On the functional completeness of simple tournaments

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Abstract. The theory of multitraces provides a new proof that any simple tournament with more than two elements is functionally complete.

A *tournament* is a finite, directed, complete graph $\langle V; E\rangle$ without multiple edges. Write $x \to y$ to indicate that $x, y \in V$ and $(x, y) \in E$. In this paper tournaments have loops on all vertices, so $x \to x$ for all $x \in V$. Associate to a tournament $\langle V; E\rangle$ an algebra $\langle V; \cdot\rangle$ with the same universe and a binary product defined by $xy = yx = x$ whenever $x \to y$. Such an algebra is also called a tournament.

In [3], P. P. Pálfy applied Rosenberg's Completeness Theorem to prove that every simple tournament is functionally complete. Here we derive the same theorem from the theory of multitraces, [2], which is a part of tame congruence theory, [1].

A finite algebra $\mathbf{A}$ is *functionally complete* if every finitary operation on its universe is a polynomial of the algebra. A *trace* of a finite simple algebra $\mathbf{A}$ is a subset of A that is minimal among subsets $T \subseteq A$ satisfying $|T| > 1$ and $T = e(A)$ for some unary polynomial e satisfying $e(e(x)) = e(x)$. A *multitrace* of a finite simple algebra $\mathbf{A}$ is a subset $M \subseteq A$ such that $M = p(T, T, \ldots, T) = p(T^n)$ for some trace T and some n-ary polynomial p. It is known that if $\mathbf{A}$ is a finite simple algebra and T and T' are traces, then there are unary polynomials f and g such that $f(T) = T'$ and $g(T') = T$, so any trace can be used in the definition of "multitrace". It is also known that if T is a trace and f is a unary polynomial whose restriction to T is nonconstant, then $f(T)$ is another trace.

It is possible to construct an algebra on a trace $T = e(A)$ by equipping T with (the restrictions to T of) all operations of the form $e(p(\mathbf{x}))$, p a polynomial operation of $\mathbf{A}$. The result is called the algebra *induced* on T by $\mathbf{A}$, and is denoted by $\mathbf{A}|_T$. It is shown in [1] that the algebras $\mathbf{A}|_T$ arising from different traces of $\mathbf{A}$ are polynomially equivalent algebras, and that they come in only five types, which are numbered **1-5**. Their polynomial equivalence types are: **1** = simple G-sets, **2** = 1-dimensional vector spaces, **3** = 2-element Boolean algebras, **4** = 2-element lattices, and **5** = 2-element semilattices.

The following specialization of Theorem 3.12 of [2] provides criteria for establishing functional completeness.

Theorem 1. *A finite algebra* $\mathbf{S}$ *is functionally complete if and only if*

(1) $\mathbf{S}$ *is simple of type* **3**, *and*

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(2) *S is a multitrace.*

Lemma 2. *Let $\mathbf{S}$ be a simple tournament with more than two elements.*

(1) *If N is a subset of S and $1 < |N| < |S|$, then there exist $x, y \in N$ and $z \in S - N$ such that $x \to z \to y$.*
(2) *$\mathbf{S}$ contains a multitrace M and an element z such that $M \cup \{z\}$ is strongly connected and $|M \cup \{z\}| > 1$. Moreover $\mathbf{S}$ has type $\mathbf{3}$.*
(3) *If M is any multitrace of $\mathbf{S}$ and $M \cup \{z\}$ is strongly connected, then $M \cup \{z\}$ is also a multitrace.*
(4) *If M is a strongly connected multitrace and $1 < |M| < |S|$, then there is an element $z \in S - M$ such that $M \cup \{z\}$ is strongly connected.*

Proof. For (1), assume instead that for every $z \in S - N$ it is the case that $x \to z$ for all $x \in N$ or $z \to x$ for all $x \in N$. Then any polynomial of the form $p(x) = sx = xs$, $s \in S$, is either constant on N or maps N into itself, implying that N is a congruence class. This is impossible if $\mathbf{S}$ is simple and $1 < |N| < |S|$. Thus there is a $z \in S - N$ such that $x \to z$ for some $x \in N$ and $z \to y$ for some $y \in N$.

For (2) start with M equal to some trace. Since the tournament multiplication is a semilattice operation on any 2-element subset, it follows from the structure of traces that M has type $\mathbf{3}, \mathbf{4}$ or $\mathbf{5}$. This implies that M has 2 elements, say $M = \{a, b\}$, where we assume $a \to b$. Since $1 < |M| = 2 < |S|$, item (1) guarantees that there is some $z \in S - M$ such that either $a \to z \to b$ or $b \to z \to a$. In the latter case, $M \cup \{z\}$ is strongly connected, establishing the first statement of (2). To complete the proof of that statement in the former case, observe that if $a \to z \to b$ then $\{a, b\}z = \{a, z\}$ is a nonsingleton polynomial image of a trace, so is another trace. Hence the set

$$N = \{s \in S \mid a \to s \text{ and } \{a, s\} \text{ is a trace}\}$$

has at least 2 elements and does not contain a. By item (1) there exist $u, v \in N$ and $z' \in S - N$ such that $u \to z' \to v$. Since $a \to u \to z'$ we have $a \neq z'$. If $a \to z'$, then $\{a, v\}z' = \{a, z'\}$ is a trace, so $z' \in N$, a contradiction. Thus we must have $z' \to a$, in which case $z' \to a \to u \to z'$ is a directed triangle containing a trace $M' = \{a, u\}$. This trace is a multitrace for which there is an element $z' \in S - M'$ such that $M' \cup \{z'\}$ is strongly connected, completing the proof of the first statement in item (2). In either case of our argument we produced a directed triangle $a \to b \to z \to a$ containing a trace $\{a, b\}$, so it is easy to see that the type of $\mathbf{S}$ is $\mathbf{3}$ (Boolean type). This is because the tournament multiplication is a semilattice operation on M while the polynomial $q(x) = ((xz)a)b$ is Boolean complementation on M.

For (3), note that if $A = p(T^m)$ and $B = q(T^n)$ are multitraces, then the complex product $AB = \{ab \mid a \in A, b \in B\}$ is also a multitrace, since $AB = r(T^{m+n})$ for $r(\mathbf{xy}) = p(\mathbf{x}) \cdot q(\mathbf{y})$. Moreover, any singleton set is a multitrace, being the image of a constant unary polynomial. Thus, if M is a multitrace, so are the complex products $M\{z\}, M(M\{z\}), M(M(M\{z\}))$, etc. We argue that this is an increasing sequence of sets which terminates at $M \cup \{z\}$ whenever $M \cup \{z\}$ is strongly connected.

Since $M \cup \{z\}$ is strongly connected, there exists $m \in M - \{z\}$ such that $z \to m$, equivalently $z = mz$. Thus, $\{z\} \subseteq M\{z\}$. Multiplying both sides of this inclusion by M repeatedly yields $M\{z\} \subseteq M(M\{z\}) = M^2\{z\}$, then $M^2\{z\} \subseteq M^3\{z\}$, etc. Thus the multitraces $M^i\{z\}$ increase with i. They are contained in $M \cup \{z\}$ since this set is a subalgebra of $\mathbf{S}$. If $X := \bigcup_i M^i\{z\}$, then $X = M^j\{z\}$ for some large j, which makes X a multitrace. By construction we have $MX = X$, so there is no directed edge from $M - X$ into X. Since $z \in X$, there can be no directed edge from the set $(M \cup \{z\}) - X = M - X$ into X. But $M \cup \{z\}$ is strongly connected and X is a nonempty subset, so this forces $M \cup \{z\} = X =$ a multitrace.

For (4), apply (1). □

Items (2) and (3) of this lemma produce a nontrivial strongly connected multitrace, while items (3) and (4) allow one to grow this multitrace without restriction until we reach S. Since the type of $\mathbf{S}$ is $\mathbf{3}$, we obtain from Theorem 1 the desired result.

Theorem 3. *A simple tournament with more than two elements is functionally complete.*

However, the advantage of multitraces is that they are a 'local' tool; they may be applied to minimal congruences as easily as to simple algebras. All the arguments of Lemma 2 apply to the setting of minimal congruences, hence:

Theorem 4. *If α is a minimal congruence of a tournament, then every α-class is a multitrace. If some α-class has at least* 3 *elements, then the type of $\langle 0, \alpha \rangle$ is* $\mathbf{3}$.

Minimal congruences of type $\mathbf{3}$ whose classes are multitraces are functionally complete in the sense that, if $U_0, U_1, \ldots, U_n$ are congruence classes, then any function $f : U_1 \times \cdots \times U_n \to U_0$ can be interpolated by a polynomial. (Theorem 3.12 of [2] proves this when all U_i are equal, but it is easy to see that the statement holds without that assumption.)

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References

[1] Hobby, D., McKenzie, R.: The Structure of Finite Algebras. Contemporary Mathematics **76**, American Mathematical Society, Providence, RI, 1988

[2] Kearnes, K.A., Kiss, E.W., Valeriote, M.A.: Minimal sets and varieties. Trans. Amer. Math. Soc. **350**, no. 1, 1–41 (1998)

[3] Pálfy, P.P.: Some functionally complete minimal clones. Acta Sci. Math. **73** no. 3-4, 487–495 (2007)

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