Let $R$ be a ring possibly without an identity. In $R$, define $a \circ b = a + b - ab$. It can be shown that this binary operation is associative, and that $(R, \circ)$ is a monoid with zero as the identity element. An element $a \in R$ is called left (resp. right) quasi-regular if $a$ has a left (resp. right) inverse in the monoid $(R, \circ)$ with identity. If $a$ is both left and right quasi-regular, we say that $a$ is quasi-regular. A set $I \subseteq R$ is called quasi-regular (resp. left or right quasi-regular) if every element of $I$ is quasi-regular (resp. left or right quasi-regular).

**Ex. 4.4** Define the Jacobson radical of $R$ by

$$\text{rad } R = \{a \in R : Ra \text{ is left quasi-regular}\}.$$ 

Show that $\text{rad } R$ is a quasi-regular ideal which contains every quasi-regular left (resp. right) ideal of $R$. (In particular, $\text{rad } R$ contains every nil left or right ideal of $R$.) Show that, if $R$ has an identity, the definition of $\text{rad } R$ here agrees with the one given in the introduction to this section.

**Solution.**

**Claim 1.** If $R$ has an identity 1, the map $\varphi : (R, \circ) \to (R, \times)$ sending $a$ to $1 - a$ is a monoid isomorphism. In this case, an element $a$ is left (right) quasi-regular if and only if $1 - a$ has a left (right) inverse with respect to ring multiplication.

**Proof.** The map is obviously a bijection with inverse $a \mapsto 1 - a$. To see that it is a homomorphism: $\varphi(a \circ b) = \varphi(a + b - ab) = 1 - a - b + ab = (1 - a)(1 - b) = \varphi(a)\varphi(b)$. The second part of the claim is trivial.

We can use this claim and the fact that in analysis we can write the inverse of $1 - a$ as an infinite geometric series to find quasi-inverses to elements in $(R, \circ)$, even when there is no identity in $R$.

**Claim 2.** If $ab$ is left quasi-regular, then so is $ba$.

**Proof.** Let $c$ be the left quasi-inverse of $ab$, i.e. $c \circ (ab) = 0$. Then using the previous claim we can find the left quasi-inverse of $ba$ to be $bca - ba$. We have to check that it is indeed a left quasi-inverse: $(bca - ba) \circ (ba) = bca - ba + ba - bcaba + baba = bca - bcaba + baba = b(c - cab + ab)a = b(c \circ (ab))a = 0$.

**Claim 3.** Any nilpotent element is quasi-regular.
Proof. Let $a$ be a nilpotent element, so $a^{2k} = 0$ for some positive integer $k$. As we have $a^{2k} = -a^{2k-1} + a^{2k-1} + a^{2k-1} a^{2k-1} = (-a^{2k-1}) \circ a^{2k-1}$, we can write recursively that $0 = a^{2k} = (-a^{2k-1}) \circ \cdots \circ (a \circ a \cdots a)$. Because the operation $\circ$ is associative, we found a left quasi-inverse to $a$. We can find a right quasi-inverse by a similar method.

Claim 4. If a left ideal is left quasi-regular, then it is quasi-regular.

Proof. Let $I \subset R$ be a left quasi-regular left ideal, and $a \in I$. Then $a$ is left quasi-regular, so $c \circ a = c + a - ca = 0$ for some $c \in R$. But then $c = ca - a \in I$, so $c$ is also left quasi-regular and $d \circ c = 0$, for some $d \in R$. As $d = d \circ c \circ a = a$, we have that $a \circ c = 0$, and thus $a$ is right quasi-regular and also quasi-regular. As $a$ was an arbitrary element of $I$, $I$ is quasi-regular.

Proof of 4.4 First we prove that $\text{rad } R$ is an ideal. Let $a, b \in \text{rad } R$. To prove that $a + b \in \text{rad } R$, we need to show that for any $r \in R$, $r(a + b) = ra + rb$ is left quasi-regular. Let $c$ be the left quasi-inverse of $ra$ which exists because $a \in \text{rad } R$. Then $c \circ (ra + rb) = c + ra + rb - cra - crb = rb - crb = (r - cr)b$. The element $(r - cr)b$ has a left quasi-inverse $d$, because $b \in \text{rad } R$, thus $d \circ c$ is a left quasi-inverse of $ra + rb$ and $a + b \in \text{rad } R$. Now let $s \in R$ be arbitrary, we need to show that $s \circ (ra + rb) = rsa$ is left quasi-regular. $Rs$ is left quasi-regular, because $Ra$ is left quasi-regular. Any element $ras$ is left quasi-regular using Claim 2 and the fact that $sra$ is left quasi-regular. So $Rs$ is also left quasi-regular and $\text{rad } R$ is an ideal.

Next we prove that $\text{rad } R$ is quasi-regular. As $\text{rad } R$ is an ideal, thus a left ideal, because of Claim 4, we only need to prove that $\text{rad } R$ is left quasi-regular. Let $a \in \text{rad } R$, thus $a^2$ has a left quasi-inverse $c$. Then $c + ca - a$ is a left quasi-inverse to $a$: $(c + ca - a) \circ a = c + ca - a - ca - ca^2 + a^2 = c + a^2 - ca^2 = c \circ (a^2) = 0$. Thus $\text{rad } R$ is quasi-regular.

Next we prove that if $I$ is a quasi-regular left ideal then $I \subset \text{rad } R$. Let $a \in I$. Then $Ra \subset I$ as $I$ is a left ideal, and so $Ra$ is quasi-regular. Thus $a \in \text{rad } R$ and $I \subset \text{rad } R$.

Now let $I$ be a quasi-regular right ideal. We need to prove that $I \subset \text{rad } R$. Let $a \in I$, then $aR \subset I$, so for any $r \in R$, $ar$ is quasi-regular. But then using Claim 2 we have that $ra$ is left quasi-regular, so $a \in \text{rad } R$ and $I \subset \text{rad } R$. Thus indeed $\text{rad } R$ contains every quasi-regular left or right ideal of $R$.

Claim 3 shows that a nil left or right ideal is quasi-regular, thus indeed $\text{rad } R$ contains every nil left or right ideal of $R$.

It remains to show that if $R$ has identity, then the two definitions of $\text{rad } R$ are equivalent. Let $J$ be the intersection of all maximal left ideals of $R$. Then by Lemma 4.1, $a \in J$ is equivalent to the statement that $1 - ra$ is left-invertible (with respect to the multiplication of $R$) for any $r \in R$. By Claim 1 this is equivalent to $ra$ being left quasi-regular for any $r \in R$. But according to the definition of $\text{rad } R$ this is equivalent to $a$ being an element of $\text{rad } R$. Thus the two definitions of the Jacobson radical are indeed equivalent.