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SHORT NOTE

The Class of Prime Semilattices is Not Finitely Axiomatizable

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In [2], R. Balbes defined a *prime semilattice* to be a meet semilattice with the property that whenever $x \not\leq y$ there is a prime filter containing x and not y. Balbes showed that a semilattice is prime if and only if the meet operation distributes over all existing finite joins. B. M. Schein stated in [4] that the class of prime semilattices is not finitely axiomatizable, but gave no proof. In [3], a problem (due to a referee of that paper) was posed which suggested a possible finite axiomatization of the class of prime semilattices. This suggestion was followed up in [5]; here Schein's statement was labeled a conjecture and an attempt was made to disprove it. The authors of [5] showed that the class of *finite* prime semilattices is finitely axiomatizable relative to the class of *finite* semilattices, but they add that "we are unable to prove [that the class of all prime semilattices is finitely axiomatizable], although we suspect that this may be so, in contrast to Schein's conjecture." Later, in [1], it was shown that the class of *well founded* prime semilattices is finitely axiomatizable relative to the class of *well founded* semilattices. In this note we verify Schein's statement by proving that the class of prime semilattices is *not* finitely axiomatizable.

Let D_n denote a first–order sentence which asserts that meet distributes over all existing n–ary joins. That is, if $y_1 \vee \cdots \vee y_n$ exists, then for each x the join $(x \wedge y_1) \vee \cdots \vee (x \wedge y_n)$ exists and equals $x \wedge (y_1 \vee \cdots \vee y_n)$. A meet semilattice is prime if and only if it satisfies D_n for all finite $n \geq 1$. The class of prime semilattices is finitely axiomatizable if and only if it is axiomatizable by the laws for meet semilattices together with finitely many of the D_n's. Since the D_n's get stronger as n increases, it suffices for us to prove that $D_n \not\Longrightarrow D_{n+1}$ for any n.

Theorem 1. *There is a meet semilattice which satisfies D_n but not D_{n+1}.*

Proof. Let $[0,1]$ denote the unit interval of the real numbers considered as a meet semilattice. We define two subsemilattices of $[0,1]^{n+1}$. The "top" part will be T, the subsemilattice $(0,1]^{n+1}$. The "bottom" will be the subsemilattice

$$B = \{(x_1, ..., x_{n+1}) \in [0,1]^{n+1} \mid \text{ at most one } x_i \text{ is nonzero}\}.$$

Our semilattice will be $E = T \cup B \cup \{s\}$ where s is an additional element. The order is as follows: $T \cup B$ has the order it inherits as a subsemilattice of $[0,1]^{n+1}$. We define s to be below every element of T and incomparable with every element of B except that $(0,\ldots,0) < s$. This order is a meet semilattice order. We let z_i denote the element (of B) which has a 1 in the i–th position and zeros elsewhere.

The following claims can be easily checked and they establish what is needed.

(i) $T \cup B$ is a subsemilattice of E and of $[0,1]^{n+1}$, and any join in E of elements from $T \cup B$ agrees with the join in $[0,1]^{n+1}$. (Hence $T \cup B$ satisfies all D_k.)

(ii) Any "nontrivial" join in E of elements from $B \cup \{s\}$ requires at least $n+1$ joinands from B. (Here a join of elements is "trivial" if the elements form a chain and "nontrivial" otherwise.)

(iii) Any "nontrivial" join in E is unchanged when s is deleted as a joinand.

(iv) $\bigvee_{i=1}^{n+1} z_i$ exists, but $s \wedge (\bigvee z_i) = s \neq (0,\ldots,0) = \bigvee (s \wedge z_i)$.

Item (iv) shows that E fails D_{n+1}. Assume that E fails D_n. Then there is a join $y_1 \vee \cdots \vee y_n$ and an element x such that $\bigvee(x \wedge y_i)$ fails to equal $x \wedge (\bigvee y_i)$. Replacing n by some m with $1 < m \leq n$ if necessary we may assume that $\bigvee y_i$ is a "nontrivial" irredundant join. In particular, by item (iii), we may assume that $s \notin \{y_1,\ldots,y_m\}$. Since $\{x, y_1, \ldots, y_m\}$ produces a failure of D_n, $E - \{s\} = T \cup B$ satisfies all D_k, and $s \notin \{y_1,\ldots,y_m\}$, we are forced to have $x = s$. We cannot have $\{y_1,\ldots,y_m\} \subseteq B \cup \{s\}$ since $\bigvee y_i$ must be "nontrivial" in order to produce a failure of the distributive law, but according to item (ii) the number of joinands is too few to be a "nontrivial" join when all joinands come from $B \cup \{s\}$. Therefore some y_i, say y_1, is in T. Since s is below all elements in T we have $s < y_1 \leq \bigvee y_i$, so $s \wedge (y_1 \vee \cdots \vee y_m) = s$. Furthermore, $s \wedge y_i \leq s$ and $s \wedge y_1 = s$. This proves that $\bigvee(s \wedge y_i)$ $(= s)$ exists and equals $s \wedge (\bigvee y_i)$. Our purported failure of D_n is not a failure after all. Thus E satisfies D_n and fails D_{n+1}. ■

References

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