GENERATING SINGULAR TRANSFORMATIONS

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Abstract. We compute the singular rank and the idempotent rank of those subsemigroups of the full transformation semigroup that contain all singular transformations.

1. Introduction

The *rank* of a semigroup is the cardinality of a least-size generating set. Let T_n denote the semigroup of all transformations of $V = \{1, \ldots, n\}$, let $S_n \subseteq T_n$ be the group of permutations in T_n, and let $\mathrm{Sing}_n = T_n - S_n$ be the subsemigroup of T_n consisting of the singular transformations. Any subsemigroup $S < T_n$ containing Sing_n is the disjoint union of Sing_n and some subgroup $G < S_n$ of permutations. Singular transformations are of no use in generating permutations, so a least-size generating set for $S = G \cup \mathrm{Sing}_n$ must be a disjoint union $M \cup N$ where $M \subseteq G$ is a least-size generating set for G, and $N \subseteq \mathrm{Sing}_n$ is a least-size set for which $\mathrm{Sing}_n \subseteq \langle G \cup N \rangle$. Therefore, we define the *singular rank* of S to be the cardinality of a least-size set $N \subseteq \mathrm{Sing}_n$ for which $\mathrm{Sing}_n \subseteq \langle G \cup N \rangle$. In this note we explain how to compute the singular rank of any subsemigroup $S < T_n$ that contains Sing_n.

The *idempotent rank* of a semigroup that is generated by idempotents is the cardinality of a least-size generating set of idempotents (cf. [4]). The semigroup Sing_n is idempotent-generated, and its idempotent rank is $n(n-1)/2$ (cf. [2, 3]). This notion of "idempotent rank" has a natural extension to subsemigroups $S < T_n$ generated by idempotents and permutations: it is the cardinality of a least-size set of idempotents needed in a generating set that consists of idempotents and permutations. Since Sing_n is idempotent-generated, any semigroup of the form $S = G \cup \mathrm{Sing}_n$ has an idempotent rank in this sense, and in this note we explain how to compute it. A special case of this computation is handled by Theorem 3.18 of [6], which proves that if $\varepsilon \in T_n$ is an idempotent of rank $n-1$, then $\mathrm{Sing}_n \subseteq \langle G \cup \{\varepsilon\} \rangle$ if and only if G acts weakly doubly transitively on $V = \{1, \ldots, n\}$. Consequently the idempotent rank of $G \cup \mathrm{Sing}_n$ is 1 if and only if G acts weakly doubly transitively on V.

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There is an alternate way to view the problems of computing the singular rank and idempotent rank of a semigroup of the form $S = G \cup \mathrm{Sing}_n$. For a group $G < S_n$ and a set $N \subseteq \mathrm{Sing}_n$ let $N^G = \{n^g = g^{-1}ng \mid n \in N, g \in G\}$ be the set of G-conjugates of elements of N. It is easy to see that $\langle N^G \rangle \subseteq \langle G \cup N \rangle$ and that these semigroups contain the same idempotents. Since Sing_n is idempotent-generated, it follows that $\mathrm{Sing}_n \subseteq \langle G \cup N \rangle$ if and only if $\langle N^G \rangle = \mathrm{Sing}_n$. Thus, the singular rank (idempotent rank) of a semigroup of the form $G \cup \mathrm{Sing}_n$ is the size of a least-size set N of singular transformations (idempotents) such that $\langle N^G \rangle = \mathrm{Sing}_n$. The stated purpose of [5] is to begin an investigation of the following question: Given $G < S_n$, what is a least-size set $N \subseteq \mathrm{Sing}_n$ such that $\langle N^G \rangle = \mathrm{Sing}_n$? Equivalently, what is the singular rank of $G \cup \mathrm{Sing}_n$? This question is not answered in [5], but will be answered here. (The result of [5] is that the singular rank of $G \cup \mathrm{Sing}_n$ is 1 if and only if G acts weakly doubly transitively on V. Thus, the results of [5] and [6] show that the singular rank of $G \cup \mathrm{Sing}_n$ is 1 if and only if the idempotent rank is 1, a fact that is very easy to establish directly. We will find in this paper that for certain groups $G < S_n$ the singular rank and idempotent rank of $G \cup \mathrm{Sing}_n$ differ.)

We introduce more notation to allow us to state our results in a simple way. Fix a subsemigroup $S < T_n$ of the form $G \cup \mathrm{Sing}_n$ with $G < S_n$. Let $E = \{\{1,2\},\{1,3\},\ldots,\{n-1,n\}\}$ be the set of doubletons of V. The pair $\langle V; E\rangle$ is a complete graph. The action of G on V induces an action of G on E according to the rule: $g(\{i,j\}) = \{g(i), g(j)\}$. Let v denote the number of vertex-orbits under the action of G, and let e denote the number of edge-orbits under the induced action.

Theorem 1.1. *If $n > 2$, and $S = G \cup \mathrm{Sing}_n$ with $G < S_n$, then the singular rank of S is e.*

The $n = 2$ case of Theorem 1.1 is covered by the next theorem, since when $n = 2$ all singular transformations are idempotent (so singular rank = idempotent rank).

Theorem 1.2. *The idempotent rank of $S = G \cup \mathrm{Sing}_n$, $G < S_n$, is either e or $e + 1$, with the two cases being distinguished in the following way:*

(i) *it is $e + 1$ if $v = 2$ and all edges connecting the two vertex-orbits belong to the same edge-orbit,*
(ii) *otherwise it is e.*

It follows from the two theorems that if one wants to generate all singular transformations with $G \cup N$ for some least-size set N of singular transformations, then for almost all groups G one can choose the elements of N to be idempotent. In the exceptional case, where $v = 2$ and edges connecting the two vertex-orbits all belong to the same edge-orbit, one can do a little better with non-idempotent transformations. However, it follows from our proof that even in the exceptional case a least-size set N can be constructed where all but one of the members are idempotent.

This paper ends with Theorem 2.9, which describes all pairs (G, I) where $G < S_n$, $I \subseteq \mathrm{Sing}_n$ is a set of idempotents, and $\langle G \cup I \rangle$ contains Sing_n. This extends Theorem 6.3.12 of [1], which describes all such pairs (G, I) with $G = \{1\}$.

2. The Proofs

In this section we prove Theorems 1.1 and 1.2 through a sequence of four lemmas. Lemmas 2.1 and 2.4 provide the necessary lower bounds for our two theorems, while Lemmas 2.6 and 2.8 provide the necessary upper bounds. The usage of the symbols $T_n, S_n, \mathrm{Sing}_n, S, G, N, I, V, E, v$, and e in this section will conform to that of the introduction.

Lemma 2.1. *If $N \subseteq \mathrm{Sing}_n$ and $\langle G \cup N \rangle$ contains Sing_n, then N must contain at least e transformations of rank $n - 1$.*

Proof. In this proof we will use the fact that a transformation in T_n has rank $n - 1$ if and only if it is constant on exactly one edge of the graph $\langle V; E \rangle$.

Assume that $\langle G \cup N \rangle$ contains Sing_n. Choose and fix an edge $\{i, j\} \in E$. Choose a transformation $\alpha \in \mathrm{Sing}_n$ of rank $n - 1$ that is constant on this edge. Since $\alpha \in \langle G \cup N \rangle$, it is possible to express α as $\tau_1 \cdots \tau_{k-1}\tau_k$ with $\tau_i \in G \cup N$. We claim that no generality is lost if we assume that $k \geq 2$, $\tau_{k-1} \in N$, and $\tau_k \in G$. To see this, note that if $\tau_k \notin G$ we can change the representation by adding $\tau_{k+1} = 1 \in G$ to the end of the product. Next, there is no need to ever choose both τ_{k-1} and τ_k from G, for if this happened we could change the representation by replacing the last two transformations by their product. Thus, we may assume that $\tau_k \in G$ and $\tau_{k-1} \notin G$. But since α is not a permutation we cannot have $k = 1$, so $k \geq 2$ and $\tau_{k-1} \in N$.

Since $\alpha = \tau_1 \cdots (\tau_{k-1}\tau_k)$ and both α and $\tau_{k-1}\tau_k$ have rank $n - 1$, it follows that α and $\tau_{k-1}\tau_k$ have the same kernel. Thus, both α and $\tau_{k-1}\tau_k$ are constant on the same edge, which is $\{i, j\}$. This implies that τ_{k-1} has rank $n-1$, and that it is constant on the edge $\{\tau_k(i), \tau_k(j)\}$. Hence, for any edge $\{i, j\}$ there is a transformation $\tau_{k-1} \in N$ of rank $n - 1$ that is constant on an edge $\{\tau_k(i), \tau_k(j)\}$ in the edge-orbit of $\{i, j\}$. It follows that the number of different $\tau_{k-1} \in N$ of rank $n - 1$ is at least as large as the number e of edge-orbits. □

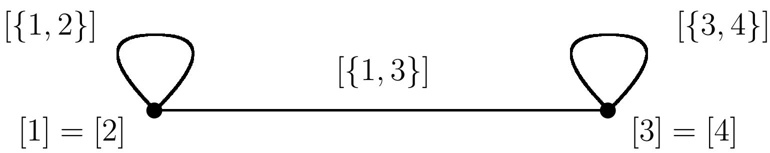
Say that a transformation τ *represents* an edge of $\langle V; E \rangle$ if τ is constant on that edge. A transformation has rank $n-1$ when it represents exactly one edge. What the proof of Lemma 2.1 shows is that if $\langle G \cup N \rangle$ contains Sing_n, then the transformations of rank $n - 1$ in N must represent at least one edge from each edge-orbit, and so there are at least e members of N of rank $n - 1$.

We will use the following notation in the upcoming definition: If a group H acts on a set X and $x \in X$, then the orbit of x is denoted $[x]$. The set of orbits of X under H is denoted X/H. Recall that G acts on both the vertices and edges of the complete graph $\langle V; E \rangle$.

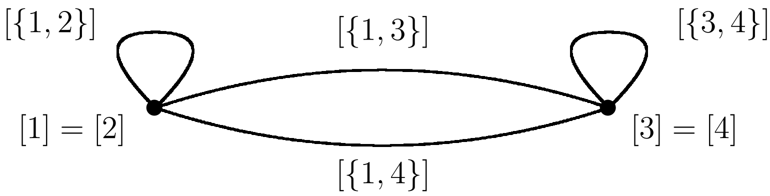
Definition 2.2. The *orbit multigraph* of G is $\langle V/G; E/G\rangle$.

The orbit multigraph of G is the (multigraph-) quotient of $\langle V; E\rangle$ modulo G. It typically has multiple edges and loops. For example, if $[i'] = [i] \neq [j] = [j']$ and $[\{i, j\}] \neq [\{i', j'\}]$, then both $[\{i, j\}]$ and $[\{i', j'\}]$ are edges in E/G that connect $[i]$ to $[j]$, but they are *different* edges. Also, if $\gamma \in G$ and $\gamma(i) \neq i$, then $[\{i, \gamma(i)\}]$ will be a loop on $[i] = [\gamma(i)]$. Of course, multiple loops on the same vertex arise frequently.

Example 2.3. Let $V = \{1, 2, 3, 4\}$ and $G < S_4$ be the group whose elements are $\{1, \alpha, \beta, \alpha\beta\}$ where $\alpha = (1\ 2)$ and $\beta = (3\ 4)$. The vertex-orbits are $[1] = \{1, 2\} = [2]$ and $[3] = \{3, 4\} = [4]$. The edge-orbits are $[\{1, 2\}] = \{\{1, 2\}\}$, $[\{3, 4\}] = \{\{3, 4\}\}$, and $[\{1, 3\}] = \{\{1, 3\}, \{1, 4\}, \{2, 3\}, \{2, 4\}\}$. Thus, the orbit multigraph of G is:



Now replace G with the subgroup $G' < G$ whose elements are $\{1, \alpha\beta\}$. There is no change in the set of vertex-orbits, but there is a change in the edge-orbits. Now they are $[\{1, 2\}] = \{\{1, 2\}\}$, $[\{3, 4\}] = \{\{3, 4\}\}$, $[\{1, 3\}] = \{\{1, 3\}, \{2, 4\}\}$, and $[\{1, 4\}] = \{\{1, 4\}, \{2, 3\}\}$. This means that the orbit multigraph of G' is:



Lemma 2.4. *Assume that $I \subseteq \mathrm{Sing}_n$ is a set of idempotents, and that $\langle G \cup I\rangle$ contains Sing_n. If the orbit multigraph of G has exactly one edge that is not a loop, $[\{k, \ell\}]$, then I must contain at least two idempotents of rank $n - 1$ that represent edges in the orbit $[\{k, \ell\}]$. Hence I contains at least $e + 1$ idempotents of rank $n - 1$.*

Proof. The last claim follows from the first, for if $\mathrm{Sing}_n \subseteq \langle G \cup I\rangle$, then by applying Lemma 2.1 to $N = I$ we see that I has at least one member of rank $n-1$ representing each edge-orbit. Therefore, if I has at least two members of rank $n - 1$ representing edges in the orbit $[\{k, \ell\}]$, then I has at least $e + 1$ members of rank $n - 1$.

To prove the first statement, assume that $\mathrm{Sing}_n \subseteq \langle G \cup I\rangle$ and that I has only one idempotent of rank $n - 1$ that represents an edge in $[\{k, \ell\}]$. By renaming k and ℓ if necessary, no generality is lost in assuming that the idempotent is

$$\varepsilon_{k\to\ell}(x) = \begin{cases} \ell & \text{if } x = k; \\ x & \text{otherwise.} \end{cases}$$

Claim 2.5. *If I' is the set of idempotents in I of rank $n-1$, then $[\ell]$ is closed under all transformations in $G \cup I'$.*

The set $[\ell]$ is closed under all elements of G because $[\ell]$ is an orbit. If $[\ell]$ were not closed under some $\varepsilon \in I'$, then there would be an element $a \in [\ell]$ such that $\varepsilon(a) \in [k]$. Then $\{a, \varepsilon(a)\}$ would be an edge of $[\{k,\ell\}]$ represented by this ε. But $\varepsilon = \varepsilon_{k\to\ell}$ is the only member of I' that represents an edge in $[\{k,\ell\}]$. Since $\varepsilon_{k\to\ell}(x) = x$ on $[\ell]$ there is no $a \in [\ell]$ with $\varepsilon_{k\to\ell}(a) \in [k]$. The claim is verified.

The property described in the claim for elements of $G \cup I'$ is inherited by elements of $\langle G \cup I' \rangle$, but it is not shared by

$$\varepsilon_{\ell\to k}(x) = \begin{cases} k & \text{if } x = \ell; \\ x & \text{otherwise,} \end{cases}$$

Thus $\varepsilon_{\ell\to k} \in \mathrm{Sing}_n \subseteq \langle G \cup I \rangle$ and $\varepsilon_{\ell\to k} \notin \langle G \cup I' \rangle$. This is a contradiction since $\varepsilon_{\ell\to k}$ has rank $n-1$ and all transformations in $\langle G \cup I \rangle$ of rank $n-1$ must lie in $\langle G \cup I' \rangle$. □

Lemma 2.6. *Assume that the orbit multigraph of G has exactly k edges that are not loops. If $k = 1$, then there is a set $I \subseteq \mathrm{Sing}_n$ of idempotents with $|I| = e + 1$ such that $\mathrm{Sing}_n \subseteq \langle G \cup I \rangle$. If $k \neq 1$, then there is a set $I \subseteq \mathrm{Sing}_n$ of idempotents with $|I| = e$ such that $\mathrm{Sing}_n \subseteq \langle G \cup I \rangle$.*

Proof. We first orient the orbit multigraph so that it is strongly connected, allowing edges to be singly-oriented or doubly-oriented. (An oriented graph or multigraph is *strongly connected* if it has a directed path between any two vertices.) We choose our orientation as follows. In the case where $k = 1$ we doubly-orient the edge that is not a loop and orient the loops arbitrarily. This is a strongly connected orientation. If $k \neq 1$, then we singly-orient all edges in such a way that the multigraph is strongly connected. To see that this is possible, note that since the orbit multigraph is the quotient by G of the complete graph $\langle V; E \rangle$ its underlying simple graph is complete. In particular, the orbit multigraph is connected. Hence, when $k = 0$ (and so all edges are loops) there is only one vertex. In this case any orientation makes it strongly connected. When $k > 1$, then since the underlying simple graph is complete there is a cycle that visits each vertex without repeating any edges. If we orient the edges in such a cycle so that it is a directed cycle, then we can orient the remaining edges and loops arbitrarily and we will have a strongly connected multigraph. Now, from each edge-orbit choose and fix exactly one edge-representative $\{i, j\}$. Using this representative, we indicate the fact that $[\{i, j\}]$ is singly-oriented from $[i]$ to $[j]$ with the notation $[i \to j]$. We use $[i \leftrightarrow j]$ to indicate that $[\{i, j\}]$ is doubly-oriented.

Now that we have described a strongly connected orientation of the orbit multigraph, we describe how to pick the set I of idempotents. We select for membership in I exactly one idempotent for each singly-oriented edge: for $[i \to j]$ we choose the unique idempotent $\varepsilon_{i\to j}$ of rank $n-1$ that satisfies $\varepsilon_{i\to j}(i) = j$. For a doubly-oriented edge $[i \leftrightarrow j]$ we choose for membership in I the two idempotents $\varepsilon_{i\to j}$ and

$\varepsilon_{j\to i}$. Note that in the situation when all edges are singly-oriented we have chosen one idempotent for each edge-orbit, so $|I| = e$. In the special situation where the orbit multigraph has exactly one edge $[i \leftrightarrow j]$ which is not a loop we have chosen two idempotents representing the edge-orbit of $\{i, j\}$ and one idempotent for all other edge-orbits, so $|I| = e + 1$.

Now we argue that $\langle G \cup I\rangle$ contains Sing_n. Define a binary relation $\sqsubseteq$ on V according to the rule: $i \sqsubseteq j$ if $i = j$ or $\varepsilon_{i\to j} \in \langle G \cup I\rangle$.

Claim 2.7. $\sqsubseteq$ *equals* $V \times V$.

For each $\varepsilon_{i\to j} \in I$ and each group element $\gamma \in G$ the idempotent $\varepsilon_{\gamma(i)\to\gamma(j)} = \gamma\varepsilon_{i\to j}\gamma^{-1}$ is in $\langle G\cup I\rangle$. Thus, $i \sqsubseteq j$ implies $\gamma(i) \sqsubseteq \gamma(j)$ for every $\gamma \in G$, so the relation $\sqsubseteq$ is invariant under (or preserved by) G. Since the idempotents in I represent every edge-orbit, it follows that for any edge $\{k, \ell\}$ either $k \sqsubseteq \ell$ or $\ell \sqsubseteq k$.

Next we argue that $\sqsubseteq$ is a transitive relation. Suppose that $i \sqsubseteq j$ and $j \sqsubseteq k$. To prove $i \sqsubseteq k$ it suffices to consider only the case where i, j and k are distinct. Therefore we have $\varepsilon_{i\to j}, \varepsilon_{j\to k} \in \langle G \cup I\rangle$. From the last paragraph we have either $\varepsilon_{i\to k} \in \langle G \cup I\rangle$ or $\varepsilon_{k\to i} \in \langle G \cup I\rangle$. In the first case there is nothing left to prove, so assume we have $\varepsilon_{k\to i} \in \langle G \cup I\rangle$. Then since $\varepsilon_{i\to j}, \varepsilon_{j\to k}, \varepsilon_{k\to i} \in \langle G \cup I\rangle$ and

$$\varepsilon_{i\to k} = \varepsilon_{i\to j}\varepsilon_{j\to k}\varepsilon_{k\to i}\varepsilon_{i\to j}\varepsilon_{j\to k}\varepsilon_{k\to i}$$

we are done. Thus, $\sqsubseteq$ is a reflexive transitive relation (i.e., a quasiorder) on V. The results of the first paragraph show that (a) any two elements of V are $\sqsubseteq$-comparable, so the associated partial order is a chain, and (b) the action of G on V is order-preserving.

Our claim that $\sqsubseteq = V \times V$ is the claim that the partial order associated to $\sqsubseteq$ has one element. To prove this, choose $t \in V$ from the top $(\sqsubseteq \cap \sqsupseteq)$-class of the quasiorder and $b \in V$ from the bottom class. Since the orientation of the orbit multigraph is strongly connected, there is a sequence $[t] = [i_1], [j_1] = [i_2], \ldots, [j_p] = [b]$ such that there is an oriented edge $[i_m \to j_m]$ for each m. Hence, $\varepsilon_{i_m\to j_m} \in I$ for each m, and so $i_m \sqsubseteq j_m$ for each m. It is also true that $j_m \sqsubseteq i_{m+1}$ for each m. To see this, note that if $j_m \not\sqsubseteq i_{m+1}$ then since all pairs of elements are $\sqsubseteq$-comparable we must have $i_{m+1} \sqsubseteq j_m$. Since $[j_m] = [i_{m+1}]$ there is a group element $\gamma \in G$ such that $\gamma(i_{m+1}) = j_m$. If the order of γ is r, then $i_{m+1} \sqsubseteq j_m = \gamma(i_{m+1}) \sqsubseteq \gamma^2(i_{m+1}) \sqsubseteq \cdots \sqsubseteq \gamma^r(i_{m+1}) = i_{m+1}$, forcing $j_m \sqsubseteq i_{m+1}$ after all. Altogether this means that $t \sqsubseteq b$, so the associated partial order has one element. This proves the claim.

It follows from the claim and the definition of $\sqsubseteq$ that $\langle G \cup I\rangle$ contains all idempotents of rank $n - 1$. It is proved in [3] that the set of idempotents of rank $n - 1$ generates Sing_n, so we are done. □

Lemma 2.8. *If* $n > 2$, *then there is a set* $N \subseteq \mathrm{Sing}_n$ *with* $|N| = e$ *such that* $\langle G \cup N\rangle$ *contains* Sing_n.

Proof. From Lemma 2.6 it follows that there is such an N consisting entirely of idempotent transformations in all situations except the situation where the orbit multigraph has exactly one edge that is not a loop. It is only in this exceptional case that we must explain how to choose N.

Assume that the edge of the orbit multigraph that is not a loop is the edge $[\{i, j\}]$. Since $n > 2$, at least one of the vertex-orbits has more than one element. Assume that $j \neq k$, but $[j] = [k]$. Since $[i] \neq [j] = [k]$ it follows that $[\{i, j\}] \neq [\{j, k\}]$.

For each edge-orbit $[\{\ell, m\}]$ that is distinct from $[\{i, j\}]$ and $[\{j, k\}]$ choose the single idempotent $\varepsilon_{\ell\to m}$ for membership in a set labeled N'. This set now has $e - 2$ elements. Let N consist of the members of N' together with the non-idempotent transformation

$$\alpha(x) = \begin{cases} j & \text{if } x = i; \\ k & \text{if } x = j; \\ x & \text{otherwise,} \end{cases}$$

and the idempotent $\varepsilon_{j\to i}$. Altogether we have chosen e transformations for N, of which $e - 1$ are idempotent.

From N we can generate $\varepsilon_{i\to j} = \alpha\varepsilon_{j\to i}$ and $\varepsilon_{j\to k} = \varepsilon_{j\to i}\alpha$. This shows that $\langle G \cup N\rangle$ contains all of the idempotents in N together with $\varepsilon_{i\to j}$ and $\varepsilon_{j\to k}$. But this set of idempotents represents every loop of the orbit multigraph and it represents the special edge $[\{i, j\}]$ in both directions, for we now have generated $\varepsilon_{i\to j}$ and we earlier chose $\varepsilon_{j\to i} \in N$. From the proof of Lemma 2.6 we see that we have enough idempotents to guarantee that $\langle G \cup N\rangle$ contains Sing_n. □

Theorem 1.1 follows immediately from Lemmas 2.1 and 2.8. Moreover, these lemmas show that a least-size set $N \subseteq \mathrm{Sing}_n$ for which $\mathrm{Sing}_n \subseteq \langle G \cup N\rangle$ can be chosen so that at most one member fails to be idempotent. Theorem 1.2 follows from Lemmas 2.4 and 2.6.

Theorem 2.9. *Given a subgroup $G < S_n$ and a subset $I \subseteq \mathrm{Sing}_n$ of idempotents, $\langle G \cup I\rangle$ contains Sing_n if and only if $\langle G \cup I'\rangle$ contains Sing_n where I' is the set of elements of I of rank $n - 1$. $\langle G \cup I'\rangle$ contains Sing_n exactly when*

(1) *every edge-orbit contains an edge represented by an element of I'; and*
(2) *the orientation induced on the orbit multigraph by I' is strongly connected.*

Proof. Since the elements of $G \cup \mathrm{Sing}_n$ of rank $\geq n - 1$ are a generating set, and to generate elements of $G \cup \mathrm{Sing}_n$ of rank $\geq n - 1$ one can only use elements of $G \cup I$ of rank $\geq n - 1$, it follows that $\langle G \cup I\rangle$ contains Sing_n if and only if $\langle G \cup I'\rangle$ generates Sing_n.

It follows from the proof of Lemma 2.6 that if conditions (1) and (2) holds, then I' contains sufficiently many idempotents for $\langle G \cup I'\rangle$ to generate Sing_n. On the other hand, if condition (1) fails to hold, then the proof of Lemma 2.1 shows that $\langle G \cup I\rangle$ will not contain Sing_n since the transformations of rank $n - 1$ in $\langle G \cup I\rangle$ represent only edges that lie in edge-orbits of elements of I'. Finally, if (1) holds but (2) fails, then $\langle G \cup I' \rangle$ cannot contain Sing_n. The argument for this is a combination of the arguments for Claims 2.5 and 2.7: If the orientation is not strongly connected, then the quasiorder $\sqsubseteq$ as defined before Claim 2.7 is not $V \times V$, but it is a G-invariant quasiorder whose associated order is a chain of more than one element. Now, using an argument like the one in Claim 2.5, one can show that if T is top $(\sqsubseteq \cap \sqsupseteq)$-class of the quasiorder, then T is closed under all elements of $G \cup I'$, but not under all idempotents of rank $n - 1$. Hence $\mathrm{Sing}_n \not\subseteq \langle G \cup I' \rangle$ when (1) holds and (2) fails. □

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