma' \vee \beta$.

We will use our minimality assumption on $I[\delta, \theta]$ and upper semimodularity to show that $\gamma' \vee \beta = \theta$. First, if $\gamma \wedge (\gamma' \vee \beta) = \gamma'$ we have a sublattice of Con **A** consisting of the congruences $\delta_1 = \gamma'$, $\alpha_1 = \alpha \vee \gamma'$, $\beta_1 = \beta \vee \gamma'$, $\gamma_1 = \gamma$ and $\theta_1 = \theta$ as the reader can easily check. Also, by upper semimodularity, $\alpha_1 \prec \beta_1$. Since the intervals $I[\alpha_1, \beta_1]$ and $I[\alpha, \beta]$ are perspective covering quotients, it follows from Lemma 6.2 of [6] that $\operatorname{typ}(\alpha_1, \beta_1) = \operatorname{typ}(\alpha, \beta) \notin \{\mathbf{1}, \mathbf{5}\}$. We conclude that $I[\delta_1, \theta_1]$ is a proper subinterval of $I[\delta, \theta]$ that satisfies all of our earlier assumptions. Of course, this is a contradiction. Hence, we cannot have $\gamma \wedge (\gamma' \vee \beta) = \gamma'$. Now, let $\delta_2 = \delta$, $\alpha_2 = \alpha$, $\beta_2 = \beta$, $\gamma_2 = \gamma \wedge (\gamma' \vee \beta)$ and $\theta_2 = (\gamma' \vee \beta)$. These elements form a sublattice of Con **A** which (one may easily check) satisfies all of our earlier assumptions and this sublattice is contained in a subinterval of $I[\delta, \theta]$. This forces $\theta = \theta_2$; that is, $\theta = \gamma' \vee \beta$.

The condition that $\theta = \gamma' \vee \beta$ and the conditions that $\theta = \alpha \vee \gamma = \beta \vee \gamma$ and $\delta = \alpha \wedge \gamma = \beta \wedge \gamma$ imply that Con **A** has the following sublattice:



In this sublattice $\alpha \prec \beta$, $\beta \prec \theta$, $\alpha \prec (\gamma' \vee \alpha)$ and $(\gamma' \vee \alpha) \prec \theta$. The coverings are guaranteed by upper semimodularity.

By perspectivity, $\operatorname{typ}(\gamma' \vee \alpha, \theta) = \operatorname{typ}(\alpha, \beta) \notin \{\mathbf{1}, \mathbf{5}\}$. This contradicts Lemma 6.3 of [6]. Our conclusion is that (a) holds.

For part (b) notice that our arguments can be dualized completely except for our invocation of Lemma 6.3 of [6]. At this point of the argument we have a lattice which is the dual of the lattice in our last diagram. Now, Lemma 6.4 of [6] can serve in place of Lemma 6.3. This completes part (b). $\square$

COROLLARY 3.5. *Let* **A** *be a finite algebra.*

(a) *If* Con **A** *is upper semimodular and* typ$\{\mathbf{A}\} \cap \{\mathbf{1}, \mathbf{5}\} = \phi$, *then* Con **A** *is modular.*

(b) *If* Con **A** *is lower semimodular and* typ$\{\mathbf{A}\} \cap \{\mathbf{1}\} = \phi$, *then* Con **A** *is modular.*

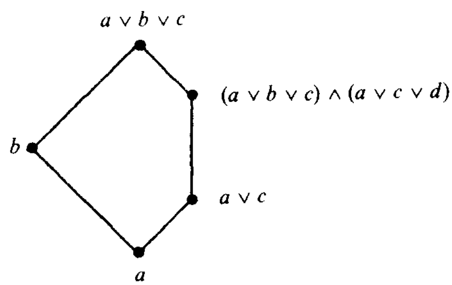
Proof. Part (a) of the theorem shows that if Con **A** is upper semimodular and typ$\{\mathbf{A}\} \cap \{\mathbf{1}, \mathbf{5}\} = \phi$, then whenever $\alpha \vee \gamma = \beta \vee \gamma$, $\alpha \wedge \gamma = \beta \wedge \gamma$ and $\alpha \leqslant \beta$ we have $\alpha = \beta$. That is, Con **A** has no sublattice isomorphic to the 5-element non-modular lattice $\mathbf{N}_5$. It is well known that this is equivalent to the condition that Con **A** is modular. This verifies part (a) of the corollary. Similarly, part (b) of the theorem implies part (b) of the corollary. □

The authors of [6] point out that if a locally finite variety $\mathscr{V}$ is congruence n-permutable for some n, or if there is a nontrivial lattice identity satisfied by all the congruence lattices of algebras in $\mathscr{V}$, then typ$\{\mathscr{V}\} \cap \{\mathbf{1}, \mathbf{5}\} = \phi$. In such a variety, the congruence lattice of a finite algebra is upper semimodular iff it is lower semimodular iff it is modular.

LEMMA 3.6. *The image of a complete homomorphism from a complete upper semimodular lattice is upper semimodular. Any convex sublattice of an upper semimodular lattice is upper semimodular. Any subdirect product of upper semimodular lattices is upper semimodular.*

By duality this lemma holds if we change "upper" to "lower" throughout.

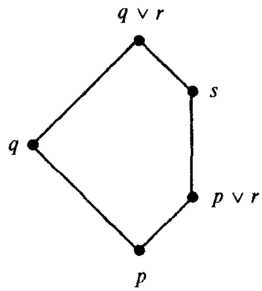
Proof. Suppose that $\lambda : \mathbf{L} \to \mathbf{K}$ is a complete lattice homomorphism from the complete lattice **L** onto the lattice **K** and that **L** is upper semimodular. Assume that **K** fails to be upper semimodular. Then we can find $w, x, y, z \in K$ such that $x \prec y$ and $x \vee z < w < y \vee z$. Choose $a, b, c, d \in L$ such that b is the least element of **L** satisfying $\lambda(b) = y$, a is the greatest element of **L** satisfying $\lambda(a) = x$ and $a \leq b$ and such that $\lambda(c) = z$, $\lambda(d) = w$. The reader can easily verify that the following is a diagram of a sublattice of **L**:



Since **L** is upper semimodular and $a \vee (a \vee c) \not\prec b \vee (a \vee c)$ we must have $a \not\prec b$. Hence, there is an element e such that $a < e < b$. By our choice of a and b, $\lambda(a) < \lambda(e) < \lambda(b)$ or $x < \lambda(e) < y$. This contradicts $x \prec y$.

For the second statement of this lemma, notice that any failure of upper semimodularity in a convex sublattice of a lattice is a failure in the whole lattice.

For the last statement, suppose that **J** is not an upper semimodular lattice. Then **J** has elements p, q, r and s such that $p \prec q$ and $p \vee r < s < q \vee r$. That is, **J** has a sublattice:



where $p \prec q$. The image of this pentagon under any surjective homomorphism φ prevents the image of **J** from being upper semimodular unless we have at least $\varphi(s) = \varphi(p \vee r)$. Hence, s and $p \vee r$ cannot be separated by homomorphisms onto upper semimodular lattices. It follows that **J** cannot be represented as a subdirect product of upper semimodular lattices. Since **J** was an arbitrarily chosen lattice failing upper semimodularity, every subdirect product of upper semimodular lattices is upper semimodular. □

The result of Corollary 3.5 raises the question of whether type **1** can occur in a locally finite congruence lower semimodular variety. The next result shows that the answer is no.

THEOREM 3.7. *Let $\mathscr{V}$ be locally finite and congruence lower semimodular. Then* $\operatorname{typ}\{\mathscr{V}\} \cap \{\mathbf{1}, \mathbf{5}\} = \phi$.

Proof. It will suffice to prove that $\mathbf{1} \notin \operatorname{typ}\{\mathscr{V}\}$. Then $\mathbf{F}_{\mathscr{V}}(4)$ is a finite algebra for which the hypotheses of Corollary 3.5 hold, so Con $\mathbf{F}_{\mathscr{V}}(4)$ is modular. By Theorem 2.1, $\mathscr{V}$ is congruence modular. This is enough to conclude that $\mathbf{5} \notin \operatorname{typ}\{\mathscr{V}\}$ as Theorem 8.5 of [6] proves that $\operatorname{typ}\{\mathscr{V}\} \subseteq \{\mathbf{2}, \mathbf{3}, \mathbf{4}\}$ for any locally finite, congruence modular variety. (On the other hand, it is possible to mimic the rest of our proof for type **5** instead of type **1** and show directly that $\mathbf{5} \notin \operatorname{typ}\{\mathscr{V}\}$.)

Now, suppose that $\mathbf{1} \in \mathrm{typ}\{\mathcal{V}\}$. Then there is a finite algebra $\mathbf{A} \in \mathcal{V}$ with a minimal congruence β such that $\mathrm{typ}(0, \beta) = \mathbf{1}$. Let U be a $\langle 0, \beta \rangle$-minimal set and let N be a trace of U. The normally-indexed algebra that $\mathbf{A}$ induces on N, $\mathbf{AI}_N$ (defined in 6.12 of [6]), is polynomially equivalent to a G-set. By Theorem 6.17 of [6], for every $C \in \mathcal{V}(\mathbf{AI}_N)$ there is an $\mathbf{A}' \in \mathcal{V}$, a congruence $\beta' \in \mathrm{Con}\,\mathbf{A}'$ and a complete homomorphism from $I[0, \beta']$ onto $\mathrm{Con}\,\mathbf{C}$. Since $\mathcal{V}$ is congruence lower semimodular we can use (the dual of) Lemma 3.6 to conclude that every $\mathbf{C}$ in $\mathcal{V}(\mathbf{AI}_N)$ is congruence lower semimodular.

Now, choose $\mathbf{C}$ in $\mathcal{V}(\mathbf{AI}_N)$ so that it is polynomially equivalent to the G-set with four 1-element orbits. Con $\mathbf{C}$ is isomorphic to Π_4, the lattice of partitions of a 4-element set. However, this lattice is not lower semimodular. For example, if the universe of $\mathbf{C}$ is $C = \{w, x, y, z\}$ and we denote certain congruences $\pi_\alpha = \theta(y, z)$, $\pi_\beta = \theta((w, x), (y, z))$ and $\pi_\gamma = \theta((w, y), (x, z))$, then

$$\pi_\gamma \prec 1 \quad \text{and} \quad 0 = \pi_\gamma \wedge \pi_\beta < \pi_\alpha < 1 \wedge \pi_\beta = \pi_\beta$$

which is a failure of lower semimodularity. This contradiction shows that $\mathbf{1} \notin \mathrm{typ}\{\mathcal{V}\}$. As we have already shown, this conclusion suffices to prove the theorem. □

We can now prove our main result.

THEOREM 3.8. *If $\mathcal{V}$ is congruence lower semimodular and $\mathbf{F}_{\mathcal{V}}(2)$ is finite, then $\mathcal{V}$ is congruence modular.*

Proof. By Theorem 2.3, $\mathcal{V}$ is congruence modular if and only if $\mathcal{V}_2 = \mathbf{HSP}(\mathbf{F}_{\mathcal{V}}(2))$ is. Now, $\mathcal{V}_2$ is a locally finite variety which is congruence lower semimodular (since it is a subvariety of $\mathcal{V}$). By Theorem 3.7, $\mathrm{typ}\{\mathcal{V}_2\} \cap \{\mathbf{1}, \mathbf{5}\} = \phi$. Now, Corollary 3.5 shows that $\mathcal{V}_2$ is congruence modular. This completes the proof. □

We cannot hope to replace "lower" by "upper" in Theorem 3.8. The variety $\mathcal{U}$ of sets is congruence upper semimodular, non-modular and $\mathrm{type}\{\mathcal{U}\} = \{\mathbf{1}\}$. The variety $\mathcal{S}$ of semilattices is congruence upper semimodular, non-modular and $\mathrm{typ}\{\mathcal{S}\} = \{\mathbf{5}\}$. The best we can hope for is the next result. Its proof is similar to the proof of Theorem 3.8.

THEOREM 3.9. *If $\mathcal{V}$ is congruence upper semimodular, $\mathbf{F}_{\mathcal{V}}(2)$ is finite and $\mathrm{typ}\{\mathbf{HSP}(\mathbf{F}_{\mathcal{V}}(2))\} \cap \{\mathbf{1}, \mathbf{5}\} = \phi$, then $\mathcal{V}$ is congruence modular.* □

COROLLARY 3.10. *If $\mathscr{V}$ is a variety such that $\mathbf{F}_{\mathscr{V}}(2)$ is finite, then the following conditions are equivalent*:

(a) *$\mathscr{V}$ is congruence upper semimodular and congruence join-semidistributive.*

(b) *$\mathscr{V}$ is congruence lower semimodular and congruence join-semidistributive.*

(c) *$\mathscr{V}$ is congruence lower semimodular and congruence meet-semidistributive.*

(d) *$\mathscr{V}$ is congruence distributive.*

Proof. Condition (d) clearly implies all the other conditions. We need to show that the other conditions imply (d). By Theorem 3.8, the conditions in (b) imply that $\mathscr{V}$ is congruence modular and congruence join-semidistributive. But any modular, join-semidistributive lattice is distributive. Hence, $\mathscr{V}$ is congruence distributive and (b) implies (d). A similar argument shows that (c) implies (d).

We will be finished if we show that (a) implies (d). By Theorem 9.11 of [6], the fact that $\mathscr{V}_2 = \mathbf{HSP}(\mathbf{F}_{\mathscr{V}}(2))$ is congruence join-semidistributive forces $\mathrm{typ}\{\mathscr{V}_2\} \cap \{\mathbf{1}, \mathbf{2}, \mathbf{5}\} = \phi$. We can use Theorem 3.9 to conclude that $\mathscr{V}$ is at least congruence modular and, since $\mathscr{V}$ is join-semidistributive, we even get that $\mathscr{V}$ is congruence distributive. $\square$

It is not true that the conditions in Corollary 3.10 are equivalent to the condition that $\mathscr{V}$ is both congruence upper semimodular and congruence meet-semidistributive. The variety of semilattices is an example of a locally finite variety that is both congruence upper semimodular and congruence meet-semidistributive, but it fails to be congruence distributive.

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